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# Rigorous analytical results on phase locking in Josephson junction ladder arrays

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Based on a previously developed analytical procedure combining ideas from the first harmonic approximation with those from the slowly varying phase we obtain some rigorous analytical results on the dynamics of two-dimensional Josephson-junction ladder arrays composed of an arbitrary number of cells. We are able to derive a general analytic expression for the reduced equations governing phase locking of the cells. While solving these reduced equations seems not to be possible in general, we were able to evaluate them up to the end for three experimentally relevant cases: (i) arrays composed of strongly damped junctions and small ring inductances without an external shunt, (ii) arrays composed of strongly damped junctions and small ring inductances with an external shunt, and (iii) arrays composed of strongly damped junctions and large ring inductances without an external shunt. [S0163-1829(98)01630-0]

## I. INTRODUCTION

Josephson-junction (JJ) arrays have been under consideration as tunable microwave radiation sources for several years now.[1,2] After some success with one-dimensional (1D) arrays[3–5] simple estimates led to the conclusion that two-dimensional (2D) arrays should be able to provide even larger output power.[6–8] However, up to now the radiation output of 2D arrays observed in experimental setups has been generally below that obtained from 1D arrays[9] and as a result, interest in this type of arrays has declined recently.

There may be several reasons responsible for this discrepancy, ranging from constructional and technological details to an insufficient theoretical understanding of phase locking in 2D Josephson-junction arrays. Our paper is devoted to the second aspect. We perform an analytical investigation of phase locking in 2D-Josephson junction arrays supported by numerical simulations carried out in parallel.

A good review on early theoretical work on 2D arrays can be found in Ref. 8. This is one of the rather few papers providing a more systematical approach to phase locking in 2D JJ arrays. Based on an analytical treatment of linearized equations Wiesenfeld *et al.* investigated the stability of the in-phase state with all junctions oscillating in phase. As a main result, they did not find any internal mechanism responsible for phase locking which seems rather plausible from our point of view, as ring inductances were neglected in their approach. In a more recent paper by Filatrella and Wiesenfeld[10] these inductances were taken into account with a corresponding coupling being observed, although the limits of the two-row coupling approximation exploited there have not been fully worked out.

There is another treatment of 2D arrays involving fabrication and simulation as well as an analytical approach.[11] Contrary to our work this investigation is mainly aimed towards modeling extended junctions, though. As a result, the authors choose a bias regime different from ours as well as weakly damped, hysteretic junctions.

Looking for a more consistent approach we developed a systematic framework for studying these types of arrays during the last years.[12–15] Our main goal was looking towards small ring inductances, including the effect of nonvanishing ring inductances as well as external flux, and to approach larger arrays step by step starting from simple configurations. The general conclusion to be drawn from these papers states that nonvanishing, small ring inductances lead to a small phase shift between the junctions oscillating within each cell but there is a tendency of the junctions to align antiphase in bias direction caused by the internal coupling of the cells.

Here, we will extend these results to larger arrays of the ladder type exploiting an analytical approximation scheme developed earlier.[15] Our approach takes into account finite (even arbitrary) ring inductances, nonvanishing external flux, an external shunt and, at least concerning the general result, a nonvanishing junction capacitance (finite McCumber parameter) as well. A short note on some strong-coupling results was published before[16] which is extended here by (i) giving a more explicit description of the method (ii) including longitudinal inductances as well as some results on nonvanishing junction capacitance (iii) including the technically more involved case of large inductances.

We start by writing down the basic dynamical equations for the array and explaining the approximation scheme in Sec. II. The lowest and first-order results presented in Sec. III give already a complete description of the dynamics within the cells while still containing a free phase parameter for each cell yet to be determined. While we are able to formulate reduced equations for these oscillation phases in Sec. IV taking into account all the parameters named above we were not able (and we doubt that it is possible at all) to present a general solution to it up to now.

Accordingly, we decided to restrict to some experimentally relevant limiting cases rendering the equations solvable: (i) small ring inductances without an external shunt (Sec. V), (ii) mall ring inductances with an external shunt (Sec. VI), (iii) large ring inductances without an external shunt (Sec. VII). Our treatment is completed by a summary of the results and some more general conclusions being presented in Sec. VIII. As a general observation, the behavior in the large-inductance case turns out to be quite different from the one observed for small ring inductances: While for small ring inductances the internal coupling gives rise to an antiphase

 

