

Energy-resolved supercurrent between two superconductors

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In this paper I study the energy-resolved supercurrent of a junction consisting of a dirty normal metal between two superconductors. I also consider a cross geometry with two additional arms connecting the above mentioned junction with two normal reservoirs at equal and opposite voltages. The dependence of the supercurrent between the two superconductors on the applied voltages is studied. [S0163-1829(98)05833-0]

The proximity effect between a normal metal and a superconductor has been discussed long ago.¹ Often it is simply described by a spatial-dependent pairing correlation function Ψ which decays from a superconductor to normal metal. However, this description is too crude to provide a proper understanding of the phenomena observed at low temperatures in the mesoscopic systems which can nowadays be prepared in the laboratories. For example, the detailed description of the energy dependence of the effective barrier conductance and diffusion coefficient²⁻⁸ is crucial in understanding the behavior of the observed conductance between a normal metal (N) and a superconductor (S) at low voltages and temperatures.⁹⁻¹²

In this paper we study the spectral current density¹³ (see also Ref. 14) of a quasi-one-dimensional SNS junction in the dirty limit. This quantity (or, more precisely, the angular average of the one defined in Ref. 13) is defined as, at energy ϵ and position x ,

$$N_J(\epsilon, x) = \langle \hat{p}_x N(\hat{p}, \epsilon, x) \rangle, \quad (1)$$

where $N(\hat{p}, \epsilon, x)$ is the density of states for momentum direction \hat{p} at energy ϵ and position x . The angular brackets denote angular average. This quantity is thus the density of states weighted by a factor proportional to the current that each state carries (in a certain direction, here \hat{x}), and thus may also be appropriately referred to as the current-carrying density of states. This is obviously a useful quantity. For example at equilibrium, the (number) supercurrent J_s can be written as

$$J_s = -2v_f \int \frac{d\epsilon}{2} N_J(\epsilon, x) h_0(\epsilon) \quad (2)$$

where $h_0(\epsilon) = \tanh(\epsilon/2T)$ and v_f is the fermi velocity. The factor of 2 includes the contribution from the two spin directions. One convenient way to interpret this formula^{14,13} (see also Refs. 15,16) is to rewrite $h_0 = (1 - 2n)$ where $n(\epsilon)$, the occupation number, is given by the Fermi function at equilibrium. For example at $T=0$ Eq. (2) can be rewritten as [using the symmetry $N_J(\epsilon) = -N_J(-\epsilon)$]

$$J_s = 2v_f \int_{-\infty}^0 d\epsilon N_J(\epsilon) \quad (3)$$

and thus can be interpreted as the current due to the occupation of negative energy states. This can also be regarded as the diamagnetic response of the superconductor if one considers the $T=0$ state as one containing no quasiparticles. Similarly at finite temperature

$$J_s(T) = J_s(T=0) + 2v_f \int_{-\infty}^{\infty} d\epsilon N_J(\epsilon) [n(\epsilon, T) - n(\epsilon, T=0)] \quad (4)$$

which can be interpreted as the sum of the diamagnetic current and the correction due to the thermal redistribution of quasiparticles. In particular an important source of the decrease of the supercurrent as the temperature increases is due to the thermal excitations of quasiparticles from $\epsilon < 0$ to $\epsilon > 0$ states, which carry opposite current.

In the dirty limit, on which this paper will concentrate, N_J can be obtained from (see the Appendix for details)

$$N_J(\epsilon, x) = -\frac{N_f l}{6} Q(\epsilon, x), \quad (5)$$

where N_f is the density of states in the normal state, l is the mean free path, and Q is given by

$$Q \equiv \frac{1}{4\pi^2} \text{Tr}[\tau_3 (\hat{g}^R \partial \hat{g}^R - \hat{g}^A \partial \hat{g}^A)]. \quad (6)$$

Here $\hat{g}^{R,A}$ are the angular averaged of the retarded and advanced components of the quasiclassical Green's function. ∂ represents spatial derivative. The equilibrium (number) supercurrent is thus given by

$$J_s = \frac{N_f D}{2} \int d\epsilon Q h_0(\epsilon), \quad (7)$$

where $D \equiv v_f l/3$ is the diffusion coefficient. For an SNS junction with no electron-electron or electron-phonon interaction in the N region, Q is independent of the position x along the junction within that region.

