Overscreened multichannel SU(N) Kondo model: Large-N solution and conformal field theory

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The multichannel Kondo model with SU(N) spin symmetry and SU(K) channel symmetry is considered. The impurity spin is chosen to transform as an antisymmetric representation of SU(N), corresponding to a fixed number of Abrikosov fermions $\Sigma_\alpha f_\alpha^\dagger f_\alpha = Q$. For more than one channel ($K>1$), and all values of N and Q, the model displays non-Fermi behavior associated with the *overscreening* of the impurity spin. Universal low-temperature thermodynamic and transport properties of this non-Fermi-liquid state are computed using conformal field theory methods. A large-N limit of the model is then considered, in which $K/N \equiv \gamma$ and $Q/N \equiv q_0$ are held fixed. Spectral densities satisfy coupled integral equations in this limit, corresponding to a (time-dependent) saddle point. A low-frequency, low-temperature analysis of these equations reveals universal scaling properties in the variable ω/T, in agreement with conformal invariance. The universal scaling form is obtained analytically and used to compute the low-temperature universal properties of the model in the large-N limit, such as the $T=0$ residual entropy and residual resistivity, and the critical exponents associated with the specific heat and susceptibility. The connections with the ''noncrossing approximation'' and the previous work of Cox and Ruckenstein are discussed. [S0163-1829(98)01920-1]

I. INTRODUCTION AND MODEL

Multichannel Kondo impurity models[1,2] have recently attracted considerable attention, for several reasons. First, in the overscreened case, they provide an explicit example of a non-Fermi-liquid ground-state. Second, these models can be studied by a variety of controlled techniques, and provide invaluable testing grounds for theoretical methods dealing with correlated electron systems. One of the most recent and fruitful development in this respect has been the conformal field-theory approach developed by Affleck and Ludwig.[3–5] Finally, multichannel models have experimental relevance to tunneling phenomena in quantum dots and two level systems,[6] and possibly also to some heavy-fermion compounds.[2]

In this paper, we consider a generalization of the multichannel Kondo model, in which the spin symmetry group is extended from SU(2) to SU(N). In addition, the model has a SU(K) symmetry among the K ''channels'' (flavors) of conduction electrons. We focus here on spin representations which are such that the model is in the non-Fermi-liquid *overscreened* regime. We shall derive in this paper several universal properties of this SU(N)$\otimes$SU(K) Kondo model in the low-temperature regime. Specifically, we obtain the zero-temperature residual entropy, the zero-temperature impurity resistivity and T matrix, and the critical exponents governing the leading low-temperature behavior of the impurity specific heat, susceptibility, and resistivity.

These results will be obtained using two different approaches. It is one of the main motivations of this paper to compare these two approaches in some detail. First (Sec. III), we apply conformal field theory (CFT) methods to study the model for general values of N and K. Then, we study the limit of large N and K, with $K/N=\gamma$ fixed. This limit was previously considered by Cox and Ruckenstein,[7] in connection with the ''noncrossing approximation'' (NCA). There is a crucial difference between our work and that of Ref. 7, however, which is that we keep track of the quantum number specifying the spin representation of the impurity by imposing a constraint (on the Abrikosov fermions representing the impurity) which also scales proportionally to N. As a result, the solution of the model at large N follows from a true saddle-point principle, with controllable fluctuations in $1/N$. Hence, a detailed quantitative comparison of the large-N limit to the CFT results can be made. The saddle-point equations are coupled integral equations similar in structure to those of the NCA, except for the different handling of the constraint. The $T=0$ impurity entropy and residual resistivity are obtained in analytical form in this paper from a low-energy analysis of these coupled integral equations and shown to agree with the large-N limit of the CFT results. We also demonstrate that the spectral functions resulting from these equations take a universal scaling form in the limit $\omega,T\rightarrow 0$, which is precisely that expected from the conformal invariance of the problem.

The Hamiltonian of the model considered in this paper reads

$$H=\sum_{\vec{p}}\sum_{i=1}^{K}\sum_{\alpha=1}^{N}\epsilon(\vec{p})c_{\vec{p}i\alpha}^\dagger c_{\vec{p}i\alpha} + J_K\sum_{A=1}^{N^2-1}S^A\sum_{\vec{p}\vec{p}'i\alpha\beta}c_{\vec{p}i\alpha}^\dagger t_{\alpha\beta}^A c_{\vec{p}'i\beta}. \qquad (1)$$

In this expression, $c_{\vec{p}i\alpha}^\dagger$ creates an electron in the conduction band, with momentum $\vec{p}$, channel (flavor) index $i=1,\ldots,K$, and SU(N) spin index $\alpha=1,\ldots,N$. The conduction electrons transform under the fundamental representation of the SU(N) group, with generators $t^A_{\alpha\beta}$ ($A=1,\ldots,N^2-1$). They interact with a localized spin degree of freedom placed at the origin $\vec{S}=\{S^A, A=1,\ldots,N^2-1\}$ which is assumed to transform under a given irreducible representation R of the SU(N) group.

In the one-channel case ($K=1$), and when R is taken to be the fundamental representation, this is the Coqblin-Schrieffer model of a conduction gas interacting with a localized atomic level with angular momentum j ($N=2j+1$).[8] In this article, we are interested in the possible non-Fermi-liquid behavior associated with the multichannel generalization ($K>1$).[1,2] We shall mostly focus on the case where R corresponds to antisymmetric tensors of Q indices, i.e., the Young tableau associated with R is made of a *single column* of Q indices. In that case, it is convenient to use an explicit representation of the localized spin in terms of N species of auxiliary fermions f_α ($\alpha=1,\ldots,N$), constrained to obey

$$\sum_{\alpha=1}^{N} f^\dagger_\alpha f_\alpha = Q \tag{2}$$

so that the N^2-1 (traceless) components of $\vec{S}$ can be represented as $S_{\alpha\beta}=f^+_\alpha f_\beta-(Q/N)\delta_{\alpha\beta}$. For these choices of R, the Hamiltonian can be written as (after a reshuffling of indices using a Fierz identity)

$$H=\sum_{\vec{p}}\sum_{i=1}^{K}\sum_{\alpha=1}^{N}\epsilon(\vec{p})c^\dagger_{\vec{p}i\alpha}c_{\vec{p}i\alpha} + J_K\sum_{\vec{p}\vec{p}'i\alpha\beta}\left(f^\dagger_\alpha f_\beta-\frac{Q}{N}\delta_{\alpha\beta}\right)c^\dagger_{\vec{p}i\beta}c_{\vec{p}'i\alpha}. \tag{3}$$

In a recent paper,[9] the case of a *symmetric* representation of the impurity spin (corresponding to a Young tableau made of a single *line* of P boxes) has been considered by two of us. In that case, a transition from overscreening to underscreening is found as a function of the ''size'' P of the impurity spin. In contrast, the antisymmetric representations considered in the present paper always lead to overscreening (except for $K=1$ which is exactly screened), as shown below. As long as only the overscreened regime is considered, the analysis of the present paper applies to symmetric representations as well, up to some straightforward replacements.

II. STRONG-COUPLING ANALYSIS

It is easily checked that a weak antiferromagnetic coupling ($J_K>0$) grows under renormalization for all K and N, and all representations R of the local spin. What is needed is a physical argument in order to determine whether the renormalization group (RG) flow takes J_K all the way to strong coupling (underscreened or exactly screened cases), or whether an intermediate non-Fermi-liquid fixed point exists (overscreened cases).

Following the Nozieres and Blandin[1] analysis of the multichannel SU(2) model, we consider the strong-coupling fixed point $J_K=+\infty$. In this limit, the impurity spin binds a certain number of conduction electrons (at most NK because of the Pauli principle). The resulting bound-state corresponds

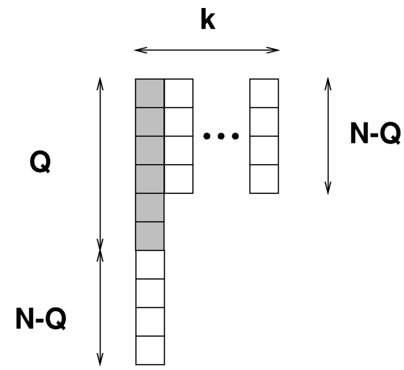

FIG. 1. Young tableau corresponding to the strong-coupling state.

to a new spin representation R_{sc} which is dictated by the minimization of the Kondo energy. For the specific choices of R above, we have proven the following.

(i) In the one-channel case ($K=1$) and for arbitrary N and Q, R_{sc} is the ''singlet'' representation [of dimension $d(R_{sc})=1$]. It is obtained by binding $N-Q$ conduction electrons to the Q pseudofermions f_α. The impurity spin is thus *exactly screened* at strong-coupling.

(ii) For all multichannel cases ($K\geqslant 2$, arbitrary N and Q), the ground state at the strong-coupling fixed point is the representation R_{sc} characterized by a rectangular Young tableau with $N-Q$ lines and $K-1$ columns. Its dimension $d(R_{sc})$ (i.e, the degeneracy of the strong-coupling bound state) is larger than the degeneracy at zero coupling, given by $d(R)=\binom{N}{Q}\equiv N!/Q!(N-Q)!$.

The Young tableau associated with the strong-coupling state in both cases is depicted in Fig. 1. The detailed proof of these statements and the explicit construction of R_{sc} are given in Appendix A. These properties are sufficient to establish the following.

(i) In the one-channel case, the strong-coupling fixed point is stable under RG, and hence the impurity spin is exactly screened by the Kondo effect.

(ii) In the multichannel case, a direct RG flow from weak to strong coupling is impossible, thereby suggesting the existence of an intermediate coupling fixed point (''overscreening'').

The connection between these statements and the above results on the nature and degeneracy of the strong-coupling bound state is clear on physical grounds. Indeed, it is not possible to flow under renormalization from a fixed point with a lower ground-state degeneracy to a fixed point with a higher one, because the effective number of degrees of freedom can only decrease under RG. Hence, no flow away from the strong-coupling fixed point is possible in the one-channel ($K=1$) case since the strong-coupling state is nondegenerate. Also, no direct flow from weak to strong coupling is possible for $K\geqslant 2$ since $d(R_{sc})>d(R)$. These statements can be made more rigorous[5] by considering the impurity entropy defined as

$$S_{imp}\equiv\lim_{T\to 0}\lim_{V\to\infty}[S(T)-S_{bulk}(T)], \tag{4}$$

where $S_{\rm bulk}$ denotes the contribution to the entropy which is proportional to the volume V (and is simply the contribution of the conduction electron gas), and care has been taken in specifying the order of the infinite-volume and zero-temperature limits. At the weak-coupling fixed point $S_{\rm imp}(J_K=0)=\ln d(R)$, while $S_{\rm imp}(J_K=\infty)=\ln d(R_{\rm sc})$ at strong-coupling. $S_{\rm imp}$ must decrease under renormalization,[5] a property which is the analog for boundary critical phenomena to Zamolodchikov's ''c theorem'' in the bulk. This suggests a RG flow of the kind indicated above. This conclusion can of course be confirmed by a perturbative calculation (in the hopping amplitude) around the strong-coupling fixed point.[1] The value of $S_{\rm imp}$ will be calculated below at the intermediate fixed point in the overscreened case, and found to be noninteger (as in the $N=2$ case[5]).

III. CONFORMAL FIELD THEORY APPROACH

Having established the existence of an intermediate fixed point for $K\geqslant 2$, we sketch some of its properties that can be obtained from conformal field-theory (CFT) methods. This is a straightforward extension to the SU(N) case of Affleck and Ludwig's approach for SU(2).[3,4] The aim of this section is not to present a complete conformal field-theory solution, but simply to derive those properties which will be compared with the large-N explicit solution given below.

In the CFT approach, the model (1) is first mapped at low energy onto a $(1+1)$-dimensional model of NK chiral fermions. At a fixed point, this model has a local conformal symmetry based on the Kac-Moody algebra $\widehat{\rm SU}_K(N)_s \otimes \widehat{\rm SU}_N(K)_f \otimes \hat{\rm U}(1)_c$ corresponding to the spin-flavor-charge decomposition of the degrees of freedom. The free-fermion spectrum at the weak-coupling fixed point can be organized in multiplets of this symmetry algebra: to each level corresponds a primary operator in the spin, flavor, and charge sectors. A major insight[3] is then that the spectrum at the infrared stable, intermediate coupling fixed point can be obtained from a ''fusion principle.'' Specifically, the spectrum is obtained by acting, in the spin sector, on the primary operator associated with a given free-fermion state, with the primary operator of the $\widehat{\rm SU}_K(N)_s$ algebra corresponding to the representation R of the impurity spin (leaving unchanged the flavor and charge sectors). The ''fusion rules'' of the algebra determine the new operators associated with each energy level at the intermediate fixed point.

This fusion principle also relates the impurity entropy $S_{\rm imp}$ as defined above to the ''modular S matrix'' S_0^R of the $\widehat{\rm SU}_K(N)$ algebra (this is the matrix which specifies the action of a modular transformation on the irreducible characters of the algebra corresponding to a given irreducible representation R). Specifically, denoting by $R=0$ the trivial (identity) representation

$$S_{\rm imp}=\ln \frac{S_0^R}{S_0^0}. \tag{5}$$

The expression of the modular S-matrix for $\widehat{\rm SU}_K(N)$ can be found in the literature.[10] We have found particularly useful to make use of an elegant formulation introduced by Douglas,[11] which is briefly explained in Appendixes A and B. For the representations R associated to a single column of length Q [corresponding to Eq. (2)], one finds using this representation

$$S_{\rm imp}=\ln \prod_{n=1}^{Q} \frac{\sin[\pi(N+1-n)/(N+K)]}{\sin[\pi n/(N+K)]}. \tag{6}$$

It is easily checked that indeed $S_{\rm imp}<\ln d(R)$ for all values of N, Q, and K. Note also that this expression correctly yields $S_{\rm imp}=0$ in the exactly screened case $K=1$ (for arbitrary N,Q).