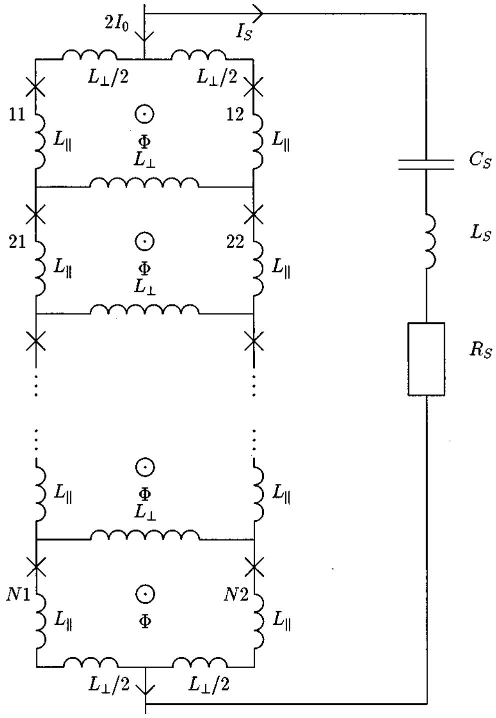


FIG. 1. A scheme of the two-dimensional ladder array under consideration. $2I_0$ is the bias current, $L_\perp$ and $L_\parallel$ are the transverse and longitudinal, respectively, inductances of the array, $\Phi$ denotes the external flux, $I_S$ is the current through the shunt and $C_S, L_S, R_S$ are the capacitive, inductive, and resistive, respectively, contribution to the shunt impedance.

(nonradiating) alignment of the voltage oscillations of neighboring junctions in the bias direction, large-inductance cells may cause in-phase oscillations of these junctions. However, for deriving more general conclusions this has to be contrasted with the fact observed earlier[2,12] that large-inductance cells lead to a relatively large phase shift in the presence of a nonvanishing external flux, while this shift is suppressed for smaller inductances. As a result, building 2D arrays providing a large radiation output will require a very special tuning of the junction as well as array parameters.

## II. BASIC EQUATIONS AND APPROXIMATION SCHEME

The 2D JJ ladder array under consideration is shown in Fig. 1. It has two junctions within each cell and no junctions transverse to the bias direction (sometimes being called a hybrid array[17]). All junctions and inductances are considered as being identical with junctions being characterized by their critical current $I_C$, normal resistance $R_N$ and capacitance $C$. For applying our perturbation scheme it proves convenient to normalize all quantities with respect to the bias current per junction $I_0$ instead of the critical current. Thus, we introduce normalized time as $s=2eR_NI_0t/\hbar$ (differing from usual Josephson physics normalization by a factor $I_0/I_C$), normalized junction capacitance as $\beta=2eR_N^2I_0C/\hbar$, and normalized ring currents as $i_k^o=(I_{k2}-I_{k1})/2I_0$ with $I_{kj}$ being the ring currents through junction $kj$ as indicated in Fig. 1. Furthermore, we define a normalized shunt current as $i_S=I_S/I_0$ with $I_S$ being the shunt current, shunt capacitance as $c_S=2eR_N^2I_0C_S/\hbar$, effective shunt inductance as $\lambda_S=2\pi I_0(L_S+NL_\parallel/2+L_\perp/2)/\Phi_0$, and shunt resistance as $r_S=R_S/R_N$ with $C_S, L_S, R_S$ being the shunt's capacitance, inductance, and resistance, respectively. Finally, $l_{\perp,\parallel}=2\pi L_{\perp,\parallel}I_0/\Phi_0$ are the normalized inductances perpendicular/parallel to the bias direction with $L_{\perp,\parallel}$ being the corresponding inductances itself and $\varphi=2\pi\Phi/\Phi_0$ is the normalized external flux with $\Phi$ being the external flux and $\Phi_0$ the magnetic flux quantum. A prime denotes the normalized time derivative with respect to $s$. In addition, we introduce the expansion parameter $b=I_C/I_0=1/i_0$.

The array will be described within the resistively capacitively shunted junction (RCSJ) model. As in previous work it proves convenient to combine the Josephson phase differences within each cell as $\Sigma_k=(\phi_{k2}+\phi_{k1})/2$ and $\Delta_k=(\phi_{k2}-\phi_{k1})/2$.

Based on these assumptions the array can be described by the following set of $3N+1$ equations:

$$\beta\Sigma_k''+\Sigma_k'+b\ \sin\Sigma_k\cos\Delta_k=1-\frac{i_S}{2}, \tag{1}$$

$$\beta\Delta_k''+\Delta_k'+b\ \sin\Delta_k\cos\Sigma_k=i_k^o, \tag{2}$$

$$\Delta_k-\frac{\varphi}{2}+(l_\parallel+l_\perp)i_k^o-\frac{l_\perp}{2}(i_{k+1}^o+i_{k-1}^o)=0, \tag{3}$$

$$\sum_{k=1}^{N}\Sigma_k''-r_Si_S'-\lambda_Si_S''-\frac{1}{c_S}i_S=0, \tag{4}$$

with $k=1\dots N$. While Eqs. (1), (2) are a direct consequence of the RCSJ model, Eq. (3) follows from the flux quantization condition and Eq. (4) from Kirchhoff's mesh rule for the loop involving the external shunt.

