# Antiferromagnetic hedgehogs with superconducting cores

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Excitations of the antiferromagnetic state that resemble antiferromagnetic hedgehogs at large distances but are predominantly superconducting inside a core region are discussed within the context of Zhang's SO(5)-symmetry-based approach to the physics of high-temperature superconducting materials. Nonsingular, in contrast with their hedgehog cousins in pure antiferromagnetism, these texture excitations are what hedgehogs become when the antiferromagnetic order parameter is permitted to ''escape'' into superconducting directions. The structure of such excitations is determined in a simple setting, and a number of their experimental implications are examined. [S0163-1829(98)04733-X]

## I. INTRODUCTION

In Zhang's SO(5)-symmetry-based approach to the physics of high-temperature superconducting materials,[1] the local state of the system at the spatial point $\mathbf{r}$ is characterized by the orientation of a five-component unit vector $\mathbf{n}(\mathbf{r})$. Orientations for which $\Sigma_{a=1}^{3}(n^a)^2=0$ are purely superconducting, the orientation of $\mathbf{n}$ in the (two-dimensional) 4-5 hyperplane determining the phase of the complex superconducting order parameter. Orientations for which $\Sigma_{a=4}^{5}(n^a)^2=0$ are purely antiferromagnetic, the orientation of $\mathbf{n}$ in the (three-dimensional) 3-4-5 hyperplane determining the direction in real space of the antiferromagnetic (i.e., Néel) vector order parameter. The novelty of Zhang's approach lies in its assembling of these two order parameters into a unified order parameter $\mathbf{n}$, and the consequent possibility of orientations of $\mathbf{n}$ that do not lie wholely in one or other of the superconducting and antiferromagnetic subspaces, instead simultaneously containing components from both subspaces and, hence, characterizing regions that are at once partially superconducting and partially antiferromagnetic. The purpose of the present paper is to point out a simple but potentially interesting property of this model: in antiferromagnetic regions of the phase diagram this model supports three-(spatial)-dimensional antiferromagnetic hedgehog configurations that find it energetically favorable to have superconducting cores, as depicted schematically in Fig. 1.

It should be noted that the subject of the present paper is, loosely speaking, conjugate to that of a recent one[2] in which it was shown that, within the SO(5) approach, the cores of vortices in the superconducting order parameter should not be singular, the mechanism for the evasion of a singularity being escape from the two superconducting dimensions into the three antiferromagnetic ones.

## II. ANTIFERROMAGNETIC HEDGEHOGS WITH SUPERCONDUCTING CORES

What we mean by antiferromagnetic hedgehog configurations with superconducting cores are energetically stationary spatial configurations of the order parameter $\mathbf{n}(\mathbf{r})$ having the following properties. (i) Far from the (arbitrarily located) center, the configuration $\mathbf{n}(\mathbf{r})$ closely resembles a (nonsuperconducting) antiferromagnetic hedgehog (i.e., a point defect in which the Néel vector points radially away from the center, or some global SO(3) rotation of this configuration) and the quantity $\sin^2\chi\,[\equiv\Sigma_{a=4}^{5}(n^a)^2]$, which measures the degree of superconducting order (without regard to its phase), is small. Correspondingly, the complement $\cos^2\chi\,[\equiv\Sigma_{a=1}^{3}(n^a)^2]$, which measures the degree of antiferromagnetic order without regard to its orientation, is close to unity. (ii) As the center of the configuration is approached, however, the order parameter escapes from dimensions 1, 2, and 3 into dimensions 4 and 5, so that superconducting order is acquired at the expense of antiferromagnetic order. Said