The behavior of Q is easiest to understand in the limit of very short junction ($E_D \equiv D/L^2 \ll \Delta$, here L is the length of the junction and Δ is the superconducting gap) and small phase difference χ . In this case Q should be the same as that

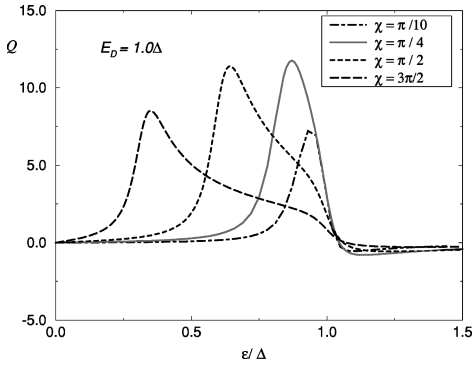


FIG. 1. Q (in units of $1/L$) for a short junction. $E_D = 1.0\Delta$. $\gamma = 0.05\Delta$.

of a bulk superconductor under a small phase gradient. The response of a dirty superconductor to a phase gradient or an external vector potential is well known.¹⁷ In this case one can show that the entire contribution to the supercurrent arises from states at $\epsilon = \Delta$, i.e., $Q \propto \delta(\epsilon - \Delta)$. In contrast, the ordinary density of states is given by $N(\epsilon) = N_f |\epsilon| / \sqrt{\epsilon^2 - \Delta^2}$. Under a small phase gradient, the gap for quasiparticle excitations persists and in particular there is no contribution to Q for energies within this gap.

An energy gap ϵ_g ($< \Delta$) also exists in general in an SNS junction (except phase difference $\chi = \pi$). This gap has been studied before in related situations.^{18–20} Associated with the existence of this (phase-dependent) gap is a relatively rapid change of \hat{g} as a function of energy (and phase difference). This has made the numerical calculation somewhat difficult. For convenience I will thus mostly concentrate on results where a small pair-breaking term γ has been included in the self-energy (see the Appendix). γ is usually chosen to be 0.05Δ , though occasionally results for $\gamma = 0$ will also be shown for comparison.

The behavior of Q for a relatively short junction is as shown in Fig. 1. At small phase differences Q is large only for ϵ near Δ . If γ were zero then Q would vanish for ϵ below a minigap ϵ_g . As the phase difference increases, the minigap decreases. Correspondingly the region of energy where Q is finite also moves down in energy, though it remains large in an energy region up to $\approx \Delta$.

For longer junctions, i.e. $L \gg \sqrt{D/\Delta}$ or, equivalently, $E_D \ll \Delta$, the behavior is somewhat different. At a given phase difference, the main region of energy where Q is significant

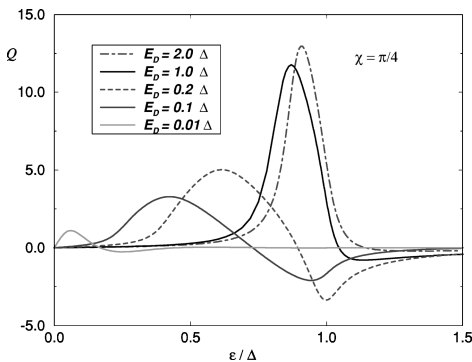


FIG. 2. Q for $\chi = \pi/4$ as a function of decreasing E_D . $\gamma = 0.05\Delta$.

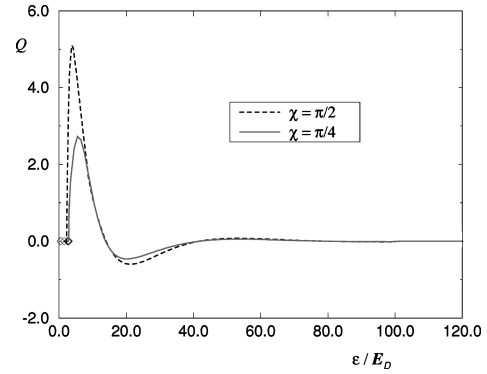


FIG. 3. Q for a long junction. $\Delta = 100E_D$. This result is for $\gamma = 0$.

is no longer of order Δ . An example for this evolution as a function of increasing length is as shown in Fig. 2. For a given phase difference, the energy where Q peaks shifts down in energy relative to Δ as L lengthens. This itself may not be surprising, and can be understood by analogy with the behavior of energy levels under a change in boundary condition in the normal state.

