Quantum integrability and exact solution of the supersymmetric U model with boundary terms

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Quantum integrability is established for the one-dimensional supersymmetric U model with boundary terms by means of the quantum inverse-scattering method. The boundary supersymmetric U chain is solved by using the coordinate-space Bethe-ansatz technique and Bethe-ansatz equations are derived. This provides us with a basis for computing the finite-size corrections to the low-lying energies in the system.
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In recent years, there has been a considerable interest in exactly solvable lattice models with boundary fields and/or interactions.[1–3] One class of such models are one-dimensional boundary strongly correlated electron systems, which is of great importance due to their promising role in theoretical condensed-matter physics and possibly in high-T_c superconductivity.[4] Boundary conditions for such systems, which are compatible with integrability in the bulk, are constructed from solutions of the (graded) reflection equations (called boundary K matrices).[1] Work in this direction has been done for the Hubbard-like models[5–8] and for the supersymmetric t-J model.[9–11]

In this paper, we study integrable open-boundary conditions for the supersymmetric U model of strongly correlated electrons introduced in Refs. 12 and 13 and extensively investigated in Refs. 14–16. We will present a boundary supersymmetric U model and show that it can be derived from the quantum inverse-scattering method by modifying and generalizing Sklyanin's arguments, thus establishing the quantum integrability of the boundary model. In doing so, we encounter the following complication: Sklyanin's definition for a boundary Hamiltonian cannot apply since the supertrace of the boundary K matrices of zero spectral parameter is equal to zero for the current case. This is related to the fact that the supersymmetric U model has been constructed from the R matrix associated with the *typical* four-dimensional irreducible representation of $gl(2|1)$. Nevertheless we manage to solve this complication by introducing a new definition for a Hamiltonian. We then solve the boundary supersymmetric U model by the coordinate-space Bethe-ansatz approach and derive the Bethe-ansatz equations.

Let $c_{j,\sigma}^{\dagger}$ and $c_{j,\sigma}$ denote fermionic creation and annihilation operators with spin σ at site j, which satisfy the anticommutation relations given by $\{c_{i,\sigma}^{\dagger},c_{j,\tau}\}=\delta_{ij}\delta_{\sigma\tau}$, where $i,j=1,2,\ldots,L$ and $\sigma,\tau=\uparrow,\downarrow$. We consider the following Hamiltonian with boundary terms

$$H=\sum_{j=1}^{L-1} H_{j,j+1}^{Q}+B_{lt}+B_{rt}, \tag{1}$$

where $H_{j,j+1}^{Q}$ is the local Hamiltonian of the supersymmetric U model introduced in Ref. 13:

$$\begin{aligned} H_{j,j+1}^{Q}=&-\sum_{\sigma} (c_{j\sigma}^{\dagger}c_{j+1\sigma}+\text{H.c.})\exp(-\tfrac{1}{2}\eta n_{j,-\sigma}\\ &-\tfrac{1}{2}\eta n_{j+1,-\sigma})+\frac{U}{2}(n_{j\uparrow}n_{j\downarrow}+n_{j+1\uparrow}n_{j+1\downarrow})\\ &+t_p(c_{j\uparrow}^{\dagger}c_{j\downarrow}^{\dagger}c_{j+1\downarrow}c_{j+1\uparrow}+\text{H.c.})+(n_j+n_{j+1}), \end{aligned} \tag{2}$$

and B_{lt}, B_{rt} are boundary terms

$$B_{lt}=-\frac{2(U+2)}{U(2-\xi_-)}\left(\frac{2}{\xi_-}n_{1\uparrow}n_{1\downarrow}-n_1\right),$$

$$B_{rt}=-\frac{2(U+2)}{U(2-\xi_+)}\left(\frac{2}{\xi_+}n_{L\uparrow}n_{L\downarrow}-n_L\right). \tag{3}$$

In the above equations, $n_{j\sigma}$ is the number density operator $n_{j\sigma}=c_{j\sigma}^{\dagger}c_{j\sigma}$, $n_j=n_{j\uparrow}+n_{j\downarrow}$ and $t_p=U/2=\pm[1-\exp(-\eta)]$; $\xi_{\pm}$ are some parameters describing boundary effects.

Some remarks are order. As is seen from Eq. (3), B_{lt} (B_{rt}) is an inhomogenous combination of two terms contributing to the left (right) boundary conditions. The physical meaning of these terms in the context of strongly correlated electrons are the following. The first term is nothing but a boundary on-site Coulomb interaction and the second term is a boundary chemical potential.