Our calculational scheme will be based on a combination of two perturbation expansions. First, we perform a small $b=1/i_0$ expansion along the lines described in Refs. 15,18. This is achieved by writing

$$\Sigma_k=\Sigma_{k,0}+b\Sigma_{k,1}, \tag{5}$$

$$\Delta_k=\Delta_{k,0}+b\Delta_{k,1}, \tag{6}$$

$$i_k^o=i_{k,0}^o+bi_{k,1}^o, \tag{7}$$

$$i_S=i_{S,0}+bi_{S,1}. \tag{8}$$

After inserting expansions (5)–(8) into the system (1)–(4), like powers of $b$ are compared to give zeroth- and first-order equations.

## III. LOWEST- AND FIRST-ORDER RESULTS

Evaluating the lowest-order equations is almost trivial giving the Josephson phase differences $\Delta_{k,0}=\varphi/2$, Ohm's law for the sums $\Sigma_{k,0}=\pi/2-\delta_k+s$ and vanishing currents $i_{S,0}=0$, $i_{k,0}^o=0$. Taking into account these lowest-order results, the first-order equations for $\Sigma_{k,1}, \Delta_{k,1}$ are inhomogeneous oscillation equations with the inhomogeneity sinusoidally oscillating with period 1. As our interest is focused on the stationary state the first-order solutions are supposed to be harmonic with the same period.

This way, a tedious but straightforward calculation leads to

$$\Sigma_k=\frac{\pi}{2}+s-\delta_k-b\frac{\cos(\varphi/2)}{1+\beta^2}[\sin(s-\delta_k)-\beta\cos(s-\delta_k)]$$
$$+b\frac{\cos(\varphi/2)}{2(1+\beta^2)|Z_S|}\sum_{j=1}^{N}[\cos(s-\delta_j-\psi_S)$$
$$+\beta\sin(s-\delta_j-\psi_S)], \tag{9}$$

$$\Delta_k=\frac{\varphi}{2}+b\sin(\varphi/2)\left(\frac{l_\perp}{2}(a_{k+1}+a_{k-1})-(l_\parallel+l_\perp)a_k\right)\cos s$$
$$+b\sin(\varphi/2)\left(\frac{l_\perp}{2}(b_{k+1}+b_{k-1})-(l_\parallel+l_\perp)b_k\right)\sin s, \tag{10}$$

and corresponding expressions for $i_k^o, i_S$ being omitted here. $|Z_S|$ and $\psi_S$ are the amount and the phase angle, respectively, of the external shunt impedance as

$$|Z_S|=\sqrt{(r_S+N/2)^2+(\lambda_S-1/c_S)^2} \tag{11}$$

and

$$\sin\psi_S=\frac{r_S+N/2}{|Z_S|},\quad \cos\psi_S=\frac{1/c_S-\lambda_S}{|Z_S|}. \tag{12}$$

At this point the oscillations within each cell are already completely determined up to the phases $\delta_k$. However, the problem is essentially complicated by the fact that the vectors $a_k$ and $b_k$ have to be determined from the linear algebraic system

$$P\boldsymbol{a}+Q\boldsymbol{b}=\boldsymbol{c},\quad -Q\boldsymbol{a}+P\boldsymbol{b}=\boldsymbol{d}, \tag{13}$$

where the tridiagonal matrices $P$ and $Q$ read

$$P=\begin{pmatrix} \beta(l_\parallel+l_\perp)-1 & -\beta l_\perp/2 & & \\ -\beta l_\perp/2 & \beta(l_\parallel+l_\perp)-1 & -\beta l_\perp/2 & 0 \\ & -\beta l_\perp/2 & \beta(l_\parallel+l_\perp)-1 & -\beta l_\perp/2 \\ 0 & \ddots & \ddots & \ddots \\ & & -\beta l_\perp/2 & \beta(l_\parallel+l_\perp)-1 \end{pmatrix}, \tag{14}$$

$$Q=\begin{pmatrix} -(l_\parallel+l_\perp) & l_\perp/2 & & \\ l_\perp/2 & -(l_\parallel+l_\perp)1 & l_\perp/2 & 0 \\ & l_\perp/2 & -(l_\parallel+l_\perp) & l_\perp/2 \\ 0 & \ddots & \ddots & \ddots \\ & & l_\perp/2 & -(l_\parallel+l_\perp) \end{pmatrix}, \tag{15}$$

while the vectors on the right hand sides of the system (13) are

$$c_k=-\sin\delta_k,\quad d_k=\cos\delta_k. \tag{16}$$

Although this is a genuine algebraical problem we are not able to write down the general solution of this system; instead, we will have to restrict ourselves to several physically relevant special cases later.