The more interesting feature is that a negative dip in Q appears at higher energies as the junction lengthens. For very long junctions, both the peak and the dip of Q move to energies of order (a few tens of times) E_D , with almost no features left near Δ (Fig. 3). This negative dip has recently been speculated to exist.²¹

In the above I have assumed that the contacts between the normal metal N and the superconducting reservoirs S are perfect. If potential barriers exist between the N and S regions, then Q decreases in magnitude, with a corresponding decrease in the energy where Q peaks. The features discussed above survives for moderate barrier resistance R_b between N and S . An example of how Q evolves as R_b increases is as shown in Fig. 4.

From the ideas presented above obviously one can affect the current flowing between the two superconducting reservoirs by changing the occupation of the quasiparticle states. Temperature is an obvious candidate. This gives the well-known reduction of the supercurrent as a function of increasing temperature. An alternative way is to create a nonequilibrium situation.²¹ Here I shall consider a steady state situation with the advantage that it is easy to analyze. The

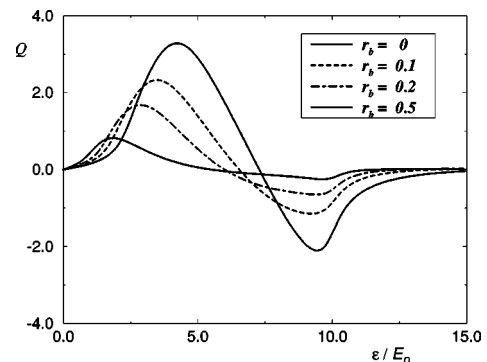


FIG. 4. Q for $\chi = \pi/4$, $\Delta = 10E_D$ as a function of increasing r_b , the ratio of the barrier resistance R_b to that of the normal metal, i.e., $r_b \equiv R_b(2N_fDS/L)$. Here S is the area. $\gamma = 0.05\Delta$.

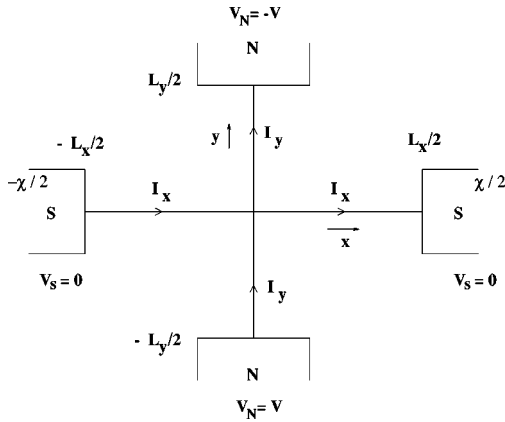


FIG. 5. The cross geometry. All “wires” connecting the reservoirs are assumed to be quasi-one dimensional.

setup is shown schematically in Fig. 5. Geometries closely related to this has been studied before^{22–27}. However, these references have concentrated on a different arrangement of voltages and/or other measurable quantities. Here I consider the case where the superconductors are at equal voltages, chosen to be zero. The normal reservoirs are at equal and opposite voltages $V_N = \pm V$. I shall study the dependence of the current between the superconducting reservoirs as a function of V .

First we should note that the presence of the side arms connected to the normal reservoirs affect the behavior of Q via the proximity effect. In order to facilitate later discussion, I plotted the quantity Q for this spatial geometry for the case $\Delta = 10E_D$ for two phase differences in Fig. 6. In this example I have assumed that the arms between the normal metal and the superconductor are symmetric and of equal length ($L_x = L_y = L$ in Fig. 5) and area S . Q is finite only for the x arms connecting the superconducting reservoirs, and is constant along them. Compared with the case without the side arms (Fig. 2), we see that the behavior of Q is somewhat different in the energy region $\epsilon < E_D$. This is because there is now no energy gap for quasiparticle excitations for any position within the N region on the cross, even for $\gamma = 0$. However, for $\epsilon > E_D$ the qualitative behavior of Q is almost the same as in the case without the side arms, except an overall reduction in magnitude.²⁸ In particular the sign change of Q remains.