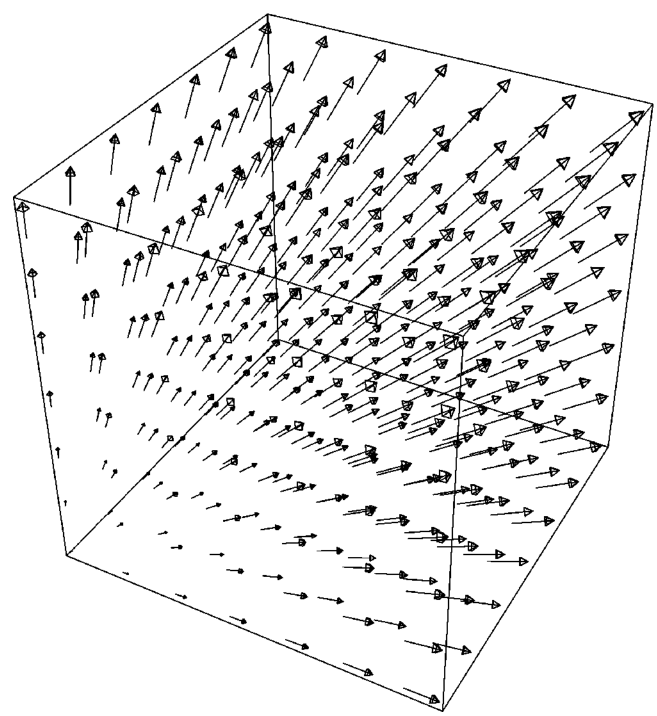

FIG. 1. An eighth of an antiferromagnetic hedgehog having a superconducting core (determined numerically). The local orientation of the vectors indicates the local orientation of the antiferromagnetism. Their local magnitude indicates the local strength of the antiferromagnetism and, hence, the local weakness of the superconductivity.

 

equivalently, the angle $\chi$ rotates from 0 to $\pi/2$ as the center of the configuration is approached. (In principle, more exotic hedgehog excitations are possible, in which the antiferromagnetic order varies more rapidly. For the sake of simplicity we shall primarily focus on the simplest class.) By this mechanism, the medium is able to remain nonsingular, and evade the (albeit finite) free-energy cost of the spatial gradient in the Néel vector (this gradient diverging as the center of the singular, purely antiferromagnetic, configuration is approached) at the expense of condensing locally into the ''wrong'' (i.e., superconducting) state. (iii) While not being stable global-energetically—the homogeneous antiferromagnetic configuration of course having a lower free energy—antiferromagnetic hedgehog configurations with superconducting cores do turn out to be energetically favorable, compared with purely antiferromagnetic hedgehogs, as we shall see below, at least when amplitude variations of the order parameter are inhibited. Presumably, such configurations are also *locally* energetically stable.[3] From the physical perspective, then, it would be quite intriguing if local regions of superconductivity were created by ''stressing'' the antiferromagnetism in regions of the phase diagram in which the stable homogeneous state is not superconducting. Moreover, the topological stability of these textures will tend to hold these ''stresses'' in place.

In the simplest version of Zhang's approach, the free energy $F$ of a three-dimensional sample in which the order parameter $\mathbf{n}(\mathbf{r})$ varies with position $\mathbf{r}$ is given by

$$F=\int d^3r\left\{\frac{\rho}{2}\sum_{\nu=1}^{3}\sum_{a=1}^{5}(\partial_\nu n^a)^2+\frac{g}{2}\sum_{a=4}^{5}(n^a)^2\right\},\quad (2.1)$$

where $\rho$ is the appropriate stiffness, $\nu$ $(=1,2,3)$ runs through the Cartesian spatial coordinates, and spatial anisotropies in the gradient term have been accommodated by coordinate rescalings. By tuning the chemical potential $\mu$ relative to its critical value $\mu_c$ (e.g., by doping), the parameter $g$ $[\propto(\mu_c^2-\mu^2)]$ is varied such that one moves from a region in which antiferromagnetic states are favored $(g>0)$ to a region in which superconducting states are favored $(g<0)$. This free energy is invariant under separate rotations on the three-dimensional antiferromagnetic and two-dimensional superconducting subspaces; invariance under arbitrary five-dimensional rotations is absent whenever $g\neq 0$. Thus, from any configuration $\mathbf{n}(\mathbf{r})$ one can obtain a configuration having the same free energy via the transformation

