Topological aspects of shift in hierarchies of the fractional quantum Hall effect

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In this paper we aim at understanding the topological aspects of the shift vectors appearing in the fractional quantum Hall effect (FQHE) of hierarchy states. The topological features of the quasiparticles are visualized through the quantization procedure by acquiring gauge theoretic extensions. These gauge fields play the role of fibers at each space-time point. The origin of this shift lies in the interaction between the gauges and effectively it is related to the departure in the Berry phase factor in the presence and absence of strong external magnetic field. In fact we have pointed out that the shift visualizes the angular momentum of the quasiparticle in every hierarchy of the FQHE states. [S0163-1829(98)05128-5]

I. INTRODUCTION

The theory of Laughlin[1] for the ground state of elementary excitations in the fractional quantum Hall effect (FQHE) is quite successful in describing the filling fractions $\nu=1/m$ and $(1-1/m)$ with an odd integer. However, an extension of Laughlin's theory is required to explain the experimentally observed higher filling fractions of the form $\nu=p/q$. It was Haldane[2] who first gave a deeper insight into these observed subsidiary fractions (2/5,2/7,3/7,3/11,4/9,...) by using a hierarchical approach. The filling fraction at the nth level of this hierarchy is given by continued fraction[3]

$$\nu=\cfrac{1}{m+\cfrac{\alpha_1}{p_1-\cfrac{\alpha_2}{p_2\cdots\alpha^n/p^n}}}. \tag{1}$$

The ground state of each level of hierarchy becomes a condensate of elementary excitations of the previous one. In view of Haldane, Laughlin, and Halperin[4] the quasiparticles are anyons obeying fractional statistics. In the view of Wilczek[5] these composite particles are formed by charged particles tied to magnetic flux tubes in two dimensions. Here the Chern-Simon (CS) gauge theory is best suited. A notable recent attempt to provide an alternative hierarchical approach is the work of Jain[6] and his collaborators. The essence of his physics lies on the formation of composite fermions where he pointed out that the FQHE of fermions is nothing but the integer QHE (IQHE) of composite fermions. A more generalized theory was given by Wen[7] and Wen and Zee[8] through a concept of *topological order* to characterize FQH liquid by symmetric integer matrix K. A new quantum number shift vector has been introduced through spin vectors due to its coupling to the curvature of space.

In a recent communication[9] we have pointed out that the hierarchical theory of Jain is equivalent to that of Haldane. This idea is similar to the other recent work of Basu and Bandyopadhyay[10] where hierarchies of FQHE have been discussed through the angular momentum aspect and final results support the findings of Jain.

Here, in this paper, we shall aim at studying the origin of the appearance of shift vectors in hierarchies of the FQHE. From the topological point of view, we shall find out the physics of the coupling between the orbital spin and the curvature of the space in a relativistic framework. We have considered here that the topological features of quasiparticles in hierarchies are visualized when through quantization it acquires two gauge-type extensions imparted by its own internal geometry and also from the strong external magnetic field.[11] Actually (coupling) interaction between these two gauges is visualized through the Berry phase in terms of the difference of magnetic charges known as the shift vector.

II. TOPOLOGICAL FEATURES OF HALL PARTICLES

In the quantum Hall effect the quantization of particles involves the appearance of gauge fields. When we consider the relativistic generalization of the stochastic quantization procedure, we find that in stochastic phase space, a relativistic quantum particle appears as a gauge theoretically extended one, so that we can write for the position and momentum operator Q_μ and P_μ (Ref. 12)

$$Q_\mu=-i\left(\frac{\partial}{\partial p_\mu}+B_\mu\right), \tag{2}$$

$$P_\mu=i\left(\frac{\partial}{\partial q_\mu}+\tilde{B}_\mu\right), \tag{3}$$

where q_μ (p_μ) denote the mean position (momentum) of the particle and $B_\mu,\tilde{B}_\mu$ correspond to gauge fields. In the case of a fermion B_μ and $\tilde{B}_\mu$ are matrix-valued SL$(2c)$ gauge fields and in the case of bosons these are just Abelian fields.

From the geometrical point of view the introduction of the additional gauge degrees of freedom in the phase space formulation deforms the symplectic structure when the one-form is given by[13]

$$W=p\,dq+\lambda \tag{4}$$

with

$$\lambda=B_\mu dp^\mu+\tilde{B}_\mu dq^\mu \tag{5}$$

for $B_\mu,\tilde{B}_\mu\in$ SL$(2c)$.