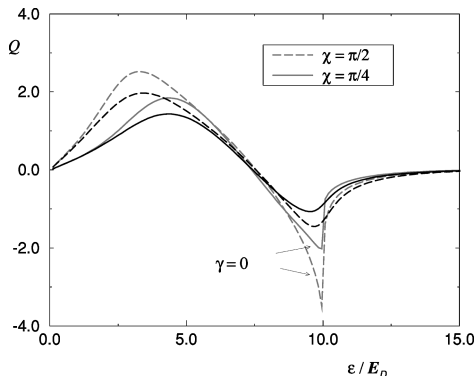


FIG. 6. The quantity Q (in units of L^{-1}) for the cross geometry of Fig. 5. $\Delta = 10E_D$. $\gamma = 0.05\Delta$. Results for $\gamma = 0$ are also shown for comparison.

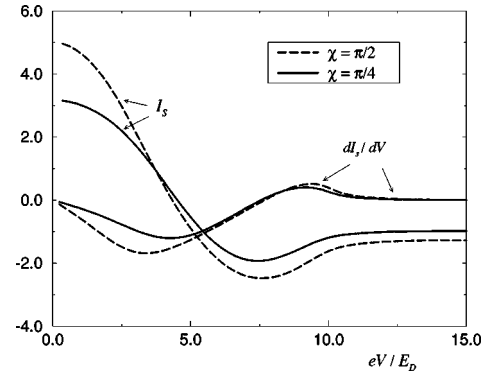


FIG. 7. dI_s/dV versus V for the cross geometry of Fig. 5. $E_D = 0.1\Delta$, $\gamma = 0.05\Delta$. Also shown are I_s as functions of V . Energies (eV) are in units of E_D and I_s is in units of $N_f E_D D S L^{-1}$.

Obviously if $V = 0$ a supercurrent I_s only flows between the superconducting reservoirs, whereas there is no current flowing in or out of the normal reservoirs. At $V \neq 0$ current is in general finite at any position on the two arms. I shall denote the currents as I_x and I_y . Neither I_x nor I_y are position dependent; moreover, the current flowing in and out of the normal reservoirs are equal on the one hand and those of the superconducting reservoirs equal on the other. One can therefore regard the current I_x (I_y) as simply flowing between the superconducting (normal) reservoirs. I shall thus continue to call I_x (I_y) the supercurrent I_s . I shall consider how this I_s is modulated by the voltage V . All results presented below are for $T = 0$.

I shall concentrate on an example in the most interesting regime, where $\Delta \sim 10E_D$. The result for dI_s/dV at $E_D = 0.1\Delta$ is as shown in Fig. 7. In this parameter range dI_s/dV at a voltage V is approximately equal to $-(N_f D S) Q$ at the corresponding energy $\epsilon = eV$. (c.f. the corresponding Q in Fig. 6) Also shown is the value of I_s at the value V , obtained by adding the integral of dI_s/dV to the equilibrium value of I_s . Note in particular that for large V , the supercurrent actually has an opposite sign than the equilibrium one, thus producing a “ π -junction.” (See Refs. 22,27,29.)

To understand dI_s/dV , it is necessary to know the behavior of the distribution functions for the quasiparticles. (see the Appendix for the technical details). I shall denote these functions on the x (y) arms as $h_{0,3}(x)$ [$h_{0,3}(y)$] etc. Since we are at $T = 0$, a small change of the voltage at V will affect only the occupation numbers at $\epsilon = \pm eV$. In Fig. 8 I have plotted the change of the distribution functions $\delta h_{0,3}$ at a relatively low energy when V is increased from below to above $eV = \epsilon$. At the S reservoirs ($x = \pm L_x/2$) $\delta h_{0,3} = 0$ by choice, whereas at the normal reservoirs ($y = \pm L_y/2$) $\delta h_3 = \pm 1$ and $\delta h_0 = -1$. The behavior of $\delta h_{0,3}$ is easy to understand in this low-energy limit, where one can ignore the superflow (Q), the coupling between the diffusion of the two distribution functions ($M_{03} = -M_{30}$ are small), and where the diffusivity for the particles ($\propto M_{33}$) reduces to that of the normal state. Thus [see Eq. (A5)] $\delta h_3(y)$ is linear in y and $\delta h_3(x) \approx 0$. Since there is an energy gap at the S reservoir, the effective diffusivity of the energy ($\propto M_{00}$) is suppressed near $x = \pm L_x/2$. Thus δh_0 only has small gradients and hence $\delta h_0 \approx -1$ everywhere except near $x = \pm L_x/2$. In the