$$\mathbf{n}\rightarrow(R^{\mathrm{A}}\oplus R^{\mathrm{S}})\cdot\mathbf{n},\quad (2.2)$$

where $R^{\mathrm{A}}$ is a $(3\times 3)$ orthogonal matrix operative in the antiferromagnetic (i.e., $a=1,2,3$) sector (i.e., a magnetization rotation) and $R^{\mathrm{S}}$ is a $(2\times 2)$ orthogonal matrix operative in the superconducting (i.e., $a=4,5$) sector (i.e., a phase rotation), and the symbol $\oplus$ indicates that the five-dimensional operator is block-diagonal and composed of one three- and one two-dimensional block.

To calculate the structure of an isolated antiferromagnetic hedgehog with a superconducting core, let us make the hypothesis that components of $\mathbf{n}(\mathbf{r})$ in this configuration can be expressed in the form

$$\begin{pmatrix} n^1\\ n^2\\ n^3\\ n^4\\ n^5\end{pmatrix}=\begin{pmatrix}\cos\chi(r)\sin\theta\cos\phi\\ \cos\chi(r)\sin\theta\sin\phi\\ \cos\chi(r)\cos\theta\\ \sin\chi(r)\\ 0\end{pmatrix}.\quad (2.3)$$

Here, $r$, $\theta$, and $\phi$ are spherical polar spatial coordinates centered on the center of the configuration, and the function $\chi(r)$, which allows for interpolation between purely antiferromagnetic and purely superconducting values of the order parameter, is assumed to depend only on the radial distance from the center. This configuration is spherically symmetric, in the sense that for it we have

$$\mathbf{n}(R^{\mathrm{A}}\cdot\mathbf{r})=(R^{\mathrm{A}}\oplus I^{\mathrm{S}})\cdot\mathbf{n}(\mathbf{r}),\quad (2.4)$$

where $I^{\mathrm{S}}$ is the identity operation in the superconducting sector. By exchanging the radial variable $r$ for the dimensionless version $t$ (i.e., the radius, measured in units of the correlation length $\xi_\pi\equiv\sqrt{\rho/g}$ for the conversion of antiferromagnetic order into superconducting order) via

$$\chi(r)\equiv X(t),\quad (2.5a)$$

$$r\equiv\sqrt{\rho/g}\,t,\quad (2.5b)$$

and inserting the configuration (2.3) into the free energy (2.1), we find that the free energy is given by

$$F=\tilde{F}\int_0^\tau dt\left\{\frac{t^2}{2}\dot{X}(t)^2+\cos^2X(t)+\frac{t^2}{2}\sin^2X(t)\right\},\quad (2.6)$$

where $\tilde{F}\equiv 4\pi g(\rho/g)^{3/2}$, the overdot denotes a derivative with respect to $t$, and $\sqrt{\rho/g}\,\tau$ is a large-distance cutoff, introduced to render finite the otherwise linearly divergent free energy. Application of the calculus of variations to the functional $F$ then leads to the stationarity condition

$$t^2\ddot{X}+2t\dot{X}+\left(1-\frac{t^2}{2}\right)\sin 2X=0.\quad (2.7)$$

The relevant solutions of Eq. (2.7) are (i) $X(t)\equiv 0$ (i.e., the pure antiferromagnetic hedgehog, unescaped into the superconducting directions) and (ii) the solution in which $X(t)$ interpolates between $\pi/2$ and 0 as $t$ varies from 0 to $\infty$ (i.e., the antiferromagnetic hedgehog with a superconducting core). The precise form of the latter solution is readily found numerically, and is shown in Fig. 2. Its asymptotic behavior is $(\frac{1}{2}\pi-X)\sim t$ (for $t\ll 1$) and $X\sim\exp(-t)$ (for $t\rightarrow\infty$). The configuration corresponding to solution (ii) is depicted in Fig. 1.