The deformation of symplectic structure is given by

$$F=D\lambda=d\lambda+1/2[\lambda,\lambda], \tag{6}$$

where F is the field strength two-form that is obtained in terms of gauge potential B_μ as

$$F_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+[B_\mu,B_\nu]. \tag{7}$$

This implies that the quantization enlarges the compact space S^2 to S^3 where S^3 acts as principle fiber bundle.

This analysis suggests that for a charged particle moving in an intense magnetic field, a process similar to quantization procedure may arise and the particle may acquire the quantum holonomy due to the topological features generated by the background magnetic field.

To this effect, following Friedman and others[14] we note that to describe a Hall particle at a filling factor $\nu=1/m$ on the surface of a 3D sphere in a radial (monopole) strong magnetic field, the topological Lagrangian will be

$$L_\theta=-\frac{\theta}{16\pi^2}\,\mathrm{Tr}^*F_{\mu\nu}F_{\mu\nu}, \tag{8}$$

where θ is the coupling constant associated with filling factor $\nu=1/m$ through Hall conductivity $g(\theta=g/c^2)$. It is noted that the term $-1/16\pi^2\ \mathrm{Tr}^*\ F_{\mu\nu}F_{\mu\nu}$ is a total divergence and can be written as $\partial_\mu\Omega_\mu$ where Ω_μ is the Chern-Simon secondary characteristic class defined by

$$\Omega_e^\mu=-1/16\pi^2\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[B_\nu F_{\alpha\beta}-2/3(B_\nu B_\alpha B_\beta)], \tag{9}$$

where the Pontryagin density is given by $P=\partial_\mu\Omega_\mu$, which gives rise to the topological index by $q=\int P d^4x$, which is the charge corresponding to the gauge field part related to the Pontryagin index q where we have

$$q=\int j_0^2 d^3x=\int \partial_\mu j_\mu^2 d^4x=\int_{\mathrm{surface}}\epsilon^{ijk}d\sigma_i F_{jk}^2 \qquad (i,j,k=1,2,3). \tag{10}$$

Visualizing F_{jk}^2 to be the magnetic-field-like components for the vector potential B_i^2, we see that q is actually associated with the magnetic pole strength for the corresponding field distribution. This helps us to identify the magnetic field B with the field strength F_{jk}^2 where we can write

$$B=-\tfrac{1}{2}\epsilon^{ij}F_{ij}^2, \tag{11}$$

where F_{ij}^2 is the second component of the SL(2,c) field strength matrix and hence corresponds to the U(1) gauge counterpart related to the magnetic field. Thus we observe that the effect of the magnetic field may be related to that of the Chern-Simons characteristic class in the effective action. In fact, the introduction of the Chern-Simons term modifies the axial vector current

$$\tilde{j}_\mu^5=j_\mu^5+2\hbar\Omega_\mu \tag{12}$$

when we have $\partial_\mu\tilde{j}_\mu^5=0$. A comparison with Eq. (12) suggests that j_μ^2 effectively represents the Chern-Simons term.[13]

It may be noted that the wave function given by $\phi(z_\mu)=\phi(x_\mu)+i\phi(\xi_\mu)$ can be treated to describe a particle moving in the external space-time having the coordinate x_μ with an attached *direction vector* ξ_μ. Thus the wave function should take into account the polar coordinates r, θ, ϕ along with the angle χ specifying the rotational orientation around the *direction vector* ξ_μ. For an extended body, θ, ϕ, and χ just represent the three Euler angles.

In an anisotropic space, these three Euler angles have their correspondence in an axisymmetric system where the anisotropy is introduced along a particular direction. It is to be noted that the axisymmetric system may be considered in studying the behavior of particles in two dimensions by putting them on the surface of this anisotropic three-dimensional manifold. In this space, the components of the linear momentum satisfy a commutation relation of the form

$$[p_i,p_j]=i\mu\epsilon_{ijk}\frac{x^k}{r^3}. \tag{13}$$

In such a space, the motion of a charged particle is equivalent to the motion of a charged particle in the field of a magnetic monopole where the conserved angular momentum is given by

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}-\mu\mathbf{r}. \tag{14}$$

In general, μ can take the values $\mu=0,\pm 1/2,\pm 1,\pm 3/2,\pm 2,\ldots$. Fierz[15] and Hurst[16] have extensively studied the spherical harmonics incorporating the terms μ. Following them, we can write

$$Y_l^{m,\mu}=(1+x)^{-(m-\mu)/2}(1-x)^{-(m+\mu)/2}\frac{d^{l-m}}{dx^{l-m}} \times[(1+x)^{l-\mu}(1-x)^{l+\mu}]e^{im\phi}e^{-i\mu\chi}, \tag{15}$$

where $x=\cos\theta$.