## IV. SLOWLY VARYING PHASE

For working out the unknown phases characterizing oscillations of junctions adjacent in the bias direction, we exploit a procedure differing from the one described in Ref. 18. Instead of investigating the second order of the small-$b$ expansion we perform a slowly varying phase analysis which has proven successful in investigating 1D arrays before.[1,2,19] Accordingly, we allow for an adiabatic time dependence of the phases,

$$\delta_k=\delta_k(s),\quad |\delta_k'|\ll 1. \tag{17}$$

To proceed, we take the time derivative of Eqs. (9), (10), insert the resulting expressions together with those for $i_k^o, i_S$ into the full system (1)–(2) and average over one period of oscillations regarding Eq. (17) (for details, see Ref. 15). This way we end up with so-called reduced equations for the phases $\delta_k$,

$$\beta_C\ddot{\delta}_k+\dot{\delta}_k=\frac{1}{2i_0}\frac{\cos^2(\varphi/2)}{1+\beta^2}+\frac{\cos^2(\varphi/2)}{4i_0|Z_S|}\sum_{i=1}^{N}\sin(\delta_k-\delta_i-\psi_S)$$
$$-\frac{\sin^2(\varphi/2)}{2i_0}\sum_{i=1}^{N}(d_kQ_{ki}a_i-c_kQ_{ki}b_i). \tag{18}$$

Here, and in the following dots denote derivatives with respect to time $\overline{s}=s/i_0$ in the conventional Josephson normalization (cf. Sec, II). The second term on the right-hand side describes the action of the external shunt while the third one characterizes interaction of the cells via internal ring currents.

In principle, the synchronization regimes of junctions adjacent in the bias direction can be worked out by solving the system (18). However, this is complicated by the circumstance that the coefficients $a_i$ and $b_i$ depending on $\delta_k$ of their own are not explicitly known but have to be determined from the system (13) which we are unable to solve in general. A possible strategy which we will pursue in the following starts by solving Eq. (13) imposing certain additional restrictions, inserting the resulting vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ together with $\boldsymbol{c}$ and $\boldsymbol{d}$ into Eq. (18) and evaluating for $\delta_k$. In general, there will be more than one solution of this system and one has to check the corresponding stability regions via a Lyapunov analysis.

## V. STRONGLY COUPLED CELLS WITHOUT EXTERNAL LOAD

Generally, we are able to solve the algebraic system (13) only for $\beta=0$ having the consequence that the following explicit results are applicable to overdamped junctions only. (Concerning some effects to be expected for $\beta\neq 0$ we refer to Ref. 15; for an earlier note on strongly coupled cells, see Ref. 16.) In the following we will additionally request $l_{\perp,\parallel}\ll 1$ corresponding to strong inductive coupling within the cells.

The strongly coupled ladder array without external load can be solved rather easily. For $\beta=0$ the matrix $P$ in Eq. (13) becomes proportional to unity. In addition, the matrix $Q$ being proportional to $l_\parallel$ and $l_\perp$, respectively, can be neglected in comparison to $P$. In this limit the system (13) admits the simple solution

$$a_k=\sin\,\delta_k,\quad b_k=-\cos\,\delta_k. \tag{19}$$

Inserting Eq. (19) into Eq. (18) leads to the reduced system

$$\dot{\delta}_k=\frac{1}{2i_0}\cos^2(\varphi/2)+\frac{l_\perp}{4i_0}\sin^2(\varphi/2)\times[\sin(\delta_k-\delta_{k-1})+\sin(\delta_k-\delta_{k+1})],\quad (k=2\ldots N-1). \tag{20}$$

The boundary cells do only have one neighbor and thus are described by different equations as

$$\dot{\delta}_1=\frac{1}{2i_0}\cos^2(\varphi/2)+\frac{l_\perp}{4i_0}\sin^2(\varphi/2)\sin(\delta_1-\delta_2), \tag{21}$$

$$\dot{\delta}_N=\frac{1}{2i_0}\cos^2(\varphi/2)+\frac{l_\perp}{4i_0}\sin^2(\varphi/2)\sin(\delta_N-\delta_{N-1}). \tag{22}$$

This way we were able to reduce the problem to the corresponding one-dimensional one being characterized by exactly the same system of equations (see, e.g. Ref. 19). This one has been solved long ago based on a Lyapunov function analysis, a matter of fact enabling us to write down the solution immediately. [The only difference from Ref. 19 is of a technical nature and results from the fact that we put the first-order frequency correction into $\delta_k(s)$ instead of writing it down explicitly as $\zeta-\zeta_0$.]

We will not go into the details of the stability analysis being identical to the one described in Ref. 19. It results in the fact that the only possible solutions are of the type

$$\delta_{k+1}-\delta_k=0,\quad \pi,\quad k=1\ldots N-1, \tag{23}$$

of which only the strictly antiphase oscillations

$$\delta_{k+1}-\delta_k=\pi \tag{24}$$

of neighboring junctions are stable against small fluctuations. Accordingly, the natural oscillation state of an unloaded small-inductance array is practically a nonradiating one.