We will establish the quantum integrability of the boundary supersymmetric U model (1) by showing that it can be derived from the quantum inverse scattering method. Let us recall that the Hamiltonian of the supersymmetric U model with the periodic boundary conditions commutes with the transfer matrix, which is the supertrace of the monodromy matrix $T(u)$,

$$T(u)=R_{0L}(u)\cdots R_{01}(u). \tag{4}$$

The explicit form of the quantum R matrix $R_{0j}(u)$ is given in Ref. 12. Here u is the spectral parameter, and the subscript 0 denotes the auxiliary superspace $V=C^{2,2}$. It should be noted that the supertrace is carried out for the auxiliary superspace V. The elements of the supermatrix $T(u)$ are the generators of two associative superalgebra $\mathcal{A}$ defined by the relations

$$R_{12}(u_1-u_2)T^1(u_1)T^2(u_2)=T^2(u_2)T^1(u_1)R_{12}(u_1-u_2), \tag{5}$$

where $X^1 \equiv X \otimes 1$, $X^2 \equiv 1 \otimes X$ for any supermatrix $X \in End(V)$. For later use, we list some useful properties enjoyed by the R matrix: (i) unitarity: $R_{12}(u)R_{21}(-u)=1$ and (ii) crossing-unitarity: $R_{12}^{st_2}(-u+2)R_{21}^{st_1}(u)=\tilde{\rho}(u)$ with $\tilde{\rho}(u)$ being a scalar function, $\tilde{\rho}(u)=u^2(2-u)^2/[(2+2\alpha-u)^2(2\alpha+u)^2]$. Throughout this letter, $\alpha=2/U$.

In order to describe integrable electronic models on open chains, we introduce an associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ defined by the R matrix $R(u_1-u_2)$ and the relations

$$R_{12}(u_1-u_2)\mathcal{T}_-^1(u_1)R_{21}(u_1+u_2)\mathcal{T}_-^2(u_2) = \mathcal{T}_-^2(u_2)R_{12}(u_1+u_2)\mathcal{T}_-^1(u_1)R_{21}(u_1-u_2), \quad (6)$$

$$R_{21}^{st_1 ist_2}(-u_1+u_2)\mathcal{T}_+^{1st_1}(u_1)R_{12}(-u_1-u_2+2)\mathcal{T}_+^{2ist_2}(u_2) = \mathcal{T}_+^{2ist_2}(u_2)R_{21}(-u_1-u_2+2)\mathcal{T}_+^{1st_1}(u_1) \times R_{12}^{st_1 ist_2}(-u_1+u_2), \quad (7)$$

respectively. Here the supertransposition st_μ ($\mu=1,2$) is only carried out in the μth factor superspace of $V\otimes V$, whereas ist_μ denotes the inverse operation of st_μ. By modifying Sklyanin's arguments,[1] one may show that the quantities $\tau(u)$ given by $\tau(u)=str[\mathcal{T}_+(u)\mathcal{T}_-(u)]$ constitute a commutative family, i.e., $[\tau(u_1),\tau(u_2)]=0$.

One can obtain a class of realizations of the superalgebras $\mathcal{T}_+$ and $\mathcal{T}_-$ by choosing $\mathcal{T}_\pm(u)$ to be of the form

$$\mathcal{T}_-(u)=T_-(u)\tilde{\mathcal{T}}_-(u)T_-^{-1}(-u),$$

$$\mathcal{T}_+^{st}(u)=T_+^{st}(u)\tilde{\mathcal{T}}_+^{st}(u)[T_+^{-1}(-u)]^{st} \quad (8)$$

with

$$T_-(u)=R_{0M}(u)\cdots R_{01}(u),$$

$$T_+(u)=R_{0L}(u)\cdots R_{0,M+1}(u), \quad \tilde{\mathcal{T}}_\pm(u)=K_\pm(u), \quad (9)$$

where $K_\pm(u)$, called boundary K matrices, are representations of $\mathcal{T}_\pm$ in Grassmann algebra.