That the angular momentum can take the value $\frac{1}{2}$ is found to be analogous to the result that a monopole charged particle composite representing a dyon satisfying the condition $e\mu=\frac{1}{2}$ has its angular momentum shifted by 1/2 unit and its statistics shift accordingly.[17] This suggests that a fermion can be viewed as a scalar particle moving with $l=\frac{1}{2}$ in an anisotropic space. The specification of the l_z value for the particle and antiparticle states then depicts it as a chiral spinor. This may be associated with a spin system when electrons are polarized in one or the other direction.

Now we note that when the angle χ depicting the rotational orientation around the *direction vector* ξ_μ attached to the space-time point x_μ is gradually changed over the closed path $0\leqslant\chi\leqslant 2\pi$ it gives rise to a phase factor in the wave function. Indeed, the angular part associated with the angle χ in the spherical harmonics $Y_l^{m,\mu}$ is given by $e^{-i\mu\chi}$ where we have

$$i\frac{\partial}{\partial\chi}e^{-i\mu\chi}=\mu e^{-i\mu\chi}. \tag{16}$$

Now when χ is changed to $\chi+\delta\chi$, we find

$$i\frac{\partial}{\partial(\chi+\delta\chi)}e^{-i\mu\chi}=i\frac{\partial}{\partial(\chi+\delta\chi)}e^{-i\mu(\chi+\delta\chi)}e^{i\mu\delta\chi}. \tag{17}$$

Thus the wave function will acquire an extra phase factor $e^{i\mu\delta\chi}$ when the angle χ is changed over the closed path $0\leqslant\chi\leqslant 2\pi$. For one such complete rotation, the wave function will acquire the phase $e^{i\mu\int_0^{2\pi}\delta\chi}=e^{i2\pi\mu}$. Thus for a closed parameter space, we have the extra phase factor $e^{i2\pi\mu}$, which represents the Berry phase. This analysis also suggests that this is related to the Pontryagin index q as is evident from Eqs. (10)–(12) and hence is associated with the chiral anomaly[18]

$$2\mu=q=\int \partial_\mu j_\mu^2 d^4x=\frac{-1}{2}\int \partial_\mu j_\mu^5 d^4x. \tag{18}$$

Thus we find that a quantum particle in an intense magnetic field will acquire the Berry phase, which is associated with the change in chirality over a closed path.

To study a quantum Hall fluid, we have considered a two-dimensional electron gas of N particles on the spherical surface of a three-dimensional sphere of large radius R in a radial (monopole) strong magnetic field. In such a 3D anisotropic space we can construct the spherical harmonics $Y_l^{m,\mu}$ with $l=1/2$, $|m|=|\mu|=1/2$ when the angular momentum is given by $\mathbf{J}=\mathbf{r}\times\mathbf{p}-\mu\mathbf{r}$. From the description of spherical harmonics a two-component spinor $\theta=\binom{u}{v}$ has been considered:

$$\begin{aligned} u&=Y_{1/2}^{1/2,1/2}=\sin\frac{\theta}{2}\exp[i(\phi-\chi)/2],\\ v&=Y_{1/2}^{-1/2,1/2}=\cos\frac{\theta}{2}\exp[-i(\phi+\chi)/2], \end{aligned} \tag{19}$$

which construct the N-particle wave function as

$$\psi_N^{(m)}=\prod_{i<j}(u_iv_j-u_jv_i)^m \tag{20}$$

where $m=1/\nu$. Evidently for odd (even) m, we will have the fermionic (bosonic) state. Since m is an integer, we can identify it, following Haldane,[2] as $m=J_{ij}=J_i+J_j$ for an N-particle system. It is noted that for $|\mu|=l=1/2$, we have $m=1$, which describes the complete filling of the lowest Landau level and corresponds to the ground state for the contribution of the factor $\mathbf{r}\times\mathbf{p}=0$. Following the Dirac quantization condition $e\mu=1/2$, the quasiparticle for $m=1$ IQH state has fermion number 1.