## VI. STRONGLY COUPLED CELLS WITH EXTERNAL LOAD

The second example being at least partly solvable up to the end concerns strongly coupled cells with an external load. In this case the system (20) gets an additional term resulting from the external load as

$$\dot{\delta}_k=\frac{1}{2i_0}\cos^2(\varphi/2)-\frac{l_\perp\sin^2(\varphi/2)}{4i_0}\times[\sin(\delta_{k-1}-\delta_k)+\sin(\delta_{k+1}-\delta_k)]+\frac{\cos^2(\varphi/2)}{4i_0|Z_S|}\sum_{i=1}^{N}\sin(\delta_k-\delta_i-\psi_S),\quad (k=2\ldots N-1), \tag{25}$$

and analogous equations for the boundary cells,

$$\dot{\delta}_1=\frac{1}{2i_0}\cos^2(\varphi/2)+\frac{l_\perp}{4i_0}\sin^2(\varphi/2)\sin(\delta_1-\delta_2)+\frac{\cos^2(\varphi/2)}{4i_0|Z_S|}\sum_{i=1}^{N}\sin(\delta_1-\delta_i-\psi_S), \tag{26}$$

$$\dot{\delta}_N=\frac{1}{2i_0}\cos^2(\varphi/2)+\frac{l_\perp}{4i_0}\sin^2(\varphi/2)\sin(\delta_N-\delta_{N-1})+\frac{\cos^2(\varphi/2)}{4i_0|Z_S|}\sum_{i=1}^{N}\sin(\delta_N-\delta_i-\psi_S). \tag{27}$$

Subtracting the phases of neighboring cells gives the system

$$\dot{\vartheta}_k=-\frac{l_\perp}{4i_0}\sin^2(\varphi/2)(\sin\,\vartheta_{k-1}-2\sin\,\vartheta_k+\sin\,\vartheta_{k+1})+\frac{\cos^2(\varphi/2)}{2i_0|Z_S|}\sum_{i=1}^{N}\cos\left(\frac{\vartheta_{k,i}}{2}+\frac{\vartheta_{k-1,i}}{2}-\psi_S\right)\sin\frac{\vartheta_k}{2}, \tag{28}$$

where we introduced difference variables as

$$\vartheta_{k,i}=\delta_{k+1}-\delta_i, \tag{29}$$

and especially for neighboring junctions

$$\vartheta_k=\vartheta_{k,k}=\delta_{k+1}-\delta_k. \tag{30}$$

It is a nontrivial result that the system can be reduced in such a way that it contains difference variables (29) only. Evidently the system (28) allows for the homogeneous mode as a static solution,

$$\vartheta_k=\delta_k=0,\quad \forall k, \tag{31}$$

and a more detailed investigation shows that antiphase oscillations are among the possible static solutions as well, at least as long as the number of cells $N$ is even. Of course, this does not exclude the existence of more solutions.

In the following we will concentrate on the stability of this uniform mode $\vartheta_k=0$. For this purpose we consider small perturbations exploiting the Lyapunov ansatz

$$\vartheta_k=0+\epsilon_k e^{\lambda s}, \quad |\epsilon_k|\ll 1. \tag{32}$$

This way we arrive at a linear algebraic system for the perturbation amplitudes $\epsilon_k$ admitting nontrivial solutions only for

$$\begin{vmatrix} x & 1 & 0 & 0 & \dots \\ 1 & x & 1 & 0 & \dots \\ 0 & 1 & x & 1 & \dots \\ 0 & 0 & 1 & x & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}_{N-1} = 0 \tag{33}$$

with

$$x=\frac{4\lambda}{l_\perp \sin^2(\varphi/2)}-2-\frac{N\cos\psi_S}{l_\perp |Z_S|}\cot^2(\varphi/2).$$

As $k$ runs from 1 to $N-1$ there are $N-1$ Lyapunov coefficients

$$\lambda=\left(1-\cos\frac{k\pi}{N}\right)\frac{l_\perp}{2i_0}\sin^2(\varphi/2)+\frac{N\cos\psi_S}{4i_0|Z_S|}\cos^2(\varphi/2) \tag{34}$$

corresponding to the $N-1$ possible relative oscillation modes of the cells. The largest (most positive) of these coefficients limits the stability of the coherent oscillation mode of junctions adjacent in the bias direction.

The first term resulting from the internal interaction is positive definite this way trying to destabilize the coherent mode. On the other hand, according to Eq. (12) the sign of the second term depends on the character of the external load. Moreover, it is interesting to note that not the ring inductances, but only their horizontal contributions determine the weight of the first term. Furthermore, because of the second term being proportional to $N$ the sign of the Lyapunov coefficients is essentially determined by the character of the external load alone for sufficiently large $N$.