The low-temperature behavior of various physical quantities can also be obtained from the CFT approach. At the intermediate coupling fixed point, the local impurity spin acquires the scaling dimension of the primary operator of the $\widehat{\rm SU}_K(N)_s$ algebra associated with the [(N^2-1)-dimensional] adjoint representation of SU(N). Its conformal dimension Δ_s [such that $\langle S(0)S(t)\rangle\sim 1/t^{2\Delta_s}$ at $T=0$] reads

$$\Delta_s=\frac{N}{N+K}. \tag{7}$$

Integrating this correlation function, this implies that the *local susceptibility* $\chi_{\rm loc}\propto\int_{\tau_0}^{1/T}\langle S(0)S(\tau)\rangle d\tau$ (corresponding to the coupling of an external field to the impurity spin *only*) diverges at low temperature when $K\geqslant N$, while it remains finite for $K<N$:

$$K\geqslant N: \chi_{\rm loc}\sim\left(\frac{1}{T}\right)^{(K-N)/(K+N)},$$

$$K=N: \chi_{\rm loc}\sim \ln 1/T,$$

$$K<N: \chi_{\rm loc}\sim {\rm const.} \tag{8}$$

Exactly at the fixed point, the singular contributions to the specific heat and *impurity susceptibility* $\chi_{\rm imp}=\chi-\chi_{\rm bulk}$ (defined by coupling a magnetic field to the total spin density) vanish.[3,12] Indeed, the singular behavior is controlled by the leading irrelevant operator compatible with all symmetries that can be generated. An obvious candidate for this operator is the spin, flavor, and charge singlet obtained by contracting the spin current with the adjoint primary operator above, $\vec{J}_{-1}\cdot\vec{\phi}$. It has dimension $1+N/(N+K)=1+\Delta_s$. This leads to a singular contribution to the impurity susceptibility (arising from perturbation theory at *second order* in the irrelevant operator[3,12]) of the same nature than for $\chi_{\rm loc}$ given in Eq. (8): $\chi_{\rm imp}\sim\chi_{\rm loc}$.

Another irrelevant operator can be constructed in the flavor sector in an analogous manner, namely, $\vec{J}^f_{-1}\cdot\vec{\phi}^f$. This operator has dimension $K/(N+K)$, which is thus lower than the above operator in the spin sector when $K<N$. This operator could *a priori* contribute to the low-temperature behavior of the specific heat, which would lead to a divergent specific heat coefficient C/T for both $K>N$ and $K<N$. On the basis of the explicit large-N calculation presented below, we believe, however, that this operator is *not generated* for model (1) when the conduction band density of state (DOS) is taken to be perfectly flat and the cutoff is taken to infinity (conformal limit), so that the specific heat ratio has the same behavior as the susceptibilities above: $C/T\sim\chi_{\rm loc}\sim\chi_{\rm imp}$. If

the model is extended to an impurity spin with internal flavor degrees of freedom, this operator will however show up (leading to $C/T \sim T^{-(N-K)/(N+K)}$ for $K<N$). It also appears (see Sec. VI B) if an Anderson model generalization of model (1) is considered away from particle-hole symmetry.

IV. SADDLE-POINT EQUATIONS IN THE LARGE-N LIMIT

We now turn to the analysis of the large-N limit of this model. This will be done by setting

$$K = N\gamma, \quad J_K = \frac{J}{N} \tag{9}$$

and taking the limit $N \to \infty$ for fixed values of γ and J, so that the number of channels is also taken to be large. In Ref. 7 (see also Ref. 2), Cox and Ruckenstein considered this limit while holding Q fixed $(Q=1)$. They obtained in this limit identical results to those of the noncrossing approximation (NCA).[13] Here, we shall proceed in a different manner by *taking Q to be large as well*:

$$Q = q_0 N. \tag{10}$$

This ensures that a true saddle point exists, with controllable fluctuations order by order in $1/N$. It will also allow us to study the dependence on the representation R of the local spin, parametrized by q_0.[14] The approach of Ref. 7 is recovered in the limit $q_0 \to 0$ (or 1).

The action corresponding to the functional integral formulation of model (3) reads

$$\begin{aligned} S = &-\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{i\alpha} c_{i\alpha}^\dagger(\tau) G_0^{-1}(\tau-\tau') c_{i\alpha}(\tau') \\ &+ \int_0^\beta d\tau \sum_\alpha f_\alpha^\dagger(\tau)\partial_\tau f_\alpha(\tau) + \int_0^\beta d\tau\, i\mu(\tau) \\ &\times\left[\sum_\alpha f_\alpha^\dagger(\tau) f_\alpha(\tau) - q_0 N\right] + \frac{J}{N}\int_0^\beta d\tau \sum_{i\alpha\beta} c_{i\alpha}^\dagger(\tau) c_{i\beta}(\tau) \\ &\times [f_\beta^\dagger(\tau) f_\alpha(\tau) - q_0\delta_{\alpha\beta}]. \end{aligned} \tag{11}$$

In this expression, the conduction electrons have been integrated out in the bulk, keeping only degrees of freedom at the impurity site. $G_0(i\omega_n) \equiv \Sigma_p 1/(i\omega_n - \epsilon_p)$ is the on-site Green's function associated with the conduction electron bath. In order to decouple the Kondo interaction, an auxiliary bosonic field $B_i(\tau)$ is introduced in each channel, and the conduction electrons can be integrated out, leaving us with the effective action

$$\begin{aligned} S_{\text{eff}} = &\int_0^\beta d\tau \sum_\alpha f_\alpha^\dagger(\tau)\partial_\tau f_\alpha(\tau) \\ &+ \int_0^\beta d\tau\, i\mu(\tau)\left[\sum_\alpha f_\alpha^\dagger(\tau) f_\alpha(\tau) - q_0 N\right] \\ &+ \frac{1}{J}\int_0^\beta d\tau \sum_i B_i^\dagger B_i + \frac{1}{N}\int_0^\beta d\tau \int_0^\beta d\tau' \\ &\times \sum_{i\alpha} B_i(\tau) f_\alpha^\dagger(\tau) G_0(\tau-\tau') B_i^\dagger(\tau') f_\alpha(\tau'). \end{aligned} \tag{12}$$

The quartic term in this expression can be decoupled formally using two bilocal fields $Q(\tau,\tau')$ and $\overline{Q}(\tau,\tau')$ conjugate to $\Sigma_i B_i^\dagger(\tau') B_i(\tau)$ and $\Sigma_\alpha f_\alpha^\dagger(\tau) f_\alpha(\tau')$, respectively, leading to the action

$$\begin{aligned} S = &\int_0^\beta d\tau \sum_\alpha f_\alpha^\dagger(\tau)\partial_\tau f_\alpha(\tau) + \frac{1}{J}\int_0^\beta d\tau \sum_i B_i^\dagger B_i \\ &+ \int_0^\beta d\tau\, i\mu(\tau)\left(\sum_\alpha f_\alpha^\dagger(\tau) f_\alpha(\tau) - q_0 N\right) \\ &- N\int\int d\tau d\tau' \overline{Q}(\tau,\tau') G_0^{-1}(\tau-\tau') Q(\tau,\tau') \\ &- \int\int d\tau d\tau'\, Q(\tau',\tau)\sum_i B_i^+(\tau) B_i(\tau') \\ &- \int\int d\tau d\tau'\, \overline{Q}(\tau,\tau')\sum_\alpha f_\alpha^+(\tau) f_\alpha(\tau'). \end{aligned} \tag{13}$$

The B and f fields can now be integrated out to yield

$$\begin{aligned} S = &-N\int\int d\tau d\tau' \overline{Q}(\tau,\tau') G_0^{-1}(\tau-\tau') Q(\tau,\tau') \\ &- Nq_0\int d\tau\, i\mu(\tau) - N\,\mathrm{Tr}\,\ln\{[-\partial_\tau - i\mu(\tau)]\delta(\tau-\tau') \\ &+ \overline{Q}(\tau,\tau')\} + K\,\mathrm{Tr}\,\ln\left[\frac{1}{J}\delta(\tau-\tau') - Q(\tau',\tau)\right]. \end{aligned} \tag{14}$$

This final form of the action involves only the three fields Q, $\overline{Q}$, and μ, and scales globally as N thanks to the scalings $K=\gamma N$ and $Q=q_0 N$. Hence, it can be solved by the saddle-point method in the large-N limit. At the saddle point, $\mu_{sp}(\tau) = i\lambda$ becomes static and purely imaginary, while $Q_{\rm sp} = Q(\tau-\tau')$ and $\overline{Q}_{\rm sp} = \overline{Q}(\tau-\tau')$ *retain time dependence* but depend only on the time difference $\tau-\tau'$ (they identify with the bosonic and fermionic self-energies, respectively).

The final form of the coupled saddle-point equations for the fermionic and bosonic Green's functions $G_f(\tau) \equiv -\langle T f(\tau) f^\dagger(0)\rangle$, $G_B(\tau) \equiv \langle T B(\tau) B^\dagger(0)\rangle$ and for the Lagrange multiplier field read:

$$\Sigma_f(\tau) = \gamma G_0(\tau) G_B(\tau), \quad \Sigma_B(\tau) = G_0(\tau) G_f(\tau), \tag{15}$$

where the self-energies Σ_f and Σ_B are defined by

$$G_f^{-1}(i\omega_n) = i\omega_n + \lambda - \Sigma_f(i\omega_n), \quad G_B^{-1}(i\nu_n) = \frac{1}{J} - \Sigma_B(i\nu_n). \tag{16}$$

In these expressions $\omega_n = (2n+1)\pi/\beta$ and $\nu_n = 2n\pi/\beta$ denote fermionic and bosonic Matsubara frequencies. Let us note that the field $B(\tau)$ is simply a commuting auxiliary field rather than a true boson (its equal-time commutator vanishes). As a result $G_B(\tau)$ is a β-periodic function, but does not share the usual other properties of a bosonic Green function (in particular at high frequency).

Finally, λ is determined by the third saddle-point equation

$$G_f(\tau=0^-) \equiv \frac{1}{\beta}\sum_n G_f(i\omega_n) e^{i\omega_n 0^+} = q_0. \tag{17}$$

These equations are identical in structure to the usual NCA equations, except for the last equation (17) which implements the constraint and allows us to keep track of the choice of representation for the impurity spin.

V. SCALING ANALYSIS AT LOW FREQUENCY AND TEMPERATURE

A. General considerations

The analysis of the NCA equations in Ref. 15 can be applied in order to find the behavior of the Green's functions in the low temperature, long time regime defined by $T_K^{-1} \ll \tau \ll \beta \to \infty$ (where T_K is the Kondo temperature). In this regime, a power-law decay of the Green's functions is found:

$$G_f(\tau) \sim \frac{A_f}{\tau^{2\Delta_f}}, \quad G_B(\tau) \sim \frac{A_B}{\tau^{2\Delta_B}}, \quad (T_K^{-1} \ll \tau \ll \beta \to \infty). \tag{18}$$

The scaling dimensions $2\Delta_f$ and $2\Delta_B$ can be determined explicitly by inserting this form into the above saddle-point equations and making a low-frequency analysis, as explained in Appendix C. This yields

$$2\Delta_f = \frac{1}{1+\gamma}, \quad 2\Delta_B = \frac{\gamma}{1+\gamma}. \tag{19}$$

The overall consistency of Eqs. (15), (16) at large time also constrains the product of amplitudes $A_f A_B$ [Eq. (C8) in Appendix C] and dictates the behavior of the self-energies [denoting by $\rho_0 = -\mathrm{Im} G_0(i0^+)/\pi$ the conduction bath density of states at the Fermi level]

$$\Sigma_f(\tau) \sim \gamma \rho_0 A_B \left(\frac{1}{\tau}\right)^{2\Delta_B+1}, \quad \Sigma_B(\tau) \sim A_f \rho_0 \left(\frac{1}{\tau}\right)^{2\Delta_f+1} \tag{20}$$

together with the sum rule

$$\Sigma_B(\omega=0, \beta=\infty) = \frac{1}{J}. \tag{21}$$

The expression (19) of the scaling dimensions Δ_f and Δ_B is in complete agreement with the CFT result. Indeed, the fermionic field transforms as the fundamental representation of the $\widehat{\mathrm{SU}}(N)_K$ spin algebra, while the auxiliary bosonic field transforms as the fundamental representation of the $\widehat{\mathrm{SU}}(K)_N$ flavor algebra, leading to $2\Delta_f = (N^2-1)/N(N+K)$ and $2\Delta_B = (K^2-1)/K(N+K)$, which agree with Eq. (19) in the large-N limit. Also, the local impurity spin correlation function, given in the large-N limit by $\langle S(0)S(\tau)\rangle \propto G_f(\tau)G_f(-\tau)$ is found to decay as $1/\tau^{2\Delta_s}$, with $\Delta_s = 2\Delta_f = 1/(1+\gamma)$ in agreement with the CFT result (7).

As will be shown below, however, these asymptotic behaviors at $T=0$ do not provide enough information to allow for the computation of the low-temperature behavior of the impurity free-energy and to determine the $T=0$ impurity entropy (4). One actually needs to determine the Green's functions in the limit $\tau, \beta \to \infty$, but for an arbitrary value of the ratio τ/β [i.e., to analyze the low-temperature, low-frequency behavior of Eqs. (15), (16) keeping the ratio ω/T fixed[16]]. It is easily seen that in this limit the Green's functions and their associated spectral densities $\rho_{f,B}(\omega) \equiv -(1/\pi)\,\mathrm{Im} G_{f,B}(\omega + i0^+)$ obey a *scaling behavior* (for $\tau, \beta \gg T_K^{-1}$, with τ/β arbitrary)

$$G_{f,B}(\tau) = A_{f,B}\beta^{-2\Delta_{f,B}} g_{f,B}\left(\frac{\tau}{\beta}\right) \tag{22a}$$

$$\rho_f(\omega) = A_f T^{2\Delta_f - 1}\phi_f\left(\frac{\omega}{T}\right), \quad \rho_B(\omega) = A_B T^{2\Delta_B-1}\phi_B\left(\frac{\omega}{T}\right). \tag{22b}$$

In these expressions, $g_{f,B}$ and $\phi_{f,B}$ are *universal scaling functions* which depend only on γ and q_0 and *not on the specific shape of the conduction band or the cutoff*. These scaling functions will now be found in explicit form.