However, if we consider the next excited state with $\mathbf{r}\times\mathbf{p}=1$, the respective angular momentum is changed to $J=3/2$ for $\mu=1/2$. This can be viewed as a system with $\mu_{\mathrm{eff}}=3/2$ having $\mathbf{r}\times\mathbf{p}=0$. Hence for the three-particle state ($N=3$) where each electron carries 3/2 angular momentum, the filling factor is fractional for $\nu=1/m=1/3$ where $m=3/2+3/2=3$. In this excited state, the fermion number of the quasiparticle is 1/3. In this way, we note that the 1/5 filling factor of FQH states corresponds to the excited state having $\mathbf{r}\times\mathbf{p}=2$. It may be pointed out that the other fractionally quantized Hall states may be associated with the parent state with the effect of majority spin.[19] With this view, over a closed path on the surface of a 3D sphere, the parallel transport of these Hall states develop the topological phase of Berry, $e^{i\phi_B}$, where

$$\phi_B=2\mu\pi\theta=\pi W_\theta=m\pi\theta, \tag{21}$$

which indicates the relationship with the Hall conductivity.

III. TOPOLOGICAL PHASES OF THE HALL PARTICLES IN HIERARCHIES

The occurrence of FQHE, in the presence of strong external orthogonal magnetic field causes a chiral symmetry breaking of fermions (Hall particles) and as a result anomaly is realized in association with the Hall conductivity when we have shown in the previous section that in quantization the Hall carrier will acquire a gauge theoretic extension induced by background magnetic field. At long distances these gauges play a key role in forming the Chern-Simons terms, which describe the effective theory of Hall fluid. In hierarchies, the effect of the interactions between the quasiparticles are visualized through the modification of the coefficients of the Chern-Simon terms. Wen and Zee[7,8] have shown that the effective Lagrangian (on suppressing indices) in $(2+1)$ dimensions after the introduction of p excitations with the parent state is

$$\mathcal{L}=\frac{K}{4\pi}B\epsilon\partial B. \tag{22}$$

Here K is a symmetric integer valued $n\times n$ matrix that defines the filling factor through Hall conductance as $\sigma_H=\Sigma K_{IJ}^{-1}$. In fact, this K matrix formalism provides a complete classification of Abelian quantum Hall fluid and is based on the same physical theory of Haldane and Halperin where in the hierarchical construction, the quasiparticles from the last condensate condense to obtain the next level hierarchical state.

Haldane[2] extended the Laughlin scheme to describe a hierarchy of fluid states by considering a 2D electron gas of N particles on the surface of a sphere having radius R, in a radial (monopole) magnetic field $B=\hbar S/eR^2(>0)$. This $2S=N_\phi$ is an integer that defines the total amount of magnetic flux through the surface. For the parent state $\nu=1/m$ the total flux is $S=(1/2)m(N-1)$. The field strength S in the first level hierarchy is

$$S(N;m,\pm p)=(1/2)m(N-1)\pm(1/2)(N/p+1), \tag{23}$$

which is formed when p (p is an even integer) excitations are added in the parent state $\nu=1/m$. These show that the filling factor satisfies a slightly complicated relation

$$2S=N_\phi=\nu^{-1}N-\mathcal{S}. \tag{24}$$

In the language of Wen and Zee,[8] this $\mathcal{S}$ is the shift, a topological quantum number. It is developed from the coupling between the orbital spin and curvature of the space. This orbital spin is the spin of the Hall particles (different from the spin of electrons) associated with the orbital angular momentum in cyclotron motion. In the thermodynamic limit, the filling factor in each Landau level is given by $\nu=N/2S\approx N/N_\phi$ when this shift becomes insignificant. On a sphere, the shift for a hierarchical state is given by

$$\mathcal{S}=\frac{1}{\nu}\sum_{IJ}(K^{-1})_{IJ}K_{JJ} \tag{25}$$

having spin $s=\frac{1}{2}K_{II}$. For a $\nu=1/m$ parent state this shift is simply $\mathcal{S}=2(n-1)+m$ with orbital spin $s=n-1+m/2$, where n identifies the Landau levels. In the effective theory, this shift can be introduced if the Lagrangian in Eq. (22) can be modified as follows:

$$\mathcal{L}=1/4\pi(KB\epsilon\partial B+2Ae\epsilon\partial B+2Cs\epsilon\partial B), \tag{26}$$

where the second term is the electromagnetic coupling and the third one is the coupling to the curvature of space.