## VII. WEAKLY COUPLED CELLS WITHOUT EXTERNAL LOAD

Treatment of weakly coupled cells is essentially more complicated than that of strongly coupled cells even in absence of an external load. Based on Eq. (14) the following considerations can be shown to hold true even for nonvanishing longitudinal inductances $l_\parallel$ as long as $l_\parallel \ll l_\perp$. In this case the system (13) is solved by expanding $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to $1/l_\perp$. This leads to the reduced equations

$$\dot{\delta}_k=\frac{1}{2i_0}+\frac{\sin^2(\varphi/2)}{2i_0 l_\perp}\sum_{j=1}^{N}\left[d_k(b_1)_k+c_k(a_1)_k\right], \quad k=1\dots N, \tag{35}$$

where the first-order expansion terms $(a_1)_k,(b_1)_k$ have to be determined from the algebraic system

$$\begin{aligned}\sum_{j=1}^{N}\overline{Q}^{(N)}_{ij}(a_1)_j&=-2d_i,\\ \sum_{j=1}^{N}\overline{Q}^{(N)}_{ij}(b_1)_j&=2c_i,\end{aligned} \tag{36}$$

where $\overline{Q}^{(N)}$ is the $N$-dimensional tridiagonal matrix

$$\overline{Q}^{(N)}=\begin{pmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ 0 & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{37}$$

For finding the general solution of the system (36) we exploit Cramers's rule, e. g.,

$$(a_1)_k=\frac{1}{D^{(N)}}R_k^{(N)}, \tag{38}$$

with the $N$-dimensional determinant

$$D^{(N)}=\det\overline{Q}^{(N)} \tag{39}$$

and

$$R_k^{(N)}=\sum_{k=1}^{N}\overline{Q}^{(N)\mathrm{adj}}_{kj}(-2d_j) \tag{40}$$

and an analogous expression for $b_k$; $\overline{Q}^{(N)\mathrm{adj}}$ denotes the adjoint of $\overline{Q}^{(N)}$.

For evaluating $D^{(N)}$ we make use of the formula

$$D^{(N)}=(N+1)(-1)^N \tag{41}$$

based on the recursion relation

$$D^{(N)}=-2D^{(N-1)}-D^{(N-2)}. \tag{42}$$

Exploiting a procedure in which the dimensions of the matrices are reduced step by step and making repeated use of Eq. (41) we arrive at

$$R_k^{(N)}=(-1)^{N-1}\left[\sum_{i=1}^{k-1}d_i(N-k+1)i+d_k(N-k+1)k+\sum_{i=k+1}^{N}d_i(N-i+1)k\right]. \tag{43}$$

Combining Eq. (38) with Eqs. (41) and (43) we obtain

$$(a_1)_k=\frac{2}{N+1}\left[\sum_{i=1}^{k-1}(N-k+1)i\cos\delta_i+(N-k+1)k\cos\delta_k+\sum_{i=k+1}^{N}(N-i+1)k\cos\delta_i\right], \tag{44}$$

$$(b_1)_k=\frac{2}{N+1}\left[\sum_{i=1}^{k-1}(N-k+1)i\sin\delta_i+(N-k+1)k\sin\delta_k+\sum_{i=k+1}^{N}(N-i+1)k\sin\delta_i\right],\tag{45}$$

and finally the compact reduced equations

$$\dot{\delta}_k=\frac{1}{2i_0}-\frac{\sin^2(\varphi/2)}{i_0 l_\perp(N+1)}\left[\sum_{i=1}^{k-1}(N-k+1)i\sin(\delta_k-\delta_i)+\sum_{i=k+1}^{N}(N-i+1)k\sin(\delta_k-\delta_i)\right],\tag{46}$$

$(k=1\ldots N)$. Again it is recommended to combine neighboring phases according to Eq. (23) leading to

$$\dot{\vartheta}_k=\frac{\sin^2(\varphi/2)}{i_0 l_\perp(N+1)}\left(\sum_{i=1}^{k-1}[(N-k)i(\sin\vartheta_{k-1,i}-\sin\vartheta_{k,i})+i\sin\vartheta_{k-1,i}]-2(N-k)k\sin\vartheta_k+\sum_{i=k+2}^{N}[(N-i+1)k(\sin\vartheta_{k-1,i}-\sin\vartheta_{k,i})-(N-i+1)\sin\vartheta_{k,i}]\right),\quad k=1\ldots N-1.\tag{47}$$

A general solution of this system lies beyond the scope of the present article. However, the stationary in-phase solution $\delta_k=0,\quad k=1\ldots N$ being the most important one for practical purposes obvious exists.