B. The particle-hole symmetric representation $q_0 = 1/2$

We shall first discuss the case where the representation R has $Q = N/2$ boxes $(q_0 = 1/2)$, for which there is a particle-hole symmetry among pseudofermions under $f_\alpha^\dagger \leftrightarrow f_\alpha$. The expression of the scaling functions $g_{f,B}$ in that case can be easily guessed from general principles of conformal invariance. The idea is that, in the limit $T_K^{-1} \ll \beta$ with τ/β fixed, the finite-temperature Green's function can be obtained from the $T=0$ Green's function by applying the conformal transformation $z = \exp(i2\pi\tau/\beta)$.[17] Applying this to the $T=0$ power-law decay given by Eq. (18), one obtains the well-known result for the scaling functions $(\tilde{\tau} \equiv \tau/\beta)$:

$$g_f(\tilde{\tau}; q_0 = 1/2) = -\left(\frac{\pi}{\sin \pi\tilde{\tau}}\right)^{2\Delta_f},$$

$$g_B(\tilde{\tau}; q_0 = 1/2) = -\left(\frac{\pi}{\sin \pi\tilde{\tau}}\right)^{2\Delta_B} \tag{23}$$

with the periodicity requirements $g_f(\tilde{\tau}+1) = -g_f(\tilde{\tau}), g_B(\tilde{\tau}+1) = g_B(\tilde{\tau})$. Note that these functions satisfy the additional symmetry $g_{f,B}(1-\tilde{\tau}) = g_{f,B}(\tilde{\tau})$ indicating that *they can only apply to the particle-hole symmetric case* $q_0 = 1/2$. The corresponding form of the scaling functions associated with the spectral densities (22b) reads, with $\tilde{\omega} \equiv \omega/T$ (after a calculation detailed in Appendix C)

$$\phi_f(\tilde{\omega}, q_0 = 1/2) = \frac{1}{\pi}(2\pi)^{2\Delta_f - 1}\cosh\frac{\tilde{\omega}}{2} \times \frac{\Gamma[\Delta_f + i(\tilde{\omega}/2\pi)]\Gamma[\Delta_f - i(\tilde{\omega}/2\pi)]}{\Gamma(2\Delta_f)}, \tag{24}$$

$$\phi_B(\tilde{\omega}, q_0 = 1/2) = \frac{1}{\pi}(2\pi)^{2\Delta_B - 1}\sinh\frac{\tilde{\omega}}{2} \times \frac{\Gamma[\Delta_B + i(\tilde{\omega}/2\pi)]\Gamma[\Delta_B - i(\tilde{\omega}/2\pi)]}{\Gamma(2\Delta_B)}.$$

Note that the $\omega\to 0$ singularity of the $T=0$ case is now recovered for $\omega \gg T$:

$$\phi_{f,B}(\tilde{\omega},q_0=1/2) \underset{\tilde{\omega}\to+\infty}{\sim} \frac{1}{\Gamma(2\Delta_{f,B})}\left(\frac{1}{\tilde{\omega}}\right)^{1-2\Delta_{f,B}}. \tag{25}$$

It is an interesting calculation, performed in detail in Appendix C, to check that indeed these scaling functions do solve the large-N equations (15),(16) at finite temperature in the scaling regime. That NCA-like integral equations obey the finite-temperature scaling properties dictated by conformal invariance has not, to our knowledge, been pointed out in the previous literature.

C. General values of q_0

1. Spectral asymmetry

Let us move to the general case of representations with $q_0 \neq \frac{1}{2}$ in which the particle-hole symmetry between pseudofermions is broken. The *exponent* of the power-law singularity in the $T=0$ spectral densities is not affected by this asymmetry. It does induce, however, *an asymmetry of the prefactors* associated with positive and negative frequencies as $\omega\to 0$. We introduce an angle θ to parametrize this asymmetry, defined such that

$$\rho_f(\omega\to 0^+)\sim h(\gamma,\theta)\frac{\sin(\pi\Delta_f+\theta)}{\omega^{1-2\Delta_f}},$$

$$\rho_f(\omega\to 0^-)\sim h(\gamma,\theta)\frac{\sin(\pi\Delta_f-\theta)}{(-\omega)^{1-2\Delta_f}}, \tag{26}$$

where $h(\gamma,\theta)$ is a constant prefactor. The explicit dependence of θ on q_0 will be derived below. This corresponds to the following analytic behavior of the Green's function in the complex frequency plane, as $z\to 0$:

$$G_f^R(z)\sim h(\gamma,\theta)\frac{e^{-i\pi\Delta_f-i\theta}}{z^{1-2\Delta_f}} \quad \mathrm{Im}\, z>0. \tag{27}$$

Equivalently, this means that the symmetry $G_f(\beta-\tau)=G_f(\tau)$ is broken, and that the scaling function $g_f(\tilde{\tau})$ must satisfy (from the behavior of its Fourier transform)

$$\frac{g_f(0^+)}{g_f(1^-)}=\frac{\sin(\pi\Delta_f+\theta)}{\sin(\pi\Delta_f-\theta)}. \tag{28}$$

We have found, by an explicit analysis of the saddle-point and constraint equations in the scaling regime, which is detailed in Appendix C, that the full scaling functions for this asymmetric case are very simply related to the symmetric ones at $q_0=1/2$, through

$$g_{f,B}(\tilde{\tau};q_0)=\frac{e^{\alpha(\tilde{\tau}-1/2)}}{\cosh(\alpha/2)}g_{f,B}(\tilde{\tau};q_0=1/2)$$
$$=-\frac{e^{\alpha(\tilde{\tau}-1/2)}}{\cosh(\alpha/2)}\left(\frac{\pi}{\sin\,\pi\tilde{\tau}}\right)^{2\Delta_{f,B}}, \tag{29}$$

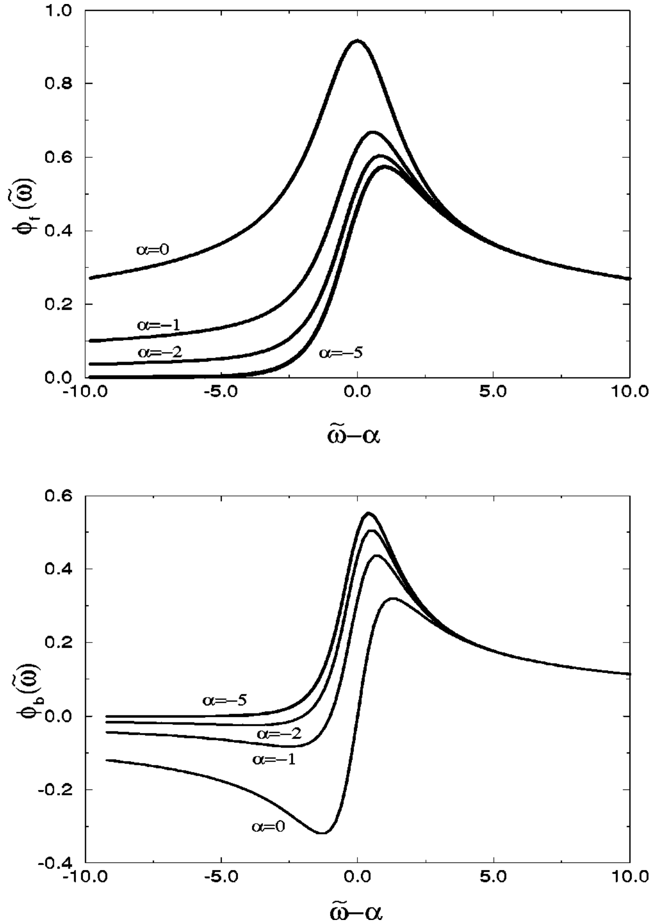

FIG. 2. Plot of ϕ_f and ϕ_b as a function of $\tilde{\omega}-\alpha$ for different values of the asymmetry parameter α: $\alpha=0$, -2, -5, $-\infty$ ($\Delta_f=0.3$).

where the parameter α is simply related to θ so as to obey Eq. (28):

$$\alpha=\ln\frac{\sin[\pi/2(1+\gamma)-\theta]}{\sin[\pi/2(1+\gamma)+\theta]}\,. \tag{30}$$

Fourier transforming, this leads to the scaling functions for the spectral densities

$$\phi_f(\tilde{\omega},q_0)=\frac{\cosh(\tilde{\omega}/2)}{\cosh[(\tilde{\omega}+\alpha)/2]\cosh(\alpha/2)}\,\phi_f(\tilde{\omega}+\alpha,q_0=1/2),$$

$$\phi_B(\tilde{\omega},q_0)=\frac{\sinh(\tilde{\omega}/2)}{\sinh[(\tilde{\omega}+\alpha)/2]\cosh(\alpha/2)}\,\phi_B(\tilde{\omega}+\alpha,q_0=1/2). \tag{31}$$

The thermal scaling function for the fermionic spectral density ϕ_f and the bosonic one ϕ_b are plotted in Fig. 2 for various values of the asymmetry parameter α. We also note the expression for the maximally asymmetric case $\alpha\to-\infty$ (corresponding, as shown below, to $q_0\to 0$, i.e., to the limit $Q \ll N$ as in the usual NCA):

$$\phi_f(\tilde{\omega}-\alpha) \underset{\alpha\to-\infty}{\sim} e^{\tilde{\omega}/2} \frac{(2\pi)^{2\Delta_f-1}\Gamma[\Delta_f+i(\tilde{\omega}/2\pi)]\Gamma[\Delta_f-i(\tilde{\omega}/2\pi)]}{\pi\Gamma(2\Delta_f)}. \tag{32}$$

We also note, for further use, the expressions of the full Green's functions in the complex frequency plane (defined by $g_{f/B}(z,\alpha)\equiv\int_{-\infty}^{+\infty}d\tilde{\omega}[\phi_{f/B}(\tilde{\omega})/(z-\tilde{\omega})]$), in the scaling regime for Im $z>0$:

$$\begin{aligned} g_f(z,\alpha) =& -\frac{2i(2\pi)^{2\Delta_f-1}}{\cosh(\alpha/2)\,\Gamma(2\Delta_f)\sin 2\pi\Delta_f} \\ & \times\Gamma\left(\Delta_f+i\frac{z+\alpha}{2\pi}\right)\Gamma\left(\Delta_f-i\frac{z+\alpha}{2\pi}\right) \\ & \times\cos\left(\pi\Delta_f-\frac{i\alpha}{2}\right)\sin\left(\pi\Delta_f+i\frac{z+\alpha}{2}\right), \end{aligned} \tag{33}$$

$$\begin{aligned} g_B(z,\alpha) =& -\frac{2(2\pi)^{2\Delta_B-1}}{\cosh(\alpha/2)\,\Gamma(2\Delta_B)\sin 2\pi\Delta_B} \\ & \times\Gamma\left(\Delta_B+i\frac{z+\alpha}{2\pi}\right)\Gamma\left(\Delta_B-i\frac{z+\alpha}{2\pi}\right) \\ & \times\sin\left(\pi\Delta_B-\frac{i\alpha}{2}\right)\sin\left(\pi\Delta_B+i\frac{z+\alpha}{2}\right). \end{aligned} \tag{34}$$

The reader interested in the details of these calculations is directed to Appendix C.

At this stage, the point which remains to be clarified is the explicit relation between the asymmetry parameter θ and the parameter q_0 specifying the representation. This is the subject of the next section.

Before turning to this point, we briefly comment on the CFT interpretation of the asymmetry parameter θ (or α) associated with the particle-hole asymmetry of the fermionic fields. The form (23) of the correlation functions at finite temperature in the scaling limit can be viewed as those of the exponential of a compact bosonic field with periodic boundary conditions. The asymmetric generalization (29) corresponds to a shifted boundary condition on the boson (i.e., to a twisted boundary condition for its exponential).

2. Relation between q_0 and θ

Let us clarify the relation between the spectral asymmetry parameter θ, and the parameter q_0 specifying the spin representation. That such a relation exists in universal form is a remarkable fact: indeed θ is *a low-energy parameter* associated with the low-frequency behavior of the spectral density, while q_0 is the total pseudofermion number related by the constraint equation (17) to an integral of the spectral density *over all frequencies*. The situation is similar to that of the Friedel sum rule in impurity models, or to Luttinger theorem in a Fermi liquid, and indeed the derivation of the relation between q_0 and θ follows similar lines.[18] It is in a sense a Friedel sum rule for the quasiparticles carrying the spin degrees of freedom (namely, the pseudofermions f_α).

We start from the constraint equation (17) written at zero-temperature as

$$q_0=-i\lim_{t\to0^+}\int\frac{d\omega}{2\pi}G_f(\omega)e^{i\omega t}. \tag{35}$$

In this expression, and below in this section, $G_f(\omega)$ and $G_B(\omega)$ denote the (Feynman) $T=0$ Green's functions while the retarded Green's functions are denoted by G^R. Using analytic continuation of Eq. (16), we have

$$\frac{\partial\ln G_f(\omega)}{\partial\omega}-G_f(\omega)\frac{\partial\Sigma_f(\omega)}{\partial\omega}=-G_f(\omega), \tag{36}$$

$$\frac{\partial\ln G_B(\omega)}{\partial\omega}-G_B(\omega)\frac{\partial\Sigma_B(\omega)}{\partial\omega}=0, \tag{37}$$

so that Eq. (35) can be rewritten as

$$\begin{aligned} q_0=i\lim_{t\to0^+}\int\frac{d\omega}{2\pi}\bigg[&\frac{\partial\ln G_f(\omega)}{\partial\omega}-G_f(\omega)\frac{\partial\Sigma_f(\omega)}{\partial\omega} \\ &-\gamma\left(\frac{\partial\ln G_B(\omega)}{\partial\omega}-G_B(\omega)\frac{\partial\Sigma_B(\omega)}{\partial\omega}\right)\bigg]e^{i\omega t}. \end{aligned} \tag{38}$$

In this expression, the bosonic part (which vanishes altogether) has been included in order to transform further the terms involving derivatives of the self-energy, using analyticity. This transformation is only possible if both fermionic and bosonic terms are considered. This is because the Luttinger-Ward functional[18] of this model involves both Green's functions. It has a simple explicit expression which reads

$$\Phi_{\rm LW}(G_{f,\alpha},G_{B,i})=\sum_{\alpha,i}\int dtG_0(t)G_{f,\alpha}(-t)G_{B,i}(t) \tag{39}$$

such that the saddle-point equations (15) are recovered by derivation

$$\Sigma_{f,\alpha}(t)=\frac{\delta\Phi_{\rm LW}}{\delta G_{f,\alpha}(-t)},\quad \Sigma_{B,i}(t)=-\frac{\delta\Phi_{\rm LW}}{\delta G_{B,i}(-t)}. \tag{40}$$