In the light of the above works, we shall here give a more meaningful idea of shift, which will be visualized from the Berry phase of the Hall particles. The appearance of this shift in the parent $\nu=1/m$ FQHE states is very trivial, which is visualized through Berry phase$=m\pi\theta$. But in hierarchies, at filling factor $\nu=1/(m+1/[p])$ (where $1/[p]=\alpha_1/p_1-a_2/p_2\ldots\alpha^n p^n$) where the interactions between the parent state and added excitations take place, the role of shifts is nontrivial.

It may be pointed out that quasiparticles in the hierarchies are formed when additional flux is attached with quantized particles, which in the effective theory is visualized through the CS term in the Lagrangian. In particular, the θ term (in Lagrangian) leads to a vortex line and the corresponding gauge field can be considered to be the field of the vortex line. With this view, the interaction between the vortices plays a primary role in forming the quasiparticle. Apart from the extension B_μ induced by external strong magnetic field each particle has its own internal extension C_μ, which visualizes the internal topology through the field strength $\tilde{F}_{\mu\nu}$ (Ref. 11)

$$\tilde{F}_{\mu\nu}=\partial_\nu C_\mu-\partial_\mu C_\nu+[C_\mu,C_\nu]. \tag{27}$$

This inherent gauge field structure is responsible for the topological features of a fermion and gives rise to quantum holonomy corresponding to the Berry phase.[19] Hence in the light of Wen's work, the effective theory of the Hall fluid can be accurately presented if we take into consideration the interaction between $C'_\mu s$ and $B'_\mu s$. In the language of differential geometry they are two fibers acting on each point of the base space S^2, which is similar to the discussion of two-dimensional vector bundles such that the total bundle is just the vector sum of two one-dimensional bundles.

In $(2+1)$ dimensions the topological Lagrangian of the Ith hierarchical state will be

$$\mathcal{L}^I=\frac{\theta}{4\pi}B_\mu\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda+\frac{\theta'}{4\pi}C_\mu\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda+\frac{\theta''}{4\pi}C^I_\mu\epsilon^{\mu\nu\lambda}\partial_\nu C_\lambda, \tag{28}$$

where θ and θ' and θ'' are the coupling constants of the B_μ-B_μ, C_μ-B_μ, and C_μ-C_μ interactions, respectively. Here each term is connected with the chiral anomaly through the Hopf invariant as follows:

$$\partial_\rho\epsilon_{\rho\mu\nu\lambda}A_\mu F_{\nu\lambda}=1/2\epsilon_{\rho\mu\nu\lambda}F_{\rho\mu}F_{\nu\lambda}. \tag{29}$$

Now we have pointed out in our earlier discussion that the quantization of a fermion that is responsible for the anomaly is associated with the gauge theoretic extension when the anomaly is related to Tr* $\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}$ where $\tilde{F}_{\mu\nu}$ is the "internal" field strength behaving like a magnetic field. We can note here that in the external magnetic field associated with the field strength $F_{\mu\nu}$ such a quasiparticle on the surface of a 3D sphere can be described by the topological Lagrangian

$$\mathcal{L}^I=-\frac{\theta}{16\pi^2}\mathrm{Tr}^*F_{\mu\nu}F_{\mu\nu}-\frac{\theta'}{16\pi^2}\mathrm{Tr}^*F_{\mu\nu}\tilde{F}_{\mu\nu}-\frac{\theta''}{16\pi^2}\mathrm{Tr}^*\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{30}$$

Here every term corresponds to a total divergence of a topological quantity, known as Chern-Simons secondary characteristics class defined by

$$\Omega^\mu{}_e=-1/16\pi^2\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[B_\nu F_{\alpha\beta}-2/3(B_\nu B_\alpha B_\beta)], \tag{31}$$

$$\tilde{\Omega}^\mu=-1/16\pi^2\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[C_\nu F_{\alpha\beta}-2/3(C_\nu B_\alpha B_\beta)], \tag{32}$$

$$\Omega^\mu{}_i=-1/16\pi^2\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}[C_\nu\tilde{F}_{\alpha\beta}-2/3(C_\nu C_\alpha C_\beta)]. \tag{33}$$