The pure existence of this solution alone does say little regarding its actual realization; instead, Lyapunov stability has to be taken into account. As all possible combinations of phases enter Eq. (47), we are forced to adopt the Lyapunov ansatz

$$\vartheta_{k,i}=\delta_{k+1}-\delta_i=0-\epsilon_{k+1,i}e^{\lambda s},\quad |\epsilon_{k+1,i}\ll 1|.\tag{48}$$

Expanding the sines and defining

$$\overline{\lambda}=\frac{i_0 l_\perp(N+1)\lambda}{\sin^2(\varphi/2)},\tag{49}$$

we obtain the following linear system $(i,k=1\ldots N-1)$:

$$(\overline{\lambda}+2Nk-2k^2)\epsilon_{k,k+1}-\sum_{i=1}^{k-1}(N-k+1)i\epsilon_{k,1}-\sum_{i=1}^{k-1}(N-k)i\epsilon_{i,k+1}+\sum_{i=k+2}^{N}(N-i+1)\epsilon_{k,i}-\sum_{i=k+2}^{N}(N-i+1)(k+1)\epsilon_{k+1,i}=0.\tag{50}$$

After some tedious algebra the characteristic equation of this system can be written as

$$\det{}_{N-1}[(\overline{\lambda}+f_k)\delta_{kj}-g_{kj}]=0\tag{51}$$

with

$$f_k=k(N-k)\left(1+\frac{N}{2}\right)\tag{52}$$

and

$$g_{kj}=\begin{cases}\dfrac{j(j+1)}{2}, & k>j,\\ 0, & k=j,\\ \dfrac{(N-j)(N-j+1)}{2}, & k<j.\end{cases}\tag{53}$$

Combining Eqs. (52) and (53) into

$$r_{jk}=f_k\delta_{jk}-g_{jk},\tag{54}$$

Eq. (51) can be rewritten as

$$\sum_{i_1\ldots i_{N-1}}\epsilon_{i_1\ldots i_{N-1}}(\overline{\lambda}\delta_{1i_1}+r_{1i_1})\times(\overline{\lambda}\delta_{2i_2}+r_{2i_2})\cdots(\overline{\lambda}\delta_{N-1\ i_{N-1}}+r_{N-1\ i_{N-1}})=0,\tag{55}$$

where $\epsilon_{i_1\ldots i_{N-1}}$ is the $N-1$-dimensional completely antisymmetrical $\epsilon$ symbol.

There is one particular term in this sum with $i_1=1,\ldots,i_{N-1}=N-1$ in which all $\overline{\lambda}$ are present,

$$(\overline{\lambda}+r_{11})(\overline{\lambda}+r_{22})(\overline{\lambda}+r_{33})\cdots(\overline{\lambda}+r_{N-1\ N-1}).\tag{56}$$

In all of the other terms, as for instance in

$$(\overline{\lambda}+r_{11})r_{23}r_{32}\cdots(\overline{\lambda}+r_{N-1\ N-1}),\tag{57}$$

there are at least two indices interchanged. One can easily see that all these terms are small in comparison to the first one if

$$f_kf_j\gg g_{kj}g_{jk},\quad\forall j,k.\tag{58}$$

Via an estimate based on Eqs. (52), (53) it can be proven that Eq. (58) is indeed fulfilled as long as $N\gg 1$. However, explicit evaluation for a three- and four-cell array prove condition (58) being valid in this case already. Our earlier investigation for a double cell[15] confirmed this result, too.

Given condition (58) being fulfilled the determinant (51) can be approximately substituted by Eq. (56). However, as the $r_{jk}$ reduce to $f_k$ for $j=k$ expression (56) can be further collapsed as

$$(\bar{\lambda}+f_1)(\bar{\lambda}+f_2)(\bar{\lambda}+f_3)\cdots(\bar{\lambda}+f_{N-1})=0. \tag{59}$$

This way we get a factorization to be easily solved by

$$\lambda_k=-\frac{\sin^2(\varphi/2)}{i_0 l(N+1)}f_k, \quad k=1\ldots N-1. \tag{60}$$

As by definition (52) all the $f_k$ are positive definite, all the Lyapunov coefficients turn out to be negative definite. This way we have shown that any two junctions adjacent in the bias direction, i.e., all junctions in the bias direction admit a stable in-phase oscillation regime.

## VIII. SUMMARY AND CONCLUSIONS

Concerning the use of ladder arrays as microwave local oscillators several general conclusions can be drawn from our results. Concerning ''pure'' ladder arrays without any external feedback we observed a strange and unexpected kind of duality. While junctions adjacent in the bias direction oscillate in phase as long as the perpendicular inductances (denoted by $l_\perp$) are large, they oscillate antiphase for small-inductance loops. On the other hand, it has been known before that the oscillation phases of junctions adjacent within a cell perpendicular to the bias current will spread as soon as an external flux enters the cells. This effect becomes more dangerous the larger the cell inductance is. Unfortunately, raising the perpendicular inductances enhances the total cell inductance as well. Thus, there seems to be only a little chance to build local oscillators based on the scheme shown in Fig. 1 without the shunt, at least as long as flux cannot be completely expelled from the array. Moreover, the oscillation state of junctions adjacent in the bias direction is always marginally stable in a flux-free environment, as can already be deduced from Eq. (18). A similar finding has been reported in Ref. 8.