We now solve Eqs. (6) and (7) for $K_+(u)$ and $K_-(u)$. For simplicity, let us restrict ourselves to the diagonal case. Then, one may check that the matrix $K_-(u)$ given by

$$K_-(u)=\frac{1}{\xi_-(2-\xi_-)}\mathrm{diag}(A_-(u),B_-(u),B_-(u),C_-(u)), \quad (10)$$

where $A_-(u)=(\xi_-+u)(2-\xi_--u)$, $B_-(u)=(\xi_--u)(2-\xi_--u)$, and $C_-(u)=(\xi_--u)(2-\xi_-+u)$ satisfy Eq. (6). The matrix $K_+(u)$ can be obtained from the isomorphism of the superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$. Indeed, given a solution $\mathcal{T}_-$ of Eq. (6), then $\mathcal{T}_+(u)$ defined by

$$\mathcal{T}_+^{st}(u)=\mathcal{T}_-(-u+1) \quad (11)$$

is a solution of Eq. (7). The proof follows from some algebraic computations upon substituting Eq. (11) into Eq. (7) and making use of the properties of the R matrix. Therefore, one may choose the boundary matrix $K_+(u)$ as

$$K_+(u)=\mathrm{diag}(A_+(u),B_+(u),B_+(u),C_+(u)) \quad (12)$$

with $A_+(u)=(2-2\alpha-\xi_+-u)(2\alpha+\xi_++u)$, $B_+(u)=(-2\alpha-\xi_++u)(2\alpha+\xi_++u)$, and $C_+(u)=(-2\alpha-\xi_++u)(2+2\alpha+\xi_+-u)$.

Now it can be shown that Hamiltonian (1) is related to the transfer matrix $\tau(u)$ (up to an unimportant additive constant):

$$H=-\frac{2(U+2)}{U}H^R,$$

$$H^R=\frac{\tau''(0)}{4(V+2W)}=\sum_{j=1}^{L-1}H_{j,j+1}^R+\frac{1}{2}K_-'^1(0) + \frac{1}{2(V+2W)}\{str_0[K_+^0(0)G_{L0}]+2\,str_0[K_+'^0(0)H_{L0}^R] + str_0[K_+^0(0)(H_{L0}^R)^2]\}, \quad (13)$$

where

$$V=str_0K_+'(0), \quad W=str_0[K_+^0(0)H_{L0}^R],$$

$$H_{i,j}^R=P_{i,j}R_{i,j}'(0), \quad G_{i,j}=P_{i,j}R_{i,j}''(0). \quad (14)$$

Here $P_{i,j}$ denotes the graded permutation operator acting on the ith and jth quantum spaces. Equation (13) implies that the boundary supersymmetric U model admits an infinite number of conserved currents that are in involution with each other, thus assuring its integrability. It should be emphasized that Hamiltonian (1) appears as the second derivative of the transfer matrix $\tau(u)$ with respect to the spectral parameter u at $u=0$. This is due to the fact that the supertrace of $K_+(0)$ is equal to zero. As we mentioned before, the reason for the zero supertrace of $K_+(0)$ is related to the fact that the quantum space is the four-dimensional *typical* irreducible representation of gl(2|1). A similar situation also occurs in the Hubbard-like models.[6]

Having established the quantum integrability of the boundary supersymmetric U model, we now solve it by using the coordinate-space Bethe-ansatz method. Let us assume that the eigenfunction of Hamiltonian (1) has the form

$$|\Psi\rangle=\sum_{\{(x_j,\sigma_j)\}}\Psi_{\sigma_1,\ldots,\sigma_N}(x_1,\ldots,x_N)c_{x_1\sigma_1}^\dagger\cdots c_{x_N\sigma_N}^\dagger|0\rangle,$$

$$\Psi_{\sigma 1,\ldots,\sigma N}(x_1,\ldots,x_N) = \sum_P \epsilon_P A_{\sigma_{Q1},\ldots,\sigma_{QN}}(k_{PQ1},\ldots,k_{PQN})\exp\left(i\sum_{j=1}^N k_{P_j}x_j\right), \quad (15)$$

where the summation is taken over all permutations and negations of $k_1,\ldots,k_N$, and Q is the permutation of the N particles such that $1\leqslant x_{Q1}\leqslant\cdots\leqslant x_{QN}\leqslant L$. The symbol ϵ_P is a sign factor ± 1 and changes its sign under each ''mutation.'' Substituting the wave function into the eigenvalue equation $H|\Psi\rangle=E|\Psi\rangle$, one gets

$$A_{\ldots,\sigma_j,\sigma_i,\ldots}(\ldots,k_j,k_i,\ldots)=S_{ij}(k_i,k_j)A_{\ldots,\sigma_i,\sigma_j,\ldots}(\ldots,k_i,k_j,\ldots),$$

$$A_{\sigma_i,\ldots}(-k_j,\ldots)=s^L(k_j;p_{1\sigma_i})A_{\sigma_i,\ldots}(k_j,\ldots),\tag{16}$$