To determine which of the solutions (i) or (ii) has the lower free energy, let us consider the quantity $\Delta F\equiv(F^{(\mathrm{ii})}-F^{(\mathrm{i})})$, where $F^{(\mathrm{i})}$ and $F^{(\mathrm{ii})}$, respectively refer to the free energy of solution (i) and of solution (ii). Then, by using Eq. (2.7), along with integration by parts, we find that

$$\Delta F=\tilde{F}\int_0^\infty dt\left\{\frac{t^2}{2}-1\right\}\{-X\cos X+\sin X\}\sin X,\quad (2.8)$$

where convergence at large $t$ permits the replacement of the upper limit by $\infty$. The numerical evaluation of this quantity

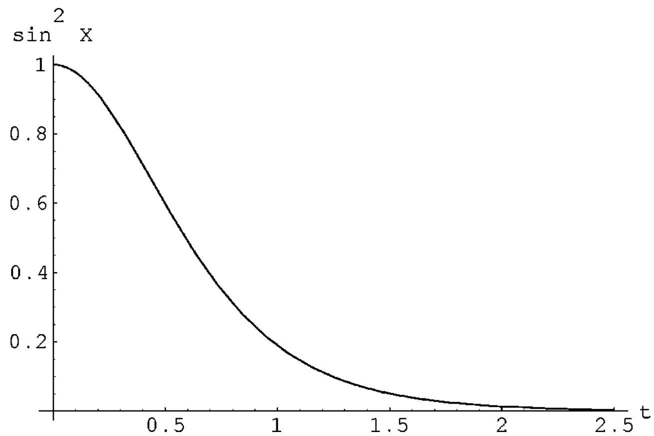


FIG. 2. The degree of superconductivity $\sin^2 X$ as a function of the scaled radial coordinate $t$ (determined numerically).

gives $\Delta F \approx -0.272\tilde{F}$. This indicates that it is energetically favorable for the order parameter in the core of an antiferromagnetic hedgehog to escape into the superconducting directions.

## III. CONSEQUENCES OF AMPLITUDE-SECTOR FLUCTUATIONS

As we have seen, in the setting of a model in which the amplitude of $\mathbf{n}(\mathbf{r})$ is constrained to be unity, there are hedgehog excitations that have superconducting cores. We now explore the issue of whether such excitations continue to exist in settings in which amplitude variations of $\mathbf{n}(\mathbf{r})$ are inhibited (i.e., are not prohibited, although they are suppressed energetically). Under such circumstances, it is possible—and may prove energetically favorable—for the core of the hedgehog to avoid antiferromagnetic gradient energy via the development of an amplitude-reduced purely antiferromagnetic core, rather than by escaping into the superconducting directions. To address this issue, we follow Arovas *et al.*[2] and consider a ''soft-spin'' generalization of the SO(5) model. Thus we consider a free energy of the form

$$F=\frac{\rho}{2}\int d^3r\{[\partial_r n(r)]^2+2r^{-2}n(r)^2\cos^2\chi(r)+n(r)^2[\partial_r\chi(r)]^2+\xi_\pi^{-2}n(r)^2\sin^2\chi(r)\}+a\int d^3r\left\{-\frac{1}{2}n(r)^2+\frac{1}{4}n(r)^4\right\}, \quad (3.1)$$

where $2a$ denotes the squared ''mass'' associated with the amplitude-sector fluctuations of $\mathbf{n}$, $n$ is the amplitude of $\mathbf{n}$, and we have restricted the discussion to (spatially) spherically symmetric configurations.