From the existence of $\Phi_{\rm LW}$, we obtain the sum rule

$$\int_{-\infty}^{\infty}d\omega\left(\Sigma_f(\omega)\frac{\partial G_f(\omega)}{\partial\omega}-\gamma\Sigma_B(\omega)\frac{\partial G_B(\omega)}{\partial\omega}\right)=0. \tag{41}$$

(Note that there is no logarithmic divergence at $\omega=0$ in this expression.) After integrating by parts and using the asymptotic behavior of the two Green's functions in order to eliminate the boundary terms, we get

$$q_0=i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\frac{\partial\ln G_f(\omega)}{\partial\omega}-\gamma\frac{\partial\ln G_B(\omega)}{\partial\omega}\right)e^{i\omega0^+}. \tag{42}$$

Since $G(\omega)=G^R(\omega)$ for $\omega>0$ and $G(\omega)=\overline{G^R(\omega)}$ for $\omega<0$ (with G^R the retarded Green's function), this can be transformed using

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\partial\ \ln\ G_{f,B}(\omega)}{\partial\omega}e^{i\omega 0^+}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\partial\ \ln\ G^R_{f,B}(\omega)}{\partial\omega}e^{i\omega 0^+}+\int_{-\infty}^{0}\frac{d\omega}{2\pi}\frac{\partial}{\partial\omega}\ln\left(\frac{\overline{G^R_{f,B}(\omega)}}{G^R_{f,B}(\omega)}\right)e^{i\omega 0^+}. \tag{43}$$

The first integrals in the right hand side can be deformed in the upper plane and their sum vanishes.[18] Thus we obtain (denoting by arg G the argument of G)

$$\pi q_0=\arg\ G_f^R(0^-)-\arg\ G_f^R(-\infty)-\gamma[\arg\ G_B^R(0^-)-\arg\ G_B^R(-\infty)]. \tag{44}$$

arg $G_f^R(0^-)$ directly follows from the parametrization (27) defining θ. It can also be read off from the behavior of the scaling function $g_f(z)$ for $z=\pm\infty$. Thus, from Eqs. (33–34) we can also read off arg $G_B^R(0^-)$:

$$\arg\ G_f^R(0^-)=\pi\Delta_f-\theta-\pi,\quad \arg\ G_B^R(0^-)=\theta-\pi\Delta_f. \tag{45}$$

Taking into account that $G_f^R(\omega)\sim_{\omega\to-\infty}1/\omega$ and that Im $G_f^R<0$ we have arg $G_f^R(-\infty)=-\pi$. Similarly, we have arg $G_B^R(-\infty)=0$. Inserting these expressions into Eq. (44), we finally obtain the desired relation between q_0 and θ (or α):

$$\frac{\theta(1+\gamma)}{\pi}=\frac{1}{2}-q_0,\quad \alpha=\ln\frac{\sin[\pi q_0/(1+\gamma)]}{\sin[\pi(1-q_0)/(1+\gamma)]}. \tag{46}$$

This, together with Eq. (33), fully determines the universal scaling form of the spectral functions in the low-frequency, low-temperature limit.

VI. PHYSICAL QUANTITIES AND COMPARISON WITH THE CFT APPROACH

A. Impurity residual entropy at $T=0$

The impurity contribution to the free-energy (per color of spin) $f_{\rm imp}=(F-F_{\rm bulk})/N$ reads, at the saddle point

$$f_{\rm imp}=q_0\lambda+T\sum_n\ \ln\ G_f(i\omega_n)-\gamma T\sum_n\ \ln\ G_B(i\nu_n)-\int_0^\beta d\tau\Sigma_f(\tau)G_f(-\tau). \tag{47}$$

This expression can be derived either directly from the saddle-point effective action (in which case the last term arises from the quadratic term in Q and $\overline{Q}$), or from the relation between the free-energy and the Luttinger-Ward functional. $F_{\rm bulk}=N^2\gamma T$ Tr ln G_0 is the free energy of the conduction electrons. In Eq. (47) the formulas Tr ln G are ambiguous. We must precisely define which regularization of these sums we consider: the actual value of the sums depends on the precise definition of the functional integral. For the fermionic field, the standard procedure of adding and substracting the contribution of a free local fermion, and introducing an oscillating term to regularize the Matsubara sum holds:

$$\mathrm{Tr}\ \ln\ G_f=-T\ \ln 2+T\sum_n\ \ln[i\omega_n G_f(i\omega_n)]e^{i\omega_n 0^+}. \tag{48}$$

The situation is somewhat less familiar for the bosonic field. As pointed out above, the latter is merely a commuting auxiliary field (rather than a true boson). We have found that the correct regularization to be used is

$$\mathrm{Tr}\ \ln\ G_B=T\lim_{N\to\infty}\sum_{n=-N}^{n=N}\ \ln[JG_B(i\omega_n)]. \tag{49}$$

The factor of J takes into account the determinant introduced by the decoupling with B, and a *symmetric* definition of the (convergent) Matsubara sum has been used. Some details and justifications about these regularizations are given in Appendix (D). We shall perform a low-temperature expansion of Eq. (47), considering successively the particle-hole symmetric ($q_0=1/2$) and asymmetric ($q_0\neq 1/2$) cases, which require rather different treatments.

1. The particle-hole symmetric point $q_0=\frac{1}{2}$

In this case $\lambda=0$, so that the first term in Eq. (47) does not contribute. Let us consider the last term in (47). Using the spectral representation of G_f and the definition of Σ_f we obtain easily (for $\lambda=0$)

$$\Psi\equiv\int_0^\beta d\tau\Sigma_f(\tau)G_f(-\tau)=\int_{-\infty}^{+\infty}d\omega\frac{\omega\rho_f(\omega)}{1+e^{\beta\omega}}. \tag{50}$$

We substract the value at $T=0$:

$$\Psi(T)-\Psi(T=0)=-\int_0^\infty d\omega\omega[\rho_f(\omega,T)-\rho_f(\omega,T=0)]+2\int_0^\infty d\omega\frac{\omega\rho_f(\omega)}{1+e^{\beta\omega}}. \tag{51}$$

In the second term, we can replace ρ_f by its scaling limit. So this term is of order $O(T^{2\Delta_f+1})$. We know the asymptotics of ϕ_f: $\phi_f(x)\sim_{x\to\infty}C_1x^{2\Delta_f-1}+C_2x^{2\Delta_f-3}$. (The term $x^{2\Delta_f-2}$ cancels due to the particle-hole symmetry.) Thus, the first term in Eq. (51) is of the form $T^{2\Delta_f+1}\int_0^\infty dxx[\phi_f(x)-C_1x^{2\Delta_f-1}]$ (the integral is convergent). We conclude that $\Psi(T)=\Psi(0)+O(T^{2\Delta_f+1})$, so that the last term in Eq. (47) does not contribute to the zero-temperature entropy in the particle-hole symmetric case.

Let us express the remaining terms in Eq. (47) as integrals over real frequencies, using the regularizations introduced above. As detailed in Appendix D, this leads to the following expression, involving the argument of the (finite-temperature) retarded Green's functions:

$$f_{\rm imp}=\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega\left\{n_F(\omega)\left[\arctan\left(\frac{G_f'(\omega)}{G_f''(\omega)}\right)-\frac{\pi}{2}\right]-\gamma n_B(\omega)\arctan\left(\frac{G_B''(\omega)}{G_B'(\omega)}\right)\right\}. \qquad (52)$$

In these expressions n_F (respectively, n_B) are the Fermi (respectively, Bose) factor.

At this point, it would seem that in order to perform a low-temperature expansion of the free-energy, one has to make a Sommerfeld expansion of the Fermi and Bose factors. This is not the case, however, for two reasons: (i) the argument of the Green's functions appearing in Eq. (52) are *not* continuous at $\omega=0$, so that a linear term in T does appear (as expected from the nonzero value of $S_{\rm imp}$) and (ii) the Green's functions have an *intrinsic* temperature dependence, and the full scaling functions computed above must be used in Eq. (52). More precisely, when computing the difference $f_{\rm imp}(T)-f_{\rm imp}(T=0)$, the leading term is obtained by replacing the Green's function by their scaling form (33).

These considerations lead to the following expression of the impurity entropy (per spin color) $s_{\rm imp}=S_{\rm imp}/N$ at zero temperature for $q_0=\frac{1}{2}$:

$$\begin{aligned}s_{\rm imp}=&-\frac{1}{\pi}\int_{-\infty}^{0}d\tilde\omega[a_f(\tilde\omega)-a_f(\tilde\omega=-\infty)]\\&-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\tilde\omega\frac{1}{e^{|\tilde\omega|}+1}\,a_f(\tilde\omega)\mathrm{sgn}(\tilde\omega)\\&-\frac{\gamma}{\pi}\int_{-\infty}^{0}d\tilde\omega[a_b(\tilde\omega)-a_b(\tilde\omega=-\infty)]\\&+\frac{\gamma}{\pi}\int_{-\infty}^{+\infty}d\tilde\omega a_b(\tilde\omega)\mathrm{sgn}(\tilde\omega)\frac{1}{e^{|\tilde\omega|}-1}\,.\end{aligned} \qquad (53)$$

In this expression, $a_{f,B}$ denotes the arguments of the scaling functions, obtained from Eq. (33):

$$a_f(\tilde\omega)\equiv\arctan\frac{g_f'(\tilde\omega)}{g_f''(\tilde\omega)}=-\arctan\left(\cot(\pi\Delta_f)\tanh\frac{\tilde\omega}{2}\right), \qquad (54)$$

$$a_B(\tilde\omega)\equiv\arctan\frac{g_B''(\tilde\omega)}{g_B'(\tilde\omega)}=\arctan\left(\tan(\pi\Delta_f)\tanh\frac{\tilde\omega}{2}\right). \qquad (55)$$

From Eq. (53), we obtain with $t=\tan\pi\Delta_f$

$$\frac{s_{\rm imp}}{\gamma+1}=-\frac{2}{\pi}\int_0^1du\left\{\frac{2\arctan t}{\pi(1-u^2)}\left[u\arctan\left(\frac{u}{t}\right)+\frac{\arctan(ut)}{u}\right]-\frac{\arctan(ut)}{u(1-u^2)}\right\}. \qquad (56)$$

To perform the integration, we note that $\partial/\partial t((\partial/\partial t)\{(1+t^2)[s_{\rm imp}/(1+\gamma)]\})=-2t/\pi(1+t^2)$ and we obtain finally the simple expression

$$s_{\rm imp}\left(q_0=\frac{1}{2}\right)=\ln 2-\frac{\gamma+1}{\pi}\int_0^{\tan[\pi/2(1+\gamma)]}\frac{\ln(1+u^2)}{(1+u^2)}du. \qquad (57)$$

This can also be rewritten, after a change of integration variable, as

$$s_{\rm imp}(q_0=1/2)\equiv\frac{1}{N}S_{\rm imp}=\frac{1+\gamma}{\pi}\left[f\left(\frac{\pi}{1+\gamma}\right)-2f\left(\frac{\pi}{2(1+\gamma)}\right)\right] \qquad (58)$$

with

$$f(x)\equiv\int_0^x\ln\sin(u)du.$$

This coincides with the large-N limit of the CFT result, Eq. (6), in the particle-hole symmetric case.

2. The general case $q_0\neq 1/2$

For $q_0\neq\frac{1}{2}$, the first term in Eq. (47) also contributes to the entropy. Indeed, as shown below, the Lagrange multiplier $\lambda(T)$ at the saddle point has a term which is linear in temperature. This stems from a very general thermodynamic relation, which is derived by taking the derivative of Z with respect to q_0 in the functional integral, leading to

$$\left\langle-\frac{1}{\beta}\int_0^\beta i\mu(\tau)\right\rangle=\frac{\partial F}{\partial q_0}, \qquad (59)$$

where the average is to be understood with the action (11). At the saddle point, we thus have $\lambda=\partial f_{\rm imp}/\partial q_0$ and in particular

$$\left.\frac{\partial\lambda}{\partial T}\right|_{T=0}=-\frac{\partial s_{\rm imp}}{\partial q_0}. \qquad (60)$$

We shall directly use this equation in order to compute the residual entropy, by calculating the linear correction in T to λ, and then integrating over q_0. This method shortcuts the full low-temperature expansion of the free energy (as done in the previous section), which actually turns out to be quite a difficult task to perform correctly for $q_0\neq 1/2$.[19] In order to calculate this linear correction, we shall relate λ to the *high-frequency behavior* of the fermion Green's function. As $\Sigma_f(i\omega_n)\to 0$ when $\omega_n\to\pm\infty$ we have

$$G_f(i\omega_n)=\frac{1}{i\omega_n}-\frac{\lambda}{(i\omega_n)^2}+O\left(\frac{1}{(i\omega_n)^2}\right). \qquad (61)$$

This shows that $-\lambda$ is the discontinuity of the derivative of $G_f(\tau)$ with respect to τ at $\tau=0$:

$$\partial_\tau G_f(0^+)+\partial_\tau G_f(\beta^-)=\int_{-\infty}^{+\infty}d\omega\ \omega\rho_f(\omega)=-\lambda. \qquad (62)$$

Let us define $g(\tau)$ by

$$G_f(\tau)=\frac{e^{\alpha(\tau/\beta-1/2)}}{\cosh\alpha/2}g(\tau), \qquad (63)$$

where α is the spectral asymmetry parameter in Eq. (46) (at this stage we emphasize that the full finite temperature, finite cutoff, Green's function G_f is considered). Equation (62) can be rewritten as

$$\lambda=\alpha T+\tanh\left(\frac{\alpha}{2}\right)[\partial_\tau g(0^+)-\partial_\tau g(\beta^-)]$$
$$-[\partial_\tau g(0^+)+\partial_\tau g(\beta^-)], \tag{64}$$

where we have used that $G_f(0^+)+G_f(\beta^-)=-1$. Denoting by $\rho_g(\omega)$ the spectral function associated with g, we have

$$\partial_\tau g(0^+)-\partial_\tau g(\beta^-)=\int d\omega\frac{\omega[\rho_g(\omega)-\rho_g(-\omega)]}{1+e^{-\beta\omega}}. \tag{65}$$