$$A_{\ldots,\sigma_i}(\ldots,-k_j)=s^R(k_j;p_{L\sigma_i})A_{\ldots,\sigma_i}(\ldots,k_j),$$

with $S_{ij}(k_i,k_j)$ being the two-particle scattering matrix and s^L,s^R the boundary scattering matrices:

$$S_{ij}(k_i,k_j)=\frac{\theta(k_i)-\theta(k_j)+ic\mathcal{P}_{ij}}{\theta(k_i)-\theta(k_j)+ic},$$

$$s^L(k_j;p_{1\sigma_i})=\frac{1-p_{1\sigma_i}e^{ik_j}}{1-p_{1\sigma_i}e^{-ik_j}},\tag{17}$$

$$s^R(k_j;p_{L\sigma_i})=\frac{1-p_{L\sigma_i}e^{-ik_j}}{1-p_{L\sigma_i}e^{ik_j}}e^{2ik_j(L+1)},$$

where

$$p_{1\sigma}\equiv p_1=-1+\frac{2(U+2)}{U}\frac{1}{2-\xi_-},$$

$$p_{L\sigma}\equiv p_L=-1+\frac{2(U+2)}{U}\frac{1}{2-\xi_+},$$

and $c=e^{\eta}-1$; $\mathcal{P}_{ij}$ is a spin permutation operator and the charge rapidities $\theta(k_j)$ are related to the single-particle quasimomenta k_j by $\theta(k)=\frac{1}{2}\tan(k/2)$.[14] Then, the diagonalization of Hamiltonian (1) reduces to solving the following matrix eigenvalue equation:

$$T_jt=t,\quad j=1,\ldots,N,\tag{18}$$

where t denotes an eigenvector on the space of the spin variables and T_j takes the form

$$T_j=S_j^-(k_j)s^L(-k_j;p_{1\sigma_j})R_j^-(k_j)R_j^+(k_j)s^R(k_j;p_{L\sigma_j})S_j^+(k_j)\tag{19}$$

with

$$S_j^+(k_j)=S_{j,N}(k_j,k_N)\ldots S_{j,j+1}(k_j,k_{j+1}),$$

$$S_j^-(k_j)=S_{j,j-1}(k_j,k_{j-1})\ldots S_{j,1}(k_j,k_1),\tag{20}$$

$$R_j^-(k_j)=S_{1,j}(k_1,-k_j)\ldots S_{j-1,j}(k_{j-1},-k_j),$$

$$R_j^+(k_j)=S_{j+1,j}(k_{j+1},-k_j)\ldots S_{N,j}(k_N,-k_j).$$

This problem may be solved using the algebraic Bethe-ansatz method. The Bethe-ansatz equations are

$$e^{ik_j2(L+1)}\zeta(k_j;p_1)\zeta(k_j;p_L)=\prod_{\beta=1}^{M}\frac{\theta_j-\lambda_\beta+i\frac{c}{2}}{\theta_j-\lambda_\beta-i\frac{c}{2}}\frac{\theta_j+\lambda_\beta+i\frac{c}{2}}{\theta_j+\lambda_\beta-i\frac{c}{2}},$$

$$\prod_{j=1}^{N}\frac{\lambda_\alpha-\theta_j+i\frac{c}{2}}{\lambda_\alpha-\theta_j-i\frac{c}{2}}\frac{\lambda_\alpha+\theta_j+i\frac{c}{2}}{\lambda_\alpha+\theta_j-i\frac{c}{2}}=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\frac{\lambda_\alpha-\lambda_\beta+ic}{\lambda_\alpha-\lambda_\beta-ic}\frac{\lambda_\alpha+\lambda_\beta+ic}{\lambda_\alpha+\lambda_\beta-ic},\tag{21}$$

where $\theta_j\equiv\theta(k_j)$ and $\zeta(k;p)=(1-pe^{-ik})/(1-pe^{ik})$. The energy eigenvalue E of the model is given by $E=-2\sum_{j=1}^{N}\cos k_j$, where we have dropped an additive constant.

In conclusion, we have studied integrable open-boundary conditions for the supersymmetric U model. Its quantum integrability follows from the fact that the Hamiltonian of the model on the open chain may be embedded into a one-parameter family of commuting transfer matrices. Moreover, the Bethe-ansatz equations are derived with the use of the coordinate-space Bethe-ansatz approach. This provides us with a basis for computing the finite-size corrections to the low-lying energies in the system, which in turn allows us to use the boundary conformal field theory technique to study the critical properties of the boundary. The details will be treated in a separate publication.

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