In the absence of amplitude fluctuations, the hedgehog with superconducting core has $n\equiv 1$ and $\chi$ varying from 0 to $\pi/2$ as the center of the texture is approached. The amplitude-reduced purely antiferromagnetic hedgehog excitation will be one for which $n$ vanishes at the center of the texture and grows to unity at large distances, and $\chi\equiv 0$. The stationarity condition for $n$, which determines the structure of the amplitude-reduced hedgehog, can be solved numerically, allowing us to obtain the function $n(r)$. Then, we may insert this back into Eq. (3.1) to obtain the free energy of the purely antiferromagnetic hedgehog with reduced-amplitude core. The free energies of the hedgehog with superconducting core and the purely antiferromagnetic hedgehog with amplitude-reduced core each must be defined with a long-distance cutoff to render them finite, but the difference between these two quantities is independent of this cutoff, and turns out to be given by

$$F_{\mathrm{AF}}-F_{\mathrm{SC}}\approx 4\pi\rho(0.272\sqrt{\rho/g}-1.454\sqrt{\rho/a}), \quad (3.2)$$

where the subscripts refer to the purely antiferromagnetic amplitude-reduced (AF) and superconducting-core (SC) hedgehogs. Thus, within the context of this model in which amplitude sector fluctuations are permitted, we find that the hedgehog with superconducting core will be energetically preferred when this quantity is positive, i.e., provided that $g<0.035a$ [or, equivalently, $\xi_\pi>5.35\xi_{\mathrm{a}}$, where $\xi_{\mathrm{a}}$ $(\equiv\sqrt{\rho/a})$ denotes the fluctuation correlation length for antiferromagnetic fluctuations]. Now, it is typical for $\xi_{\mathrm{a}}$ to be on the order of a lattice spacing for the cuprate materials, whereas $\xi_\pi$ is expected to grow as the superconducting phase boundary is approached from the antiferromagnetic state. Thus, one should anticipate that over a substantial portion of the antiferromagnetic part of the phase diagram, antiferromagnetic hedgehog excitations will have escaped-superconducting (rather than amplitude-reduced purely antiferromagnetic) cores.

## IV. TOPOLOGICAL CLASSIFICATION OF HEDGEHOG EXCITATIONS

We now turn to the issue of the topological classification of hedgehog excitations having superconducting cores, these excitations being nonsingular textures of the order-parameter field $\mathbf{n}(\mathbf{r})$. In pure antiferromagnets, the existence of singular, hedgehog point-defect excitations is expressed, mathematically, by the statement $\Pi_2(S_2)=Z$.[4,5] What this means is that mappings (provided by order-parameter configurations) of spheres in real space into the antiferromagnetic order-parameter space $S_2$ fall into homotopically inequivalent classes labeled by the integers (and combine according to integer arithmetic). Within the SO(5) approach, however, the nonsingular hedgehog texture excitations having superconducting cores are described by order parameter configurations $\mathbf{n}(\mathbf{r})$ that lie in the antiferromagnetic subspace $S_2$ at large distances from the core, but escape into the full order-parameter space $S_4$, as the core is approached. In order to complete the classification of these textures, then, we should ascertain whether or not there exist homotopically inequivalent textures that, at large distances, are homotopically equivalent and lie in the antiferromagnetic subspace $S_2$. The appropriate mathematical machinery for this task involves *relative homotopy groups* and *exact homotopy sequences*.[5,6] To implement this machinery, we consider mappings of cubes (in real space) such that the surface of the cube is mapped into $S_2$ whereas the interior of the cube is mapped into $S_4$. Such mappings are classified according to the relative homotopy group $\Pi_3(S_4,S_2)$. This group is readily computed by making use of the exact sequence of homomorphisms:

$$\Pi_3(S_4)\xrightarrow{\beta_3}\Pi_3(S_4,S_2)\xrightarrow{\gamma_3}\Pi_2(S_2)\xrightarrow{\alpha_2}\Pi_2(S_4). \tag{4.1}$$