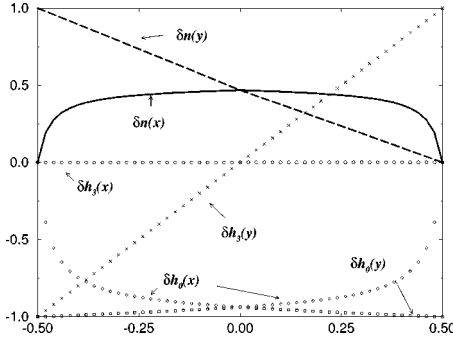


FIG. 8. The distribution functions at $\epsilon=0.24E_D$ as functions of x/L or y/L for the cross geometry with parameters as in the last figure. $\chi=\pi/4$.

language of the more familiar occupation number $n(\epsilon)=\{1-[h_0(\epsilon)+h_3(\epsilon)]\}/2$, in this $\epsilon\rightarrow 0$ limit $\delta n(y)$ is linear in y and thus $\delta n=1/2$ at $(x,y)=(0,0)$. $\delta n(x)$ is almost constant and $\approx 1/2$ near $x\approx 0$ and only changes rapidly to 0 near the S reservoirs. A finite ϵ provides a correction to the above picture as can be also seen from Fig. 8.³⁰ The values of δn at $\epsilon=-eV$ can be obtained by symmetry since $n(-\epsilon)=\{1+h_0(\epsilon)-h_3(\epsilon)\}/2$. At this energy $\delta n(y)$ changes from 0 at $y=-L_y/2$ to -1 at $y=L_y/2$ and $\delta n(x)\approx -1/2$ near the center of the cross.

If $\delta n(x)$ were exactly $\pm 1/2$ at $\epsilon=\pm eV$ and if one ignores the fact that δn is actually x dependent, with Eq. (A6) [or the equivalence of Eq. (4)] for the current it is obvious that dI_s/dV will be equal to $-N_f D Q S$ at the corresponding energy. If this is the case then at large V the current I_s would be exactly zero. However, the actual current consists of both the supercurrent and the contributions from the gradients of distribution functions [see Eq. (A6)]. Moreover $\delta n(x)$ is not exactly $\pm 1/2$ even at $x=0$ when ϵ is finite. Thus the above approximation becomes worse as the energy increases, making in general the magnitude of dI_s/dV somewhat smaller than that of $N_f D Q S$. In particular the positive hump of dI_s/dV at large V (near $10E_D$ in this particular example) is smaller than the corresponding dip in Q near that energy. Hence at large voltages I_s becomes negative as noted above.³¹

In conclusion, in this paper I studied the current-carrying density of states of a junction consisting of a dirty normal metal between two superconductors. I have also considered the dependence of the supercurrent between the two superconductors on the applied voltages at the normal reservoirs of a cross geometry.

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APPENDIX

In this appendix I summarize some basic equations for easy reference (see, e.g., Ref. 6) The basic equation to be solved is the Usadel equation

$$[\epsilon\tau_3, \check{g}] + \frac{D}{\pi} \partial_\mu (\check{g} \partial_\mu \check{g}) = 0 \quad (\text{A1})$$

together with the normalization condition $\check{g}^2 = -\pi^2 \check{1}$ governing the angular averaged matrix Green's function \check{g} which in turn has $\hat{g}^{R,A,K}$, the retarded, advanced, and Keldysh matrix Green's functions, as its components. Here ϵ is the energy. The pair breaking mentioned in the text is simulated by $\epsilon \rightarrow \epsilon + i\gamma$ where $\gamma > 0$.³²

\hat{g}^R can be parametrized as $-i\pi(\cos\theta\tau_3 - \sin\theta\cos\phi\sigma_2\tau_1 + \sin\theta\sin\phi\sigma_2\tau_2)$. \hat{g}^A can be related to \hat{g}^R by symmetry. The variables θ and ϕ obey the differential equations

$$2i(\epsilon+i\gamma)\sin\theta + D[\partial_x^2\theta - \sin\theta\cos\theta(\partial\phi)^2] = 0 \quad (\text{A2})$$

and

$$\partial(\sin^2\theta\partial\phi) = 0 \quad (\text{A3})$$

with the boundary conditions that they assume their equilibrium values at the reservoirs. For a normal reservoir $\theta=0$, while at a superconducting reservoir $\cos\theta = -i(\epsilon+i\gamma)/D$ where $D \equiv \sqrt{\Delta^2 - (\epsilon+i\gamma)^2}$. ($\gamma \rightarrow 0_+$ if pair breaking is not included.)