In the scaling limit, the spectral function ρ_g must become *particle-hole symmetric* [since the effect of the particle-hole asymmetry in this limit is entirely captured by α in Eq. (63)], and must coincide with $A_f T^{2\Delta_f-1}\phi_f(\omega/T;q_0=1/2)$. Hence, following the same reasoning than for Ψ above, the term in Eq. (65) is of order const$+O(T^{2\Delta_f+1})$. Thus we have

$$\left.\frac{\partial\lambda}{\partial T}\right|_{T=0}=\alpha-\left.\frac{\partial A}{\partial T}\right|_{T=0}, \tag{66}$$

where $A=\partial_\tau g(0^+)+\partial_\tau g(\beta^-)$ is the discontinuity of the derivative $\partial_\tau g$. A reflects the particle-hole asymmetry of g and thus vanishes in the scaling limit. Actually the derivative $\partial A/\partial T$ also vanishes as $T\to 0$ as we now show. Consider first sending the bare cutoff to infinity (along with J) so as to keep the Kondo temperature fixed. In this limit A takes the form: $A=Tf(T/T_K)$. The low-energy scaling limit, in which Eq. (29) holds, can be reached by fixing T and sending T_K to infinity. Since g must become particle-hole symmetric in this limit, this implies that $f(x)$ vanishes at small x. Hence, taking a derivative with respect to temperature, of $A=Tf(T/T_K)$ we find that $\partial A/\partial T|_{T=0}=0$. Thus we finally obtain

$$\frac{\partial s_{\rm imp}}{\partial q_0}=-\left.\frac{\partial\lambda}{\partial T}\right|_{T=0}=-\alpha, \tag{67}$$

where $\alpha(q_0)$ is given in Eq. (46). Integrating this equation over q_0 [taking into account as a boundary condition the value of $s_{\rm imp}(q_0=1/2)$ obtained above], we finally derive the expression of the entropy

$$s_{\rm imp}\equiv\frac{1}{N}S_{\rm imp}=\frac{1+\gamma}{\pi}\left[f\left(\frac{\pi}{1+\gamma}\right)-f\left(\frac{\pi}{1+\gamma}(1-q_0)\right)\right.$$
$$\left.-f\left(\frac{\pi}{1+\gamma}q_0\right)\right]. \tag{68}$$

with, as above, $f(x)\equiv\int_0^x \ln\sin(u)du$. The expression (68) coincides precisely with the large N limit of the CFT result (6).[20] A plot of the residual entropy and of the asymmetry parameter α as a function of q_0 is displayed in Fig. 3. $s_{\rm imp}$ is maximal at $q_0=1/2$ and vanishes as $q_0\to 0$ as expected.

In this section, we have discovered that the spectral asymmetry parameter (''twist'') α shares a fairly simple relation with the residual entropy, given by Eq. (67). These are two

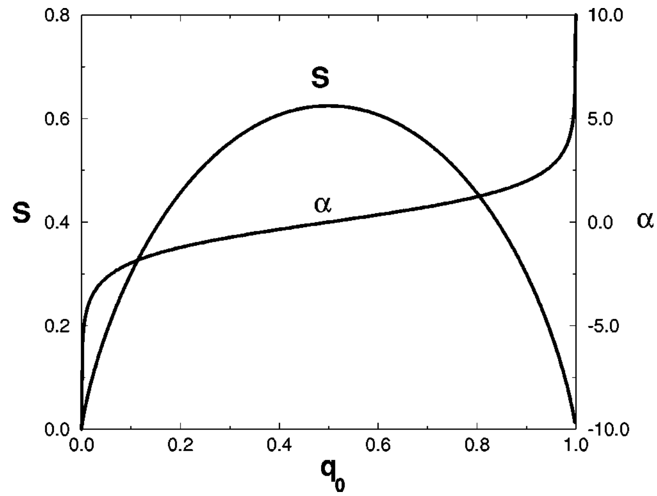

FIG. 3. Residual entropy $s_{\rm imp}$ and α vs q_0 for $\gamma=1.5$.

universal quantities, characteristic of the fixed point. Remarkably, α also coincides with the term proportional to T in λ (while λ itself is nonuniversal, its linear term in T is). It is tempting to speculate that a deeper interpretation of these facts is still to be found.

B. Internal energy and specific heat

The low-temperature behavior of the internal energy in the large-N limit can be obtained by two different methods. We shall briefly describe both since they emphasize different and complementary aspects of the physics.

In the first method, we use the effective action in the form (11), *before* the decoupling with the auxiliary bosonic field $B_i(\tau)$ is made. We thus have a quartic interaction vertex between the conduction electrons at the origin and the Abrikosov fermions representing the quasiparticles in the spin sector, which reads

$$\frac{J}{N}\sum_{1\leqslant\alpha,\beta\leqslant N}\left(f_\beta^\dagger f_\alpha-\frac{Q}{N}\delta_{\alpha\beta}\right)\sum_{i=1}^{K}c_{i\alpha}^\dagger c_{i\beta}. \tag{69}$$

One can then perform a skeleton expansion of the free-energy functional in terms of the *interacting* Green's functions for the pseudofermions and the conduction electrons $G_f(\tau)$ and $G_c(\tau)$. The first-order (Hartree) contribution to this functional vanishes because the spin operator in Eq. (69) is written in a traceless manner. The next contribution, at second order, yields the most singular contribution at low temperature and reads

$$\Delta E\propto J^2\int_0^\beta d\tau G_c(\tau)^2 G_f(-\tau)^2. \tag{70}$$

At the saddle point, the interacting conduction electron Green's function is $G_c(\tau)\propto G_f(\tau)G_B(-\tau)$, and hence its dominant long-time behavior is $G_c(\tau)\sim 1/\tau$. Inserting this, together with $G_f(\tau)\sim 1/\tau^{2\Delta_f}$ in Eq. (70), we see that the leading low-temperature behavior to the energy reads $\Delta E\propto c_1T^{4\Delta_f+1}+c_2T^2+\cdots$, and hence to the specific heat coefficient

$$\gamma>1:\ C/T\sim T^{4\Delta_f-1}\sim\left(\frac{1}{T}\right)^{(\gamma-1)/(\gamma+1)},$$
$$\gamma=1:\ C/T\sim\ln 1/T, \tag{71}$$
$$\gamma<1:\ C/T\sim\text{ const}$$

which agrees with the CFT result described above. We note that there is a quite precise connection between this calculation and the CFT approach: the operator appearing in the Kondo interaction (69) acquires conformal dimension $2(\Delta_c+\Delta_f)=1+2\Delta_f$ and has the appropriate structure of the scalar product of a spin current with the operator $S_{\alpha\beta}$ (transforming as the adjoint). Therefore, it is the large-N version of the leading irrelevant operator associated with the spin sector, as described in Sec. III. It is satisfying that the leading low-T behavior comes from the second-order contribution of this operator in this formalism as well. We note that for the simple Kondo model (1), in the scaling limit, the analogous irrelevant operator in the flavor sector *does not* show up in the calculation of the energy in the large-N solution. We shall comment further on this point below.

The second method to investigate the internal energy is to push the low-temperature expansion of the free energy to higher orders. To this end, we need to compute higher-order terms in the expansion (22a) of the Green's functions in the scaling regime. This computation is detailed in Appendix C 3, and leads to

$$G_f(\tau)=A_f\beta^{-2\Delta_f}g_f\left(\frac{\tau}{\beta}\right)+\beta^{-4\Delta_f}g_f^{(2)}\left(\frac{\tau}{\beta}\right)+\beta^{-6\Delta_f}g_f^{(3)}\left(\frac{\tau}{\beta}\right)+\cdots, \tag{72}$$

$$G_B(\tau)=A_B\beta^{-2\Delta_B}g_B\left(\frac{\tau}{\beta}\right)+\beta^{-1}g_B^{(2)}\left(\frac{\tau}{\beta}\right)+\beta^{-1-2\Delta_f}g_B^{(3)}\left(\frac{\tau}{\beta}\right)+\cdots. \tag{73}$$

Let us emphasize that the exponents appearing in this expansion *are not symmetric between the bosonic and fermionic degrees of freedom*. This is because we are dealing with the Kondo model for which the auxiliary field (bosonic) propagator has no frequency dependence in the noninteracting theory. Also, the expansion given in Eq. (72) assumes a perfectly flat conduction band in the limit of an infinite bandwith (conformal limit). Using this expansion into the expression (52) of the free energy leads to a specific heat coefficient $C/T\sim c_0T^{2\Delta_f-1}+c_1T^{4\Delta_f-1}+c_2+\cdots$. The coefficient c_0 actually vanishes, so that the behavior in Eq. (71) is recovered. The vanishing of c_0 was clear in the first approach, where it followed from the absence of Hartree terms. In the CFT approach, it is associated with the fact that the leading irrelevant operator does not contribute to the free energy at first order. The vanishing of c_0 implies nontrivial sum rules relating the scaling functions $g_{f,B}$ and $g_{f,B}^{(2)}$ (which we have not attempted to check explicitly).

We also note that this behavior of the specific heat is modified when an Anderson model version of the present model is considered (as in Ref. 7). Because the noninteracting slave boson propagator has a frequency dependence, the exponents of the second-order terms as written in Eq. (72) are only correct for $\gamma>1$ for the Anderson model. For $\gamma<1$, the term $\beta^{-4\Delta_f}g_f^{(2)}(\tau/\beta)$ is replaced by $\beta^{-1}g_f^{(2)}(\tau/\beta)$, while $\beta^{-1}g_B^{(2)}(\tau/\beta)$ is replaced by $\beta^{-4\Delta_b}g_B^{(2)}(\tau/\beta)$. As a result, one finds a diverging specific heat coefficient *in both cases*, with $C/T\sim T^{-(\gamma-1)/(\gamma+1)}$ for $\gamma>1$ and $C/T\sim T^{-(1-\gamma)/(\gamma+1)}$ for $\gamma<1$. The behavior for $\gamma<1$ is due to the leading irrelevant operator in the flavor sector. Similarly, for $\gamma<1$ in the Anderson model, the susceptibility associated with the *flavor (channel) sector* χ_f is found to diverge,[7] so that a finite Wilson ratio can still be defined as $T\chi_f/C$ for $\gamma<1$.

C. Resistivity and *T* matrix

In order to discuss transport properties, we define a scattering T matrix for the conduction electrons in the usual manner (for a single impurity):

$$G(\vec{k},\vec{k}',\omega+i0^+)=G_0(\vec{k},\omega+i0^+)\delta_{\vec{k},\vec{k}'}+G_0(\vec{k},\omega+i0^+)\times T(\omega)G_0(\vec{k}',\omega+i0^+), \tag{74}$$

where G and G_0 denote the interacting and noninteracting conduction-electron Green's functions, respectively. Taking a flat particle-hole symmetric band for the conduction electron and denoting by ρ_0 the local noninteracting density of states, this yields the local conduction electron Green's function in the form

$$G(\omega+i0^+)\equiv\sum_{\vec{k},\vec{k}'}G(\vec{k},\vec{k}')=-i\pi\rho_0[1-i\pi\rho_0T(\omega)]. \tag{75}$$

Following Ref. 4, we parametrize the zero-frequency limit of the T matrix in terms of a scattering amplitude S^1 as

$$T(\omega=0)\equiv-\frac{i}{2\pi\rho_0}(1-S^1) \tag{76}$$

so that, the zero frequency electron Green's function reads

$$G(i0^+)=-i\pi\rho_0\frac{1+S^1}{2}. \tag{77}$$

$S^1=1$ corresponds to no scattering at all, while $S^1=-1$ corresponds to maximal unitary scattering (i.e., $\pi/2$ phase shift and vanishing conduction electron density of states at the impurity site). In the overscreened case, as noted in Ref. 4, S^1 is in general such that $|S^1|<1$, reflecting the non-Fermi-liquid nature of the model, and the fact that the actual quasiparticles bear no resemblance to the original electrons. In addition here, we shall find the feature that S^1 is in fact a *complex number* for nonparticle-hole symmetric spin representations (i.e., $q_0\neq 1/2$).

We first derive an expression for S^1 for arbitrary N, K and spin representation $Q=Nq_0$ by generalizing to SU(N) the CFT approach of Ref. 4. There, it was shown that S^1 can be expressed as a ratio of elements of the modular S matrix $S_{\alpha,\beta}$ of the $\widehat{\mathrm{SU}}(N)_K$ algebra. Denoting by 0 the identity representation, by F the fundamental representation (corresponding to a Young tableau with a single box), and by R the representation in which the impurity lives (Young tableau with a single column of Q boxes), one has[3,4]

$$S^1=\frac{S_{F,R}/S_{0,R}}{S_{F,0}/S_{0,0}}. \tag{78}$$

The evaluation of these elements of the modular S matrix can be done along the same lines as the conformal field theory calculation of the entropy, described above. Some details are given in Appendix B. The result is:

$$S^1=\frac{\sin[(N+1)\pi/(N+K)]\exp\{-i[\pi(1-2q_0)/(N+K)]\}-\sin[\pi/(N+K)]\exp\{-i[\pi(N+1)(1-2q_0)/(N+K)]\}}{\sin[\pi N/(N+K)]}. \tag{79}$$

Notice that S^1 has both real and imaginary parts in the absence of particle hole symmetry $q_0\neq\frac{1}{2}$.

Let us take the large-N limit of this expression, with $K/N=\gamma$ fixed. This reads, to first nontrivial order

$$S^1=1+\frac{\pi}{N(1+\gamma)}\left[\cot\frac{\pi}{1+\gamma}-\frac{\cos[\pi(1-2q_0)/(1+\gamma)]}{\sin[\pi/(1+\gamma)]}\right] -\frac{i\pi}{N(1+\gamma)}\left[1-2q_0-\frac{\sin[\pi(1-2q_0)/(1+\gamma)]}{\sin[\pi/(1+\gamma)]}\right]. \tag{80}$$

We now show how to recover this expression from an analysis of the integral equations of the direct large-N solution. Coupling an external source to the conduction electrons in the functional integral formulation of the model, it is easily seen that the conduction electron T matrix is given, in the large-N limit, by

$$T(\omega)\ =\ \frac{1}{N}\mathcal{G}(\omega+i0^+), \tag{81}$$

where $\mathcal{G}$ denotes the convolution of the fermion and auxiliary boson Green's function

$$\mathcal{G}(\tau)=G_f(\tau)G_B(-\tau). \tag{82}$$

Hence, we have at $T=0$

$$S^1=1-\frac{2i\pi\rho_0}{N}\mathcal{G}(i0^+). \tag{83}$$

We first make use of the scaling limit of the two Green's functions, given by Eq. (29), and obtain the scaling form of $\mathcal{G}$:

$$\mathcal{G}(\tau)=G_f(\tau)G_B(\beta-\tau)=\frac{\pi A_fA_B}{\beta\cosh^2(\alpha/2)\sin(\pi\tau/\beta)}+\cdots. \tag{84}$$

We note that, *in this scaling limit*, the particle-hole asymmetry of the impurity Green's function has been lost altogether: α has cancelled completely in the τ dependence of $\mathcal{G}$ in this limit and is only present in the prefactor. Thus, only Re S^1 can be extracted from the scaling limit, while Im S^1 requires a more sophisticated analysis. Equation (84) implies $\mathrm{Im}\,\mathcal{G}=A_fA_B\pi$, and hence $\mathrm{Re}\,S^1-1=2\pi^2A_fA_B\rho_0/[N\cosh^2(\alpha/2)]$. We make use of the expression (C8) derived in Appendix C for the product of amplitudes A_fA_B and obtain

$$\mathrm{Re}\,S^1-1=-\frac{\pi}{(1+\gamma)N}\mathrm{Re}\tan\left(\pi\Delta_f-\frac{i\alpha}{2}\right). \tag{85}$$

After expressing α in terms of q_0 using Eq. (46) as

$$\tanh\left(\frac{\alpha}{2}\right)=-\cot(\pi\Delta_f)\tan\left(\frac{(1-2q_0)\pi}{2(1+\gamma)}\right), \tag{86}$$

Re S^1 coincides with the real part of Eq. (80).