However, decoupling radiation out of the array requires an external load anyway. We obtained detailed and rigorous results for arrays with an external shunt. While for sufficiently small inductances phase splitting within the cells will remain small, Eq. (34) shows several parameters to adjust (in a way) coupling junctions adjacent in the bias direction in phase as well. The most important fact to notice is that a sufficiently inductively dominated external load ($1/c_S \ll \lambda_S$) will render any ladder array oscillating in-phase as long as $\varphi \neq \pi$. This fact has not found much attention in most experiments performed with 2D arrays so far. Moreover, for sufficiently long arrays ($N \gg 1$) the second term in Eq. (34) will always dominate over the first one. Thus, for sufficiently long arrays it is actually not required that the inductive contribution of the impedance of the external load must be very large compared to the capacitive one, as long as the first one dominates over the second one at all. In addition, as $\lambda$ involves $L_\parallel$ as well, the longitudinal inductances have a tendency of driving the cells in phase, too.

| $l_0 \Leftrightarrow i_0$ | **2.0** | **1.8** | **1.5** | **1.3** |
|---|---|---|---|---|
| **10.0** | ↑↑↑↑ | $\leadsto$ | ↑↑↓↓ | ↑↑↓↓ |
| **5.0** | $\leadsto$ | ↑↑↓↓ | ↑↑↓↓ | $\leadsto$ |
| **3.0** | ↑↓↓↑ | ↑↑↓↓ | $\leadsto$ | $\leadsto$ |
| **2.0** | ↑↑↓↓ | $\leadsto$ | ↑↓↓↑ | ↑↓↓↑ |
| **1.5** | $\leadsto$ | ↑↑↓↓ | ↑↙↘↑ | ↑↙↘↑ |
| **1.0** | $\leadsto$ | ↑↗↙↑ | ↑↓↗↓ | ↑↓↑↓ |
| **0.5** | ↑↓→↘ | ↑↙↗↖ | ↑↓↑↓ | ↑↓↑↓ |

FIG. 2. Phase-locked states of a 2D array consisting of four cells in the bias direction. Plotted here are the results from a numerical simulation. $i_0=I_0/I_C$ is the bias current and $l_0=2\pi L I_C/\Phi_0$ is the ring inductance both being normalized in conventional Josephson units.

While we were able to derive reduced equations for the oscillation phases, including non-vanishing McCumber parameters, a detailed treatment of junction capacitance effects was beyond the scope of the present investigation. However, previous investigations of a two-cell array[15] have already indicated the tendency of inductances $\beta_C \lesssim 0.8$ driving junctions adjacent in the bias direction in phase while still avoiding hysteresis.

To summarize, our results suggest the following design criteria for realizing the in-phase oscillation mode in 2D Josephson-junction arrays of the ladder type: (i) It is recommended to make ring inductances small, $l_0=2\pi L I_C/\Phi_0 \lesssim 0.5$, for aligning oscillations within the cells in phase. (ii) The external load required for decoupling radiation from the array must be inductive in character, $1/c_S < \lambda_S$. (iii) The ladder should be sufficiently long for the external load dominating the Lyapunov coefficients and driving junctions in the bias direction in phase. (iv) Junctions should have a small, but nonvanishing McCumber parameter $\beta_c \lesssim 0.8$.

Another possible design can only be guessed from the current results but needs further investigation. If it is possible to expel external flux completely from the cell, junctions within each cell will oscillate in phase in any case. In an unloaded array the ''natural'' oscillation state of the junctions in the bias direction was found to be in phase. According to our small-inductance results and having in mind that the internal/external contributions to Lyapunov coefficients do not mix up, addition of an inductive external load can be expected to improve this alignment, at best. Thus, in a completely shielded environment weakly coupled cells with an inductive external load might work as well.

Finally, we would like to mention that a numerical simulation of the problem was performed in parallel. The results for a four-cell array can be seen in Fig. 2. For sufficiently large $i_0 \gg 1$ (being the prerequisite for our approximation in general) as well as sufficiently large ring inductances $l_0=2(l_\parallel+l_\perp)\gg 1$, all four junctions adjacent in the bias direction are found to oscillate strictly in phase. On the other hand, for small ring inductances $l_0<1$ with at least $i_0>1$, adjacent junctions are observed to oscillate antiphase. One additional conclusion to be drawn from this figure says that outside of these limiting cases the behavior turns out to be quite involved. There are also additional oscillation regimes with pairs of adjacent junctions oscillating in phase while the pairs oscillate antiphase relative to each other. There are even parameter regions where there is no synchronization at all pointing to marginally stable oscillation regimes.

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