Assuming a particular choice of coupling $\theta=\theta'=\theta''$ in the Lagrangian the topological part of the action in $(3+1)$ dimensions becomes

$$W_\theta=2(\mu_e+\mu_i+\tilde{\mu})\theta, \tag{34}$$

where μ_e, μ_i, and $\tilde{\mu}$ are the corresponding magnetic (charges) that are connected with the respective charges through the Dirac quantization conditions and Pontryagin density:

$$2\mu=q=\int\partial_\mu\Omega_\mu d^4x. \tag{35}$$

It has been pointed out earlier[11,19] that over a closed path on the surface of a 3D sphere the parallel transport of a Hall hierarchy state develops the topological phase of Berry:

$$\phi_B=\pi W_\theta=2\pi\tilde{\mu}_{\mathrm{eff}}\theta=2(\mu_{\mathrm{eff}}+\tilde{\mu})\pi\theta=2\pi(\mu_e+\mu_i+\tilde{\mu})\theta. \tag{36}$$

Here the first term is associated with the Berry phase factor of Hall particles due to external magnetic field μ_e. The second term gives rise to the inherent Berry phase factor μ_i associated with the chiral anomaly of a free electron (in the absence of an external magnetic field) and the third one effectively relates the coupling of the external field with the internal one, which gives rise to the phase factor $\tilde{\mu}$. This μ_{eff} actually visualizes the filling factor through the relation $\nu=n/2\mu_{\mathrm{eff}}$, where n denotes the Landau level.[10] In fact this μ_{eff} satisfies the Dirac quantization condition

$$e'\mu_{\mathrm{eff}}=\frac{n}{2}. \tag{37}$$

Each quasiparticle in the nth Landau level having charge e' behaves as a composite fermion in the higher Landau level. It will behave as a fermion in the ground state following the Dirac condition

$$\tilde{e}\tilde{\mu}=\pm 1/2, \tag{38}$$

which can be obtained from Eq. (37) as follows:

$$\tilde{e}\left(\mu_{\rm eff}-\frac{n\pm 1}{2}\right)=\pm\frac{1}{2}. \tag{39}$$

This implies that $(n\pm 1)/2$ is the magnetic strength μ' of the added quanta, which is associated with the excitations developed due to change of Landau levels. Here for $\mu'=\pm 1/2$, $\pm 3/2$,... the quanta behave like fermions and for $\mu'=\pm 1$, ± 2,... they show bosonic behavior. We can identify here that this change in μ has been visualized through shift $\mathcal{S}$ by the relation

$$2\tilde{\mu}=\mathcal{S}=2\mu_{\rm eff}-(n\pm 1)=\frac{n}{\nu}-(n\pm 1), \tag{40}$$

where $n=1,2,3,\ldots$ denotes the hierarchy levels.

In view of this the shift on a sphere can be related to the factor $2\tilde{\mu}=2(\mu_{\rm eff}-\mu')$. It shows the deviation from the Berry phase factor of a free electron where the term $n\pm 1/2$ can be associated with the second term, which corresponds to the Berry phase factor μ'.

In the light of Jain formalism Basu and Bandyopadhyay[10] pointed out that FQHE states can be depicted by the filling factor

$$\nu=\frac{n}{2\mu_{\rm eff}}=\frac{n}{2mn\pm 1}, \tag{41}$$

where $2\mu_{\rm eff}\pm 1$ is an even integer given by $2mn$ for n being any integer and $\pm$ implies the orientation of the vortex line. This leads the shift to be in the form

$$\mathcal{S}=(2mn\pm 1)-(n\pm 1). \tag{42}$$

An equivalent scheme of Haldane's hierarchy gives rise to the filling factor of continued fraction $\nu=(m\pm 1/[P])^{-1}$, where $[P]$ can be written as P_1/P_2. This causes the shift to be written in the form

$$\mathcal{S}=\frac{mP_1\pm P_2}{P_1}\,n-(n\pm 1). \tag{43}$$

These results are identical with that obtained in the K-matrix scheme as well as the Jain scheme. It also implies that shifts obtained from the two different hierarchy schemes (Jain and Haldane) yield identical values. It effectively represents the deviation from the inherent Berry phase factor associated with the chiral anomaly in the absence of external magnetic field.