We now consider Im S^1, for which we need to go beyond the scaling limit and use global properties of the Green's functions. First expressing Σ_f as a convolution on the imaginary axis and using $\partial_\omega G_0(i\omega)\to-2i\pi\rho_0\delta(\omega)$ in the limit of a flat particle-hole symmetric band we obtain

$$-i\rho_0\gamma2\pi\mathcal{G}(i\nu)=\mathcal{A}+\mathcal{B}(\nu) \tag{87}$$

with the definitions

$$\mathcal{A}=\int d\omega G_f(i\omega)\partial_\omega\Sigma_f(i\omega), \tag{88}$$

$$\mathcal{B}(\nu)=\int d\omega[G_f(i\omega+i\nu)\partial_\omega\Sigma_f(i\omega)-G_f(i\omega)\partial_\omega\Sigma_f(i\omega)]. \tag{89}$$

In the limit of vanishing ν, $\mathcal{B}(\nu)$ can be calculated from the scaling limit of G_f. We obtain

$$\mathcal{B}(\nu\to 0^+) = \frac{\gamma}{1+\gamma}\left((-e^{-i\pi(1-2q_0)/(1+\gamma)} + e^{-2i\pi\Delta_f}) \frac{\pi}{\sin 2\pi\Delta_f} + i\pi \right). \tag{90}$$

On the other hand, $\mathcal{A}$ contains high-frequency information that is lost in the scaling limit. We find

$$\mathcal{A} = -\frac{i\pi\gamma}{1+\gamma}(1-2q_0). \tag{91}$$

The details of these calculations are provided in Appendix E.

Combining Eqs. (91), (90), (87), (83) we find agreement with the large N limit of S_1 in Eq. (80). For a dilute array of impurities (of concentration $n_{\rm imp}$), the conduction electron-self energy is given by $\Sigma(\omega + i0^+) \simeq n_{\rm imp} T(\omega)$, to lowest order in $n_{\rm imp}$. As shown above, $T(\omega)$ is given in the large-N approach by the Fourier transform of $G_f(\tau)G_B(-\tau)$. The expansion (72) yields the long-time behavior $G_f(\tau) \sim A_f/\tau^{2\Delta_f} + A_f^{(2)}/\tau^{4\Delta_f} + \cdots$ and $G_B(\tau) \sim A_B/\tau^{2\Delta_B} + A_B^{(2)}/\tau + \cdots$. From the fact that $2\Delta_f + 2\Delta_B = 1$, this implies $G_f(\tau)G_B(\tau) \sim 1/\tau + 1/\tau^{1+2\Delta_f} + \cdots$. Hence the resistivity behaves as

$$\rho(T) \sim n_{\rm imp}\rho_u \left(\frac{1 - \mathrm{Re}\, S^1}{2} - cT^{2\Delta_f} + \cdots \right), \tag{92}$$

where ρ_u is the impurity resistivity in the unitary limit. For the same reasons as above, the Anderson model result would lead to an exponent $2\Delta_B$ in the regime $\gamma < 1$.[7]

D. The limit of a large number of channels ($\gamma \to \infty$)

We finally emphasize that all the expressions derived above greatly simplify in the limit of a large number of channels $\gamma \to \infty$. This is expected, since in this limit the non-Fermi-liquid intermediate coupling fixed point becomes perturbatively accessible from the weak-coupling one.[1,21] The physics of the fixed point can be viewed as an almost free spin of ''size'' $Q = Nq_0$ weakly coupled to the conduction electrons. Indeed the large-γ expansion of the entropy (53), the Green function G_f (33), and the twist α (46) are

$$S_{\rm imp} = -[q_0 \ln q_0 + (1-q_0)\ln(1-q_0)] - \frac{\pi^2 q_0(1-q_0)}{6\gamma^2} + \cdots,$$

$$g_f(\tilde{\tau}) = -\frac{e^{\alpha(\tilde{\tau}-1/2)}}{\cosh(\alpha/2)}\left[1 + \frac{1}{\gamma} \ln\frac{\pi}{\sin \pi\tilde{\tau}} + \cdots \right], \tag{93}$$

$$\rho_f(\omega) = \frac{1}{T}\delta\left(\frac{\omega}{T} + \alpha \right) + \cdots,$$

$$\alpha = \ln \frac{q_0}{1-q_0} + \cdots,$$

and the leading terms in these expansions are given by the corresponding quantities for a free spin of size Nq_0. Moreover the scattering matrix S^1 and resistivity have the following expansion:

$$\mathrm{Re}\, S^1 = 1 - \frac{2\pi^2 q_0(1-q_0)}{N\gamma^2}\cdots, \quad \mathrm{Im}\, S^1 = O\left(\frac{1}{\gamma^3}\right),$$

$$\rho(T=0)/(n_{\rm imp}\rho_u) = \frac{1}{N} q_0(1-q_0)\frac{\pi^2}{\gamma^2} + \cdots, \tag{94}$$

while the anomalous dimensions read $\Delta_S = 2\Delta_f = 1/\gamma - 1/\gamma^2 + \cdots$, $2\Delta_B = 1 - 1/\gamma + 1/\gamma^2 + \cdots$.

VII. CONCLUSION

In this paper, we have focused on the *non-Fermi-liquid overscreened regime* of the $SU(N)\times SU(K)$ multichannel Kondo model. This model has actually a wider range of possible behavior, which become apparent when other kinds of representations of the impurity spin are considered. In a recent short paper,[9] two of us have studied *fully symmetric* representations corresponding to Young tableaus with a single line of P boxes. (This amounts to considering *Schwinger bosons* in place of the Abrikosov fermions used in the present work.) It was demonstrated that, in that case, a transition occurs as a function of the size P of the impurity spin, from overscreening (for $P<K$) to underscreening (for $P>K$), with an exactly screened point in between ($P=K$). The large-N analysis of the overscreened regime $P<K$ is essentially identical to that presented in the present paper for antisymmetric representations.

Obviously, an interesting open problem is to understand the physics of the model for more general impurity spin representations, involving both ''bosonic'' and ''fermionic'' degrees of freedom (corresponding respectively to the horizontal and vertical directions in the associated Young tableau). CFT methods are a precious guide in achieving this goal. In particular, the formulas and rules given in Appendixes A and B allow for an easy derivation of the impurity $T=0$ residual entropy and zero-frequency T matrix, using Affleck and Ludwig's fusion principle and the identification of these quantities in terms of modular S matrices.

An open question which certainly deserves further study is to identify which of these more general spin representations are such that a direct large-N solution of the model can be found. This question has obvious potential applications to the multi-impurity problem and Kondo lattice models.

During the course of this study, we learned of a work by A. Jerez, N. Andrei and G. Zaránd on the same model using the Bethe Ansatz method. Our results and conclusions agree when a comparison is possible (in particular for the impurity residual entropy and low-temperature behavior of physical quantities).

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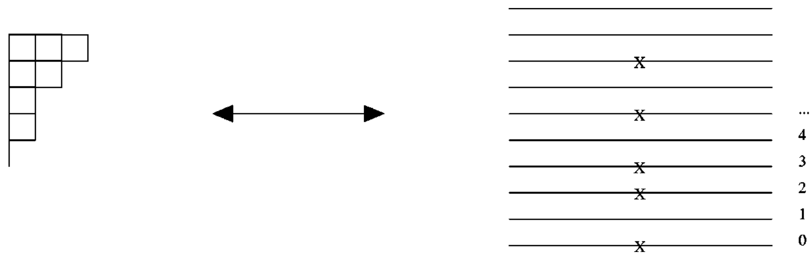

FIG. 4. An example of an SU(N) Young tableau (for $N=5$) and its associated fermionic representation.

APPENDIX A: THE STRONG COUPLING STATE

We now describe in more detail the proof of the statements in Sec. II about the nature and degeneracy of the strong-coupling state $R_{\rm sc}$. For a general reference on the group theory material used in this appendix, the reader is referred e.g., to Ref. 22. Let us note $\mathcal{N}_Y$ the number of electrons brought on the impurity site and by Y the Young tableau with $\mathcal{N}_Y$ boxes associated with the representation in which the conduction electrons on the impurity site combine. Because of the Pauli principle, *the length of any of its lines must be smaller than* K (and hence $\mathcal{N}_Y$ must be smaller than NK). Indeed, we must antisymmetrize the wave function separately for each flavor.

The Kondo energy is given by

$$E=J_K\sum_{\alpha\beta} S_{\alpha\beta}S'_{\beta\alpha} \tag{A1}$$

with

$$S_{\alpha\beta}=f^\dagger_\alpha f_\beta-\frac{Q}{N}\delta_{\alpha\beta}\quad S'_{\alpha\beta}=c^\dagger_\alpha c_\beta \tag{A2}$$

in which f denotes the pseudofermion and c the conduction electrons at the impurity site. We can introduce the linear combinations

$$T_{\alpha\beta}=\frac{S_{\alpha\beta}+S_{\beta\alpha}}{\sqrt{2}},\quad \alpha>\beta,\quad T_{\alpha\beta}=\frac{S_{\alpha\beta}-S_{\beta\alpha}}{i\sqrt{2}},$$
$$\alpha<\beta,\quad T_{\alpha\beta}=S_{\alpha\alpha},\quad \alpha=\beta, \tag{A3}$$

such that

$$\sum_{\alpha\beta} S_{\alpha\beta}S_{\beta\alpha}=\sum_{\alpha\beta} T^2_{\alpha\beta}. \tag{A4}$$

This leads to the following expression of the Kondo energy:

$$\frac{2E}{J_K}=C_2(R_{\rm sc})-C_2(Y)-C_2(R) \tag{A5}$$

in which $C_2(Z)$ denotes the quadratic Casimir operator of the representation Z. The representation $R_{\rm sc}$ is the specific component of $Y\otimes R$ associated with the bound state formed by the impurity spin and the conduction electrons at strong coupling. We recall that R is a column of length Q in this paper. We have to minimize E over all possible choices of Y and of $R_{\rm sc}$.

First let us recall that for a general representation Y, C_2 is given by

$$C_2=\frac{1}{N}\;{}^t\vec{n}A^{-1}\left(\frac{\vec{n}}{2}+1\right), \tag{A6}$$

where $n_i(1\leqslant i\leqslant N-1)$ is the number of columns with length i in the Young tableau Y and A is the Cartan matrix of the SU(N) group.[22] Let us denote by $f_j(1\leqslant j\leqslant N)$ the length of the line j in the tableau. Then we have

$$C_2=\frac{1}{N}\left[\frac{1}{2}\sum_{j=1}^{N}(f_j-j+N)^2-\left(\frac{1}{2N}\mathcal{N}_Y^2+\frac{N-1}{2}\mathcal{N}_Y\right)\right.$$
$$\left.-\frac{N(N-1)(2N-1)}{12}\right], \tag{A7}$$

with $\mathcal{N}_Y=\Sigma_{j=1}^N f_j$ is the number of boxes of Y. Note that with this definition, all f_j's can be shifted by the same constant without changing the representation (this is because a column of length N can be removed without changing the representation). Equation (A7) can be given a simple interpretation in terms of N "particles" occupying a set of fermionic levels. This interpretation was introduced in a slightly different form by Douglas.[11] Let $p_j=f_j-j+N$ be the position of the particle j. Because Y is a Young tableau, the particles are ordered and cannot be on the same level. Figure 4 gives an example of the construction of the diagram associated with a simple Young tableau.

A simple construction of all allowed Young tableaus appearing in the tensor product $Y\otimes R$ (Ref. 22) can be given in this fermionic language. Starting with the diagram associated with Y, we choose Q particles and raise each of them by one level beginning with the one in the highest level. (We note that, in the fermionic interpretation, adding a box to line i corresponds to raising the ith particle up by one level.) An example is given in Fig. 5.

Let us denote by p_i the positions of the N particles in Y, and by p'_i the new positions in a given allowed component of $Y\otimes R$. The Kondo energy is given by

$$E=\frac{1}{4N}\left(\sum_{j=1}^{N}(p'^2_i-p_i^2)-\frac{\mathcal{N}_Y}{2N}-\frac{2N-1}{8}-C_2(R)\right). \tag{A8}$$

The last two terms are constant (R is held fixed) and can be dropped in the minimization process.

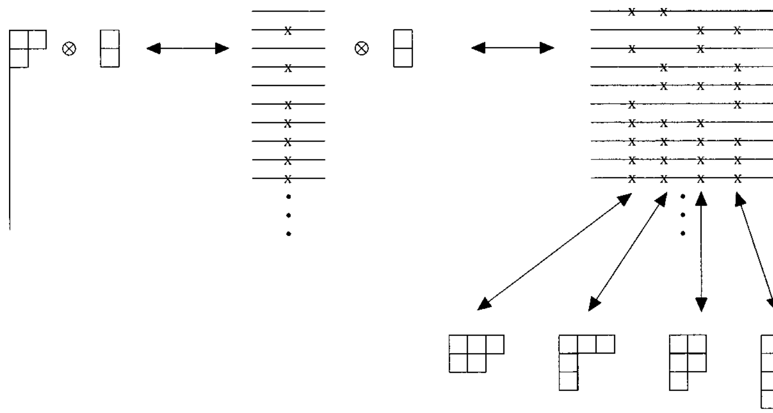

FIG. 5. An example of the general composition rule explained in the text. We show the fermionic diagram associated with Y, the resulting fermionic diagrams and their transcription in terms of Young tableaus. (N is arbitrary in this example as shown by the dots).