Here, $\beta_3$, $\gamma_3$, and $\alpha_2$ denote mappings of the elements of the previous group in the sequence to elements of the following group, that, in general, are not isomorphic.[7] Now, as $\Pi_3(S_4)$ and $\Pi_2(S_4)$ are both the trivial group, the homomorphism $\gamma_3$ is, in fact, an isomorphism,[5,6] from which it follows that $\Pi_3(S_4,S_2)\cong\Pi_2(S_2)\cong Z$ and, thus, we find that there is no structure in $\Pi_3(S_4,S_2)$ beyond what was already present in $\Pi_2(S_2)$. The physical consequence of this result is that while hedgehog excitations fall into homotopically inequivalent classes, the possible nonsingular superconducting cores of a given class of hedgehog are homotopically equivalent to one another.

## V. RELATED STRUCTURES IN OTHER CONDENSED STATES

The notion of the conversion of singularities into textures via the escaping of order-parameters into additional directions has been realized in several other condensed matter settings. For example, (uniaxially) nematic liquid-crystalline media have long been known to exhibit a structure closely related to antiferromagnetic hedgehog configurations with superconducting cores. When confined to a cylinder that imposes homeotropic (i.e., perpendicular) boundary conditions on the nematic alignment, the system can evade the threading of the cylinder by a singularity because the order parameter orientation can escape from the radial plane into the axial direction.[8] This mechanism remains energetically favorable even for diamagnetic nematics in an axial magnetic field (for which escape costs condensation energy).

Superfluid $^3$He is another system that provides a rich array of topologically interesting textures.[9] The example having the most relevance to the present paper is that of hedgehog excitations in $^3$He-B. The order parameter for $^3$He-B is a complex-valued $3\times3$ matrix of the form $A_{\mu\nu}=e^{i\phi}R_{\mu\nu}(\hat{\mathbf{n}},\theta)$, where $\phi$ is a phase angle and $R_{\mu\nu}$ is a rotation matrix about the unit vector $\hat{\mathbf{n}}$ by an angle $\theta$. On long length scales, $\theta$ becomes fixed, due to a dipolar coupling, acquiring the value $\theta_L$, known as the Leggett angle, so that the low-energy degrees of freedom are expressed by the possible values of $\phi$ and the directions of $\hat{\mathbf{n}}$. Thus, the order-parameter space $G$ is given by $G=\mathrm{U}(1)\times S_2$, so that $\Pi_2(G)=\Pi_2(S_2)=Z$, so that the system may form hedgehogs with the unit vector $\hat{\mathbf{n}}$. On short length scales, however, $\theta$ can vary, so that the order-parameter space is effectively enlarged to U(1)×SO(3). We see that, as $\Pi_2[\mathrm{SO}(3)]=0$, over short distances there are no topologically stable point defects, so that hedgehogs in $^3$He-B have nonsingular cores.

A similar effect occurs in nematic liquid crystals,[10] where on large length scales the system is uniaxial, so that the relevant order-parameter space is $RP_2$ (i.e., the real projective plane constructed by identifying opposite points on $S_2$). On short length scales, however, the order-parameter space is enlarged to $S_4$, so that disclinations in the nematic order have nonsingular cores.

## VI. EXPERIMENTAL SIGNATURES OF HEDGEHOGS; CONCLUDING REMARKS

We now briefly consider some issues associated with antiferromagnetic hedgehogs having superconducting cores that might be relevant to experiments. These excitations should be present after performing a quench, from high temperature or high magnetic field, into the antiferromagnetic state. The number of excitations per unit volume should be higher for more rapid quenches. As time proceeds after the quench, the number of hedgehog excitations can decrease via their mutual annihilation, although this would require collisions of two or more hedgehogs. Presumably, this process can occur relatively slowly, at least at sufficiently low temperatures, so that one may anticipate regimes in which these excitations, once created, remain long enough for their consequences to be detected.