Now we will study the angular momentum of the quasiparticles from the topological aspects of the shifts $\mathcal{S}$. Considering the hierarchies $3/7\rightarrow 4/9\rightarrow 5/11\rightarrow 6/13$ of the state 2/5, the daughter state of 1/3 parent state in the first Landau level we can write

for $\nu=\frac{2}{5}$, $e'\frac{5}{2}=1=1/2+1/2$ or $e'(5/2-1/2)=1/2$ or $\mathcal{S}=2\tilde{\mu}=4$

for $\nu=\frac{3}{7}$, $e'\frac{7}{2}=3/2=1+1/2$ or $e'(7/2-1)=1/2$ or $\mathcal{S}=5$

for $\nu=\frac{4}{9}$, $e'\frac{9}{2}=2$, or $e'(9/2-3/2)=1/2$ or $\mathcal{S}=6$

for $\nu=\frac{5}{11}$, $e'\frac{11}{2}=\frac{5}{2}$, or $e'(11/2-2)=1/2$ or $\mathcal{S}=7$

for $\nu=\frac{6}{13}$, $e'\frac{13}{2}=3$, or $e'(13/2-5/2)$ or $\mathcal{S}=8$.

It is transparent from the above that these shifts actually visualize the resultant angular momentum $J_{ij}=J_i+J_j$ of the quasiparticles in the respective hierarchy state where J_i and J_j denote the angular momentum of the parent ($\nu=1/m$) state and added particle, respectively. Earlier we have pointed out that the filling fractions of the type $\nu=1/3,1/5,1/7,\ldots$ are associated with higher angular momentum $r\times p=1,2,3,\ldots$, respectively. In this case, from Eq. (14) the respective angular momentum will have values $J=3/2,5/2,7/2,\ldots$. Here we have found that the 2/5 state has angular momentum

$$J_{ij}=J_i+J_j=5/2-1/2$$

in the 2nd Landau level (LL), which implies that the added quantum is a fermion having angular momentum $J_j=1/2$. In the 3rd LL the angular momentum of the quasiparticle will be

$$J_{ij}=7/2-1,$$

where a boson having $\mu=1$ is added to form the 3/7 state. In this way we find that the shift in every hierarchy state can shed light on the angular momentum acquired by the quasiparticle in that state.

Finally, we would like to express the wave functions of the FQHE state having filling factor

$$\nu=\frac{p}{q}=\frac{n}{2\mu_{\rm eff}}=\frac{n}{n(m-1)\pm 1}$$

in terms of spherical harmonics $Y_l^{m,\mu}$. In the absence of disorder, Jain[20] has constructed the incompressible state $\Psi_\nu{}^m$ as

$$\Psi_\nu(R)^m=\Phi_n(R)[\Phi_1(R)]^{m-1} \tag{44}$$

where $m=$odd and $n=$integer for $\Psi_\nu{}^m$ anisymmetric. Here $\Phi_n(R)$ is the N-particle antisymmetric wave functions for n filled LL's in a fixed area Ω. In particular the Laughlin state at $\nu=1/m$ is a special example of such a state with $n=1$ when $\Psi_\nu(R)^m=\Phi_1(R)^m$. It should be noted that in the presence of disorder $\Phi_n(R)$ is the Slater determinant for noninteracting electrons at filling factor n. States of the above form are grouped into a family depending on the values of m. Thus $m=1$ are integer states, states with $m=3$ are the same family of the Laughlin $\nu=1/3$ state, etc. Hence this implies that any FQHE state can be expressed in terms of the IQHE state and as a result the wave functions of the filling factors $\nu=1/3,1/5,\ldots$ will be

$$\Psi_3(R)=\Phi_1(R)^{1/3},\quad \Psi_5(R)=\Phi_1(R)^{1/5}. \tag{45}$$

They are the primary states, which further help to have the hierarchical states

$$\Psi_{2/5}(R)=\Phi_2(R)\Phi_1(R)^{3-1}=\Phi_1(R)^{5/2}, \qquad (46)$$

$$\Psi_{3/7}(R)=\Phi_1(R)^{7/3}. \qquad (47)$$

In our previous works we have defined the wave functions for the FQHE state having $\nu=1/m$, which we have shown in Eq. (20). Now we want to generalize it for the FQHE hierarchy state of filling factor $\nu=[(m-1)+1/n]^{-1}$. Identifying the Hall particles moving on the surface of a 3D sphere by the spinor $\theta=\binom{u}{v}$ where