The p_i's can be decomposed in two sets: those for which $p_i' = p_i$ (we have $N-Q$ of them) and those for which $p_i' = p_i + 1$ (Q of them). Let us denote by P the sum of the latter ones. We have

$$E = \frac{1}{4N}\left(Q + 2P - \frac{\mathcal{N}_Y}{2N} - \frac{2N-1}{8} - C_2(R) \right). \tag{A9}$$

Thus the lowest energy is achieved for the smallest possible value of P. Since a given shift $p \to p+1$ can only appear once in the sum (because double occupancies are forbidden and a given particle cannot be raised twice), the absolute minimum is obtained when we sum on all the lowest Q shifts. This implies that the diagram associated with Y has Q particles on the Q lowest levels (from 0 to $Q-1$) and none on the Qth level.

The upper part of the diagram (above level Q) is then determined by the maximization of $\mathcal{N}_Y$. Going back to the language of Young tableaus, the minimum is thus achieved when Y is a rectangle of height $N-Q$ and width K, and $R_{\rm sc}$ is given by the same tableau with the first column removed.

Two cases must thus be distinguished

(i) For $(K=1)$ and for arbitrary N and Q, $R_{\rm sc}$ is the trivial (singlet) representation [of dimension $d(R_{\rm sc}) = 1$].

(ii) For ($K \geq 2$, arbitrary N and Q) the dimension $d(R_{\rm sc})$ is larger than the dimension of R. Indeed, denoting by $d_K(R_{\rm sc})$ the dimension $R_{\rm sc}$ for K channels, we have the recursion relation (from the "hook law"[23])

$$\frac{d_{K+1}}{d_K} = \frac{(N+K)(N+K-1)\cdots(N+K-Q+1)}{(Q+K)(Q+K-1)\cdots(Q+K+1-Q)} > 1 \tag{A10}$$

(because $Q<N$). It increases with K. The $K=2$ case is just a column of length $N-Q$ which has the dimension of R. Moreover the inequality is strict for $K>2$.

APPENDIX B: CONSTRUCTION OF MODULAR *S*-MATRICES

If two representations R and R' correspond to fermion configurations with positions $\{p_1, \ldots, p_N\}$ and $\{p_1', \ldots, p_N'\}$ (See Appendix A), respectively, then the modular S-matrix element is

$$S_{R,R'} = C_{N,K} e^{-2\pi i N \overline{p}\,\overline{p'}/(N+K)} \det[e^{2\pi i p_i p'_j/(N+K)}] \tag{B1}$$

with $\overline{p} = \Sigma_i p_i / N$, $\overline{p'} = \Sigma_j p_j'/N$ and $C_{N,K}$ is a constant which depends only on N and K. Since for the trivial representation 0, the p's are the consequent integers $0,1,\ldots,N-2,N-1$, $S_{0,R}$ involves a determinant of the form

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_1 & z_2 & \cdots & z_{N-1} & z_N \\ z_1^2 & z_2^2 & \cdots & z_{N-1}^2 & z_N^2 \\ \vdots & \vdots & & \vdots & \vdots \\ z_1^{N-2} & z_2^{N-2} & \cdots & z_{N-1}^{N-2} & z_N^{N-2} \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_{N-1}^{N-1} & z_N^{N-1} \end{vmatrix},$$

where $z_j = e^{2\pi i p_j/(N+K)}$, $\{p_1, \ldots, p_N\}$ being the positions of fermions corresponding to the representation R. This is just the Van Der Monde determinant $\Delta(z) = \Pi_{i<j}(z_i - z_j)$.

To calculate the T matrix, we also need to know the S-matrix element between the fundamental representation F and an arbitrary representation R. For F, the positions of the fermions are $0,1,2,\ldots,N-2,N$. Therefore $S_{F,R}$ involves the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_1 & z_2 & \cdots & z_{N-1} & z_N \\ z_1^2 & z_2^2 & \cdots & z_{N-1}^2 & z_N^2 \\ \vdots & \vdots & & \vdots & \vdots \\ z_1^{N-2} & z_2^{N-2} & \cdots & z_{N-1}^{N-2} & z_N^{N-2} \\ z_1^{N} & z_2^{N} & \cdots & z_{N-1}^{N} & z_N^{N} \end{vmatrix}.$$

This determinant has the same antisymmetry property in z_i's as the Van Der Monde determinant. However, the present determinant is one order higher than $\Delta(z)$ as a homogeneous polynomial in z's. A little reflection shows it is $(\Sigma_i z_i)\Delta(z)$.

Finally we find

$$\frac{S_{F,R}}{S_{0,R}} = e^{-2\pi i \overline{p}/(N+K)} \sum_{j=1}^{N} e^{2\pi i p_j/(N+K)}. \tag{B2}$$

Using this formula we deduce Eq. (79) from Eq. (78).

APPENDIX C: SOLUTION OF THE SADDLE-POINT EQUATIONS IN THE SCALING REGIME

In this appendix we solve Eq. (15) in the scaling regime as explained in Sec. V A and obtain the scaled spectral densities and Green functions.

1. Scaling functions

First we show that Eq. (29) is the solution of the saddle point equations in the scaling regime. We deal with an arbitrary q_0. Let us denote by $\sigma_{f,B}$ the scaling function of the fermionic and bosonic self-energies

$$\Sigma_{f,B}(\tau)=A_{B,f}\ \beta^{-2\Delta_{B,f}-1}\sigma_{f,B}\left(\frac{\tau}{\beta}\right). \tag{C1}$$

G_0 is the local Green function for the conduction electron. Its density of states does not depend on T. So its scaling form is

$$G_0(\tau)=-\frac{\rho_0\pi}{\beta\sin(\pi\tau/\beta)}, \tag{C2}$$

with $\rho_0=-(1/\pi)\mathrm{Im}\,G_0(\omega=0)$. Using this formula and Eq. (15), $\sigma_{f,B}$ are related to $g_{f,B}$. We insert the scaling form (22a) into Eq. (16). Matching the power in β leads to $2\Delta_f+2\Delta_B=1$ and

$$A_f^{-1}g_f^{-1}(i\overline{\omega_n})=[\lambda-\Sigma_f(i\omega_0)]\beta^{2\Delta_B}-A_B[\sigma_f(i\overline{\omega_n})-\sigma_f(i\overline{\omega_0})],$$

$$A_B^{-1}g_B^{-1}(i\overline{\nu_n})=\left(\frac{1}{J}-\Sigma_B(0)\right)\beta^{2\Delta_f}-A_f[\sigma_B(i\overline{\nu_n})-\sigma_B(0)] \tag{C3}$$

with $i\overline{\omega_n}=i(2n+1)\pi$ and $i\overline{\nu_n}=i2n\pi$. The term $i\omega_n$ in Eq. (16) vanishes in this scaling limit because $\Delta_{f,B}<1$. We assume that at zero temperature

$$\lambda-\Sigma_f(0)=\frac{1}{J}-\Sigma_B(0)=0 \tag{C4}$$

so β disappears of these equations at lower order. Only $\sigma_f(i\overline{\omega_n})-\sigma_f(i\overline{\omega_0})$ and $\sigma_B(i\overline{\nu_n})-\sigma_B(0)$ have a scaling form.

We insert our ansatz into Eq. (C3) with the following Fourier transform formulas [which follow from Ref. 24, Eq. (3.631)]:

$$g_f(i\overline{\omega_n})=\frac{(2\pi)^{2\Delta_f}T^{2\Delta_f-1}i(-1)^{n+1}\Gamma(1-2\Delta_f)}{\cosh(\alpha/2)\Gamma(1-\Delta_f-\overline{\omega_n}/2\pi+i\alpha/2\pi)\Gamma(1-\Delta_f+\overline{\omega_n}/2\pi-i\alpha/2\pi)}, \tag{C5a}$$

$$g_B(i\overline{\nu_n})=\frac{(2\pi)^{2\Delta_B}T^{2\Delta_B-1}(-1)^{n+1}\Gamma(1-2\Delta_B)}{\cosh(\alpha/2)\Gamma(1-\Delta_B-\overline{\nu_n}/2\pi+i\alpha/2\pi)\Gamma(1-\Delta_B+\overline{\nu_n}/2\pi-i\alpha/2\pi)}, \tag{C5b}$$

$$\sigma_f(i\overline{\omega_n})-\sigma_f(i\overline{\omega_0})=\frac{i\gamma\rho_0(2\pi)^{2\Delta_B+1}T^{2\Delta_B}(-1)^{n+1}\Gamma(-2\Delta_B)}{\cosh(\alpha/2)\Gamma(\frac{1}{2}-\Delta_B-\overline{\omega_n}/2\pi+i\alpha/2\pi)\Gamma(\frac{1}{2}-\Delta_B+\overline{\omega_n}/2\pi-i\alpha/2\pi)}-(n=0), \tag{C5c}$$

$$\sigma_B(i\overline{\nu_n})-\sigma_B(0)=\frac{\rho_0(2\pi)^{2\Delta_f+1}T^{2\Delta_f}(-1)^{n+1}\Gamma(-2\Delta_f)}{\cosh(\alpha/2)\Gamma(\frac{1}{2}-\Delta_f-\overline{\nu_n}/2\pi+i\alpha/2\pi)\Gamma(\frac{1}{2}-\Delta_f+\overline{\nu_n}/2\pi-i\alpha/2\pi)}-(n=0). \tag{C5d}$$

We see that Eq. (29) is the solution of Eq. (C3) provided that the following conditions are met.

(i) The precise form of the cancelation (C4) at finite temperature is (at leading order in T)

$$\frac{1}{J_K}-\Sigma_B(0)=\frac{\rho_0A_f(2\pi)^{2\Delta_f+1}T^{2\Delta_f}\Gamma(-2\Delta_f)}{\cosh(\alpha/2)\Gamma(\frac{1}{2}-\Delta_f+i\alpha/2\pi)\Gamma(1/2-\Delta_f-i\alpha/2\pi)}, \tag{C6}$$

$$\lambda-\Sigma_f(i\omega_0)=\frac{i\gamma\rho_0A_B(2\pi)^{2\Delta_B+1}T^{2\Delta_B}\Gamma(-2\Delta_B)}{\cosh(\alpha/2)\Gamma(-\Delta_B+i\alpha/2\pi)\Gamma(1-\Delta_B-i\alpha/2\pi)}. \tag{C7}$$

(ii) Equation (19) is obeyed: $2\Delta_f=1/(1+\gamma)$, $2\Delta_B=\gamma/(1+\gamma)$.

(iii) We have the relation between amplitudes:

$$2\Delta_B=-2\gamma A_fA_B\rho_0\Gamma(1-2\Delta_B)\Gamma(2\Delta_B)\times\frac{|\sin(\pi\Delta_B-i\alpha/2)|^2}{\cosh^2(\alpha/2)}. \tag{C8}$$

In our scaling forms, α is the same for the fermionic and the bosonic function. One can check easily that for a more general ansatz with α_f and α_B the saddle-point equations imply $\alpha_f=\alpha_B$.

2. Spectral densities

We calculate now the scaled spectral density from the above scaling function. Denoting $\zeta=-1$ for fermions and $\zeta=1$ for bosons we have the general formula

$$G_{f,B}(\tau)=-\int_{-\infty}^{+\infty}\frac{e^{-\tau\varepsilon}}{1-\zeta e^{-\beta\varepsilon}}\rho_{f,B}(\varepsilon)d\varepsilon \quad 0\leqslant\tau\leqslant\beta. \tag{C9}$$

In the scaling regime, we have to solve

$$\frac{e^{\alpha(x-1/2)}}{\cosh(\alpha/2)}\left(\frac{\pi}{\sin(\pi x)}\right)^{2\Delta_{f,B}} =\int_{-\infty}^{+\infty}\frac{e^{-xu}}{1-\zeta e^{-u}}\phi_{f,B}(u)du \quad 0\leqslant x\leqslant 1. \tag{C10}$$

Setting $t=i(x-1/2)$ we see it is sufficient to solve

$$\frac{e^{-i\alpha t}}{\cosh(\alpha/2)}\left(\frac{\pi}{\cosh(\pi t)}\right)^{2\Delta_{f,B}} =\int_{-\infty}^{+\infty}\frac{e^{itu}}{e^{u/2}-\zeta e^{-u/2}}\phi_{f,B}(u)du \quad |\mathrm{Im}\ t|<\frac{1}{2}. \tag{C11}$$

Due to the properties of Fourier transformation, we can just solve for $\alpha=0$, and obtain the solution for arbitrary α with Eq. (31). With

$$\int_{-\infty}^{+\infty}dt\left(\frac{\pi}{\cosh(\pi t)}\right)^{\Delta}e^{-itu} =(2\pi)^{\Delta-1}\frac{\Gamma(\Delta/2+iu/2\pi)\Gamma(\Delta/2-iu/2\pi)}{\Gamma(\Delta)} \quad \begin{cases}0<\Delta<1\\ u \text{ real,}\end{cases} \tag{C12}$$

[see formula (3.313.2) of Ref. 24), we find the result given in the text (24):

$$\phi_f(\tilde{\omega},q_0=1/2)=\frac{1}{\pi}(2\pi)^{2\Delta_f-1}\cosh\frac{\tilde{\omega}}{2} \times\frac{\Gamma[\Delta_f+i\tilde{\omega}/2\pi]\Gamma[\Delta_f-i\tilde{\omega}/2\pi]}{\Gamma(2\Delta_f)}$$

$$\phi_B(\tilde{\omega},q_0=1/2)=\frac{1}{\pi}(2\pi)^{2\Delta_B-1}\sinh\frac{\tilde{\omega}}{2} \times\frac{\Gamma[\Delta_B+i\tilde{\omega}/2\pi]\Gamma[\Delta_B-i\tilde{\omega}/2\pi]}{\Gamma(2\Delta_B)}. \tag{C13}$$

The asymptotic behavior follows from formula (8.328) of Ref. 24.

We then derive the full Green function by taking the Hilbert transform

$$g(z)=\int_{-\infty}^{+\infty}dx\,\frac{\phi(x)}{z-x}. \tag{C14}$$

We find Eq. (33) using the following.