What experimental signatures might antiferromagnetic hedgehogs with superconducting cores yield? Let us suppose that a sufficiently high density of such excitations can be created, and that this density can be maintained for a sufficiently long time. Then one may crudely regard the excitations as providing a set of randomly located, randomly phased, superconducting inclusions.[11] These inclusions would not be unlike Aslamazov-Larkin paraconducting fluctuations,[12] except that they would be ''externally'' maintained and, therefore, could be much longer lived. One might hope that these inclusions would be detectable in electrical conductivity experiments, their presence leading to an enhancement of the conductivity. (One would need to account for scattering from the antiferromagnetic hedgehogs which, presumably, diminishes the conductivity.) This enhancement should be suppressed by magnetic fields, and by the decay of the excitations. Similarly, one might also envisage observing Andreev reflection from the superconducting inclusions (although capacitive charging effects may suppress this effect[13]).

An externally applied magnetic field will be partially screened by these inclusions, leading to a negative contribution to the magnetic susceptibility. To estimate the size of this effect, we approximate the hedgehog cores to be uniformly superconducting and spherical in shape.[14] In the regime where the London penetration depth $\lambda$ is much longer than the core radius $\xi$, this leads to a diamagnetic susceptibility contribution $\chi=-\delta\xi^5/40\pi\lambda^2$, where $\delta$ is the number of excitations per unit volume. One might also hope that the presence of antiferromagnetic hedgehogs with superconducting cores would be detectable via probes such as nuclear magnetic resonance, electromagnetic absorption and hedgehog/antihedgehog pair creation and, perhaps fancifully, scanning tunneling microscopy (e.g., with a magnetic tip).

In addition, these excitations should leave their fingerprint on the (staggered) magnetic structure factor $S(k)$, this factor being determined by $\mathbf{N}(\mathbf{k})$ (i.e., the Fourier transform of the antiferromagnetic Néel vector at the probing wave vector $\mathbf{k}$) via

$$S(k)\equiv V^{-1}\mathbf{N}(\mathbf{k})\cdot\mathbf{N}(\mathbf{k}), \tag{6.1}$$

where $V$ is the volume of the system. Specifically, for length scales that are long compared with the core size $\xi_\pi$ but short compared to the spacing between the hedgehogs $\delta^{-1/3}$, we expect $S(k)$ to have the conventional hedgehog form. However, for length scales that are short compared with the core size but long compared with the lattice spacing, we expect a reduction in $S(k)$ (and hence scattering), owing to the diminution of the Fourier amplitude of the antiferromagnetic moment at these scales. The computed hedgehog structure does indeed realize this scenario:

$$S(k)\sim\begin{cases}\delta\xi^6(k\xi)^{-6} & \text{for } \delta^{-1/3}\ll k\ll\xi_\pi^{-1},\\ \delta\xi^6(k\xi)^{-10} & \text{for } \xi_\pi^{-1}\ll k.\end{cases}\tag{6.2}$$

However, it should be noted that, being sensitive only to antiferromagnetic order, this particular probe does not directly ascertain whether or not the cores of hedgehog excitations are superconducting (except via the dependence of the size of the cores on the location of the system in the phase diagram).

If detected in experiments, antiferromagnetic hedgehogs with superconducting cores would provide striking evidence in support of Zhang's SO(5) approach to the physics of high-temperature superconducting materials. Their presence would corroborate the notion that superconducting excitations are essential low-energy excitations of the antiferromagnetic state. Moreover, it would prove rather intriguing to have at hand a physical system in which superconductivity is induced by the distortion of a thermodynamically preferred nonsuperconducting state.

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[7] This means that the homomorphisms are not necessarily *onto* (i.e., do not map onto every element of the following group) and are not necessarily *one-to-one* (i.e., some elements of one group may be mapped into more than one element of the following group).

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[13] We thank Boris Altshuler for alerting us to the issue of charging effects.

[14] For a discussion see Chap. 2 of the book by Tinkham cited in Ref. 12.