$$u=Y_{1/2}^{1/2,1/2}=\sin\frac{\theta}{2}\exp[i(\phi-\chi)/2],$$
$$v=Y_{1/2}^{-1/2,1/2}=\cos\frac{\theta}{2}\exp[-i(\phi+\chi)/2] \qquad (48)$$

and denoting $\Phi_1(z)=(u_i v_j-u_j v_i)$ the relativistic wave functions $\Psi_\nu(R)^m=\Phi_n(R)[\Phi_1(R)]^{m-1}$ will be

$$\Psi_\nu(R)^m=\Phi_n(R)[\Phi_1(R)]^{m-1}$$
$$=\Phi_n(z)[\Phi_1(z)]^{m-1}=\Phi_1(z)^{2m+1/n}=[\Phi_1(z)]^{1/\nu}. \qquad (49)$$

This shows that in the absence of disorder the relativistic generalized wave function for $\nu=1/[(m-1)+1/n]$ is similar to that of the $\nu=1/m$.

Here if we consider the choice $\chi_i=\chi_j=\chi$ then our wave function for an $n=1$ IQHE state will be

$$\Phi_1=\prod(\tilde{u}_i\tilde{v}_j-\tilde{u}_i\tilde{v}_j)\prod e^{-i\chi_k/2}, \qquad (50)$$

where $\tilde{u}=\sin(\theta/2)e^{i\phi/2}$ and $\tilde{v}=\cos(\theta/2)e^{i\phi/2}$.

This helps one to express the wave function in Eq. (49) as

$$\Psi_\nu(R)^m=[\Phi_1(z)]^{1/\nu}$$
$$=\prod(\tilde{u}_i\tilde{v}_j-\tilde{u}_i\tilde{v}_j)^{\nu^{-1}}\prod e^{-i\chi_k/2\nu}$$
$$=\prod(\tilde{u}_i\tilde{v}_j-\tilde{u}_i\tilde{v}_j)^{\nu^{-1}}\prod e^{-i(\mu_{\rm eff}/n)\chi_k}. \qquad (51)$$

This leads to the generalized wave function of FQHE fluid in the relativistic framework to conceive the χ-dependent term visualizing $\mu_{\rm eff}$, which in fact defines the resultant chirality of the hierarchical state. The term $e^{-i\mu_{\rm eff}\chi_k}$ physically denotes the resultant topological phase of the quasiparticle over a closed path in the thermodynamic limit. In this case the shift becomes insignificant. Normally from our above analysis we note that the shift is effectively related to the departure in the Berry phase factor associated with the chiral anomaly in the presence and absence of the external magnetic field. In fact the topological phase of the composite particle is related to the inherent Berry phase, which when in the presence of external magnetic field is accompanied with the AB-type phase becomes $\tilde{\mu}_{\rm eff}$ as in Eq. (36) visualizing the shift through the term $\tilde{\mu}$.

IV. DISCUSSION

It is known[19] that in the presence of an intense magnetic field, the space-time structure gets modified in such a way that a ''direction vector'' is attached to each space-time point. In addition, the quantization visualizes the topological features of the fermion by acquiring gauge theoretic extensions through the appearance of the Berry phase.[18] This implies that apart from the self-extension of the quantized particle, the external anisotropy (magnetic field) attaches a vortex at each space-time point. As a result the combined effect gives rise to a composite particle or quasiparticle in the hierarchies of FQHE. The shift $\mathcal{S}$, appearing in every filling factor, is originated from the interaction between these two gauges. In the presence of an external magnetic field, the Berry phase factor changes, which is manifested in FQHE and the shift effectively represents the departure from the fermion number of a free fermion induced by the external strong magnetic field. In the generalized wave function this shift remains insignificant for a number of particles being large (thermodynamic limit). On the other hand we have pointed out that the shift visualizes the angular momentum of the quasiparticle in the hierarchies of the FQHE states.

We can add here that in a flat space, when the Hall system is represented by a Z_p spin system, the anomaly vanishes and hence we cannot define a conserved charge, indicating that the shift here is zero.[9] In view of this, shift can be associated with the curvature of space, and it vanishes when the curvature associated with the quantization procedure vanishes in a certain geometrical setup.

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