(i) The representation

$$\frac{1}{z-u}=-i\int_0^{+\infty}e^{i\lambda(z-u)}d\lambda \quad \mathrm{Im}\,z>0. \tag{C15}$$

(ii) The Fourier formula which inverses Eq. (C12).

(iii) The formula

$$\int_0^{+\infty}dx\,\frac{e^{izx}}{[\sinh(\pi x/\beta)]^{\Delta}}=2^{\Delta-1}\frac{\beta}{\pi}\,\frac{\Gamma(\Delta/2-i\beta z/2\pi)\Gamma(1-\Delta)}{\Gamma[1-\Delta/2-i(\beta z/2\pi)]} \quad \begin{cases}0<\Delta<1\\ z \text{ real,}\end{cases} \tag{C16}$$

which results from formula (3.112.1) of Ref. 24.

Finally, we comment on the treatment of the constraint equation (17) in our derivation of the scaling functions. The relation between α and q_0 has been derived from a Luttinger sum rule, which holds at zero temperature. So one may worry whether the scaling form does satisfy the leading low-temperature corrections to the $T=0$ constraint equation. We show now that this is actually the case. Starting from Eq. (17) written as:

$$\int_{-\infty}^{+\infty}d\omega\rho_f(\omega,T)n_F(\omega)=q_0 \tag{C17}$$

we substract the relation at $T=0$ and take into account the asymptotic behavior of ϕ_f given by Eq. (25) to obtain

$$\int_{-\infty}^{0}dx\left(\phi_f(x)-\frac{e^{\alpha/2}|x|^{2\Delta_f-1}}{\cosh(\alpha/2)\,\Gamma(2\Delta_f)}\right) +\int_{-\infty}^{\infty}dx\ \frac{\mathrm{sgn}\ x\,\phi_f(x)}{e^{|x|}+1}=0. \tag{C18}$$

It is a rather strong constraint on the scaling function ϕ_f that this equation should hold, and it is satisfying that the explicit form obtained for ϕ_f does satisfy Eq. (C18). This proves that ϕ_f is really a solution of the full system (15),(16),(17) in the scaling regime *at fixed* q_0.

3. Higher-order terms in the scaling expansion

Here, we give some indications on the derivation the expansion in Eq. (72). Let us start from the long-time expansion for $T_K^{-1} \ll \tau \ll \beta$

$$G_f(\tau) \sim \frac{A_f}{\tau^{2\Delta_f}} + \frac{A_f^{(2)}}{\tau^{\alpha}}, \quad G_B(\tau) \sim \frac{A_B}{\tau^{2\Delta_B}} + \frac{A_B^{(2)}}{\tau^{\lambda}}, \tag{C19}$$

in which α and λ are exponents to be determined below. Then we have

$$G_f(\omega) \sim A_f C_{2\Delta f-1} \omega^{2\Delta_f-1} + A_f^{(2)} C_{\alpha-1} \omega^{\alpha-1}, \tag{C20}$$

with $C_\Delta = \int dt\, (e^{it}/t^{\Delta+1})$, and a similar expression for G_B. We can then deduce the expansions of Σ_f and Σ_B, and insert them into the saddle-point equation. We find

$$\frac{\omega^{1-2\Delta_f}}{A_f C_{2\Delta_f-1}} - \frac{A_f^{(2)} C_{\alpha-1}}{A_f^2 C_{2\Delta_f-1}^2}\, \omega^{\alpha+1-4\Delta_f} = \omega - \frac{\gamma C_{2\Delta_B} A_B \rho_0}{\pi}\, \omega^{2\Delta_B} - \frac{\gamma C_\lambda A_B^{(2)} \rho_0}{\pi}\, \omega^{\lambda}, \tag{C21}$$

$$\frac{\omega^{1-2\Delta_B}}{A_B C_{2\Delta_B-1}} - \frac{A_B^{(2)} C_{\lambda-1}}{A_B^2 C_{2\Delta_B-1}^2}\, \omega^{\lambda+1-4\Delta_B} = -\frac{C_{2\Delta_f} A_f \rho_0}{\pi}\, \omega^{2\Delta_f} - \frac{C_\alpha A_f^{(2)} \rho_0}{\pi}\, \omega^{\alpha}.$$

The first order yields

$$-\pi = \rho_0 A_f A_B C_{2\Delta_f} C_{2\Delta_B-1} = \gamma \rho_0 A_f A_B C_{2\Delta_B} C_{2\Delta_f-1}, \tag{C22}$$

which, using

$$C_{\Delta-1} \propto \Delta C_\Delta, \tag{C23}$$

gives Eq. (19) again. The second equation leads to $\lambda = \alpha + 1 - 4\Delta_f$.

First suppose $\lambda < 1$: in this case we must drop the ω term but we have

$$\frac{C_{\alpha-1} C_{\lambda-1}}{C_{2\Delta_B-1} C_{2\Delta_f-1}} = \frac{C_\alpha C_\lambda}{C_{2\Delta_B} C_{2\Delta_f}}, \tag{C24}$$

which implies $\alpha = 2\Delta_f$ or $\alpha = 2\Delta_f - 1$ [taking Eq. (C23) into account]. So this possibility must be rejected. Finally we are lead to $\lambda = 1$ and $\alpha = 4\Delta_f$. The higher order corrections can be dealt with in a similar manner. Restoring the scaling functions, this leads to Eq. (72).

APPENDIX D: CALCULATION OF THE RESIDUAL ENTROPY

1. The formula of the free energy

We first give a few more details on the regularization in Eq. (47). We will check that Eq. (49) is the right formula for the pseudoboson.

In the following we will denote by $\mathrm{Tr}_\pm$ the regularization with $e^{i\omega_n 0^\pm}$ and by $\mathrm{Tr}_{\mathrm{sym}}$ the regularization of Eq. (49). We note that $\mathrm{Tr}_{\mathrm{sym}} = (\mathrm{Tr}_+ + \mathrm{Tr}_-)/2$ as can be checked explicitly using a spectral representation of the function to be summed.

Let us introduce the following notation for any quantity A (function of λ): $\Delta_\lambda A = A^\lambda - A^{-\lambda}$. As the free energy is particle-hole symmetric, we have

$$-\lambda = \Delta_\lambda(T\,\mathrm{Tr}\ln G_f) - \gamma \Delta_\lambda(T\,\mathrm{Tr}\ln G_B). \tag{D1}$$

Let us consider

$$\phi(i\omega_n) = \ln\left(\frac{i\omega_n + \lambda - \Sigma_f^{\lambda}(i\omega_n)}{i\omega_n - \lambda - \Sigma_f^{-\lambda}(i\omega_n)} \right) \tag{D2}$$

such that

$$\Delta_\lambda(T\,\mathrm{Tr}_+ \ln G_f) = -\phi(\tau = 0^-). \tag{D3}$$

As ϕ is particle-hole symmetric, we have $\phi(\tau=0^+) = -\phi(\tau=0^-)$. As its asymptotic behavior is $\phi(i\omega_n) \sim 2\lambda/i\omega_n$, its discontinuity is $\phi(\tau=0^+) - \phi(\tau=0^-) = -2\lambda$.

We obtain

$$\Delta_\lambda(T\,\mathrm{Tr}_+ \ln G_f) = -\lambda. \tag{D4}$$

This implies that the bosonic term does not contribute to Eq. (D1). But there is an analogous relation for the boson: we first calculate the discontinuity of Σ_B from the saddle-point equations, use an analogous function ϕ, and obtain $\Delta_\lambda(T\,\mathrm{Tr}_\pm \ln G_B) = \mp(1-2q_0)J/2$. So we find

$$\Delta_\lambda(T\,\mathrm{Tr}_{\mathrm{sym}} \ln G_B) = 0. \tag{D5}$$

So we have checked that Eq. (49) is the right regularization for the bosonic term.

2. Derivation of Eq. (52)

We consider first the fermionic term. Let $G_0(i\omega_n) = 1/i\omega_n$ be the Green function of free electrons. We have

$$\begin{aligned} T\,\mathrm{Tr}_+ \ln G_f &= -T \ln 2 - \frac{1}{\pi}\int_{\mathcal{R}} d\omega (\mathrm{Im}\ln G_f - \mathrm{Im}\ln G_0) \times n_F(\omega) \\ &= -T\ln 2 + \frac{1}{\pi}\int_{\mathcal{R}} d\omega \left(\arctan\frac{G_f'(x)}{G_f''(x)} + \frac{\pi}{2} - \pi\theta(-x) \right) n_F(\omega) \end{aligned} \tag{D6}$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega \left(\arctan\frac{G_f'(x)}{G_f''(x)} - \frac{\pi}{2} \right) n_F(\omega). \tag{D7}$$

The bosonic term is obtained by an analogous calculation. In the particle hole symmetric case considered in the text the three regularizations for the bosonic term are equivalent. We have

$$-T\,\mathrm{Tr}_{\mathrm{sym}}\ln G_B=-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega\ \mathrm{Im}\ \ln[JG(\omega)]n_B(\omega)$$
$$=-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega\ \arctan\frac{G_B''(x)}{G_B'(x)}n_B(\omega). \tag{D8}$$

Finally we find the formula quoted in the text, Eq. (52).

APPENDIX E: SOME DETAILS OF THE T-MATRIX CALCULATION

In this appendix, we calculate $\mathcal{A}$ and $\mathcal{B}(0^+)$.

1. Computation of $\mathcal{A}$

Using the definition of Σ and introducing a oscillating term to regulate the two integrals, we have

$$\mathcal{A}=\int d\omega\ G_f(i\omega)\partial_\omega\Sigma_f(i\omega) \tag{E1}$$

$$=i\int_{-\infty}^{\infty}d\omega\ G_f(i\omega)e^{i\omega 0^+}$$
$$+\int_{-\infty}^{\infty}d\omega\ \partial_\omega\ln G_f(i\omega)e^{i\omega 0^+}. \tag{E2}$$

The first term is $2i\pi q_0$. Using Eq. (46) and

$$\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega 0^+}}{\omega}=i\pi \tag{E3}$$

(the integral is to be understood as a principal part), we have with $\psi(z)=\ln[zG_f(z)]$

$$\mathcal{A}=-2i\theta(1+\gamma)+\int_{i\mathcal{R}}dz(\partial_z\psi)(z) \tag{E4}$$

$$=-2i\theta(1+\gamma)-\lim_{\epsilon\to 0}2i\ \mathrm{Im}[\psi(i\epsilon)-\psi(i\infty)] \tag{E5}$$

$$=-2i\gamma\theta. \tag{E6}$$

We used $\psi(i\infty)=0$ and $\psi(i\epsilon)\sim\ln(A\epsilon^{2\Delta_f})-i\theta$ with A a real constant. Finally we find Eq. (91).

2. Computation of $\mathcal{B}(0^+)$

$$\mathcal{B}(\nu)=i\int d\omega[G_f(i\omega+i\nu)-G_f(i\omega)]$$
$$-\int d\omega[G_f(i\omega+i\nu)-G_f(i\omega)]\partial_\omega G_f^{-1}(i\omega). \tag{E7}$$

We replace G_f by the scaling function g_f. The second term is of order 1 whereas the first is $O(T^{2\Delta_f})$ and can be neglected. We have then

$$\mathcal{B}(\nu)=-\int dx\{g_f[i(x+1)\tilde{\nu}]-g_f(ix\tilde{\nu})\}\partial_x g_f^{-1}(ix\tilde{\nu}) \tag{E8}$$

with $\tilde{\nu}=\nu/T$. We want $\mathcal{B}(\nu=0^+,T=0)$ which is obtained by taking the limit $\tilde{\nu}\to+\infty$ in the previous scaling limit of $\mathcal{B}$. To perform this limit we use $g_f(\overline{z})=\overline{g_f(z)}$ and the following expansion for g:

$$g_f(ix)\underset{x\to+\infty}{\sim}cAx^{2\Delta_f-1}\ \text{ with }\ A=i\cosh\left(\frac{\alpha}{2}+i\pi\Delta_f\right), \tag{E9}$$

where c is a real constant [Eq. (E9) is obtained directly from Eq. (33)]. We find

$$-\mathcal{B}(0^+)=(1-2\Delta_f)\Bigg[-\int_{-\infty}^{-1}dx\frac{e^{ix0^+}}{|x|^{2\Delta_f}|x+1|^{1-2\Delta_f}}$$
$$-\frac{A}{\overline{A}}\int_{-1}^{0}dx\frac{e^{ix0^+}}{|x|^{2\Delta_f}(x+1)^{1-2\Delta_f}}$$
$$+\int_0^\infty dx\frac{e^{ix0^+}}{x^{2\Delta_f}(x+1)^{1-2\Delta_f}}-\int dx\frac{e^{ix0^+}}{x}\Bigg]. \tag{E10}$$

The last term a principal part and is given by Eq. (E3). We then use the following identity:

$$0=\int_{\mathcal{R}+i0^+}dz\frac{e^{iz0^+}}{z^{2\Delta_f}(z+1)^{1-2\Delta_f}}$$
$$=-\int_{-\infty}^{-1}dx\frac{e^{ix0^+}}{|x|^{2\Delta_f}|x+1|^{1-2\Delta_f}}$$
$$+e^{-2i\pi\Delta_f}\int_{-1}^{0}dx\frac{e^{ix0^+}}{|x|^{2\Delta_f}(x+1)^{1-2\Delta_f}}$$
$$+\int_0^\infty dx\frac{e^{ix0^+}}{x^{2\Delta_f}(x+1)^{1-2\Delta_f}}. \tag{E11}$$

We find

$$\mathcal{B}(0^+)=(1-2\Delta_F)\Bigg[\left(\frac{A}{\overline{A}}+e^{-2i\pi\Delta_f}\right)$$
$$\times\int_{-1}^{0}d\omega\frac{e^{ix0^+}}{|x|^{2\Delta_f}(x+1)^{1-2\Delta_f}}+i\pi\Bigg]$$
$$=\frac{\gamma}{1+\gamma}\left[\left(\frac{A}{\overline{A}}+e^{-2i\pi\Delta_f}\right)\frac{\pi}{\sin 2\pi\Delta_f}+i\pi\right]. \tag{E12}$$

A simple calculation with Eq. (E9) shows

$$\frac{A}{\overline{A}}=-e^{-i\pi(1-2q_0)/(1+\gamma)} \tag{E13}$$

and we find Eq. (90).

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