Q , related to the current-carrying density of states as discussed in the text, is given by

$$Q = 2 \text{Im}[\sin^2\theta\partial\phi]. \quad (\text{A4})$$

It is thus then position independent within any wire by Eq. (A3), a result which can also be directly obtained from the definition (6) for Q and by taking the appropriate trace of Eq. (A1). Q obeys the symmetry $Q(-\epsilon) = -Q(\epsilon)$.

\hat{g}^K is expressed via the distribution function \hat{h} as $\hat{g}^R\hat{h} - \hat{h}\hat{g}^A$ where \hat{h} can be chosen diagonal: $\hat{h} = h_0\hat{\tau}_0 + h_3\hat{\tau}_3$. The distribution functions obey the equations

$$\partial[Qh_0 + (M_{33}\partial h_3 + M_{30}\partial h_0)] = 0 \quad (\text{A5})$$

and the equation with $0 \leftrightarrow 3$. These two equations express respectively the conservation of particle and energy at each individual energy (due to the absence of interactions). The (real) M_{ij} coefficients are defined by $M_{ij} \equiv \delta_{ij} + (1/4\pi^2) \text{Tr}[\hat{g}^A\tau_i\hat{g}^R\tau_j]$.

The distribution functions at the reservoirs are given by their equilibrium values. At voltage V , $h_0(\epsilon) = \{\tanh[(\epsilon+eV)/2T] + \tanh[(\epsilon-eV)/2T]\}/2$ and $h_3(\epsilon) = \{\tanh[(\epsilon-eV)/2T] - \tanh[(\epsilon+eV)/2T]\}/2$. Thus, at $T=0$, when the voltage sweeps through the corresponding energy $\epsilon=eV$, the distribution functions at $y=-L_y/2$ change by $\delta h_0 = -1$ and $\delta h_3 = -1$. At the point where the voltage is $-V$ ($y=L_y/2$), $\delta h_0 = -1$, and $\delta h_3 = 1$ (see Fig. 8).

The total number current density is given by

$$J^N = \frac{N_f D}{2} \int d\epsilon [Qh_0 + (M_{33}\partial h_3 + M_{30}\partial h_0)]. \quad (\text{A6})$$

The three terms represent, respectively, the contributions from occupation of current-carrying states, ordinary diffusion (with a modified diffusion coefficient), and an extra contribution due to broken particle-hole symmetry.

If a potential barrier exists, there will be discontinuities of the parameters θ , ϕ across the barrier. The appropriate

boundary conditions are derived from³³

$$(2N_f D S) \check{g} \partial_\mu \check{g} = \frac{1}{2R_b} [\check{g}(x_{b-}), \check{g}(x_{b+})] \quad (\text{A7})$$

where R_b is the resistance of the barrier at x_b .

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- ³⁰This correction in particular becomes large if $\epsilon \gg \Delta$. It is easy to see that for the specific example given where the x and y arms have identical resistivity and length, then $\delta n(x \approx 0) \rightarrow \pm 1/4$ for $\epsilon \gg \Delta$.
- ³¹From the arguments above it is clear that for the approximate equality between dI_S/dV and $-N_f D Q S$ at low energies to hold, it is essential that the two y arms connecting the junction to the N reservoirs are symmetric. Neither $L_x = L_y$, nor the equality of the resistivity between the x and y arms are necessary. In particular, one can for example decrease I_y by increasing the resistivity or the lengths of the y arms, though this will decrease the width of the energy region where the above mentioned approximation remains good.
- ³²This γ which I have included is somewhat different from magnetic scattering. For magnetic scattering the term that should be added is actually $i\gamma \cos\theta$. In this case a minigap ϵ_g will still exist, where for energies below ϵ_g , the Usadel equations allow solutions with $\text{Re } \theta = \pi/2$ and $\text{Im } \phi = 0$, with $i\gamma \cos\theta$ real.
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