Five-loop additive renormalization in the ϕ^4 theory and amplitude functions of the minimally renormalized specific heat in three dimensions

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We present an analytic five-loop calculation for the additive renormalization constant $A(u,\epsilon)$ and the associated renormalization-group function $B(u)$ of the specific heat of the $O(n)$ symmetric ϕ^4 theory within the minimal subtraction scheme. We show that this calculation does not require new five-loop integrations but can be performed on the basis of the previous five-loop calculation of the four-point vertex function combined with an appropriate identification of symmetry factors of vacuum diagrams. We also determine the amplitude function $F_+(u)$ of the specific heat in three dimensions for $n=1,2,3$ above T_c and $F_-(u)$ for $n=1$ below T_c up to five-loop order, without using the $\epsilon=4-d$ expansion. Accurate results are obtained from Borel resummations of $B(u)$ for $n=1,2,3$ and of the amplitude functions for $n=1$. Previous conjectures regarding the smallness of the resummed higher-order contributions are confirmed. Combining our results for $B(u)$ and $F_+(u)$ for $n=1,2,3$ with those of a recent three-loop calculation of $F_-(u)$ for general n in $d=3$ dimensions we calculate Borel resummed universal amplitude ratios A^+/A^- for $n=1,2,3$. Our result for $A^+/A^-=1.056\pm0.004$ for $n=2$ is significantly more accurate than the previous result obtained from the ϵ expansion up to $O(\epsilon^2)$ and agrees well with the high-precision experimental result $A^+/A^-=1.054\pm0.001$ for ^{4}He near the superfluid transition obtained from a recent experiment in space. [S0163-1829(98)07229-4]

I. INTRODUCTION

One of the fundamental achievements of the renormalization-group (RG) theory of critical phenomena is the identification of universality classes in terms of the dimensionality d of the system and the number n of components of the order parameter.[1] Specifically, RG theory predicts that the critical exponents, certain amplitude ratios, and scaling functions are universal quantities that do not depend, e.g., on the strength of the interaction or on thermodynamic variables (such as the pressure). The superfluid transition of ^{4}He belongs to the $d=3$, $n=2$ universality class and provides a unique opportunity for an experimental test of the universality prediction by means of measurements of the critical behavior at various pressures P along the λ line $T_\lambda(P)$. Early tests have been performed by Ahlers and collaborators and consistency with the universality prediction was found within the experimental resolution.[2] At a significantly higher level of accuracy, the superfluid density and the specific heat (or, equivalently, thermal expansion coefficient) above and below $T_\lambda(P)$ are planned to be measured in the Superfluid Universality Experiment[3] under microgravity conditions or at reduced gravity in the low-gravity simulator.[4] As demonstrated recently,[5] this would allow one to perform measurements up to $|t|\simeq10^{-9}$ in the reduced temperature $t=[T-T_\lambda(P)]/T_\lambda(P)$.

On the theoretical side, the corresponding challenge is to calculate as accurately as possible the properties of the $O(n)$ symmetric ϕ^4 model in three dimensions. To extract the leading critical exponents from the experimental data and to demonstrate their universality at a highly quantitative level requires detailed knowledge on the ingredients of a nonlinear RG analysis.[6] They include not only the well-known RG exponent functions of the ϕ^4 model whose fixed point values determine the critical exponents but also the less well-known amplitude functions[7–12] which contain the information about universal ratios of leading and subleading amplitudes.[1]

The existing theoretical predictions on the critical exponents[13] within the minimal subtraction scheme[14,15] are based on field-theoretic calculations to five-loop order[16–19] and Borel resummation. By contrast, the present theoretical knowledge of the amplitude ratios for $n>1$ below T_c is based only on low-order (mainly one- and two-loop) calculations which imply an uncertainty at the level of at least 10–30%.[1] It has therefore been proposed[20] to significantly reduce this uncertainty by performing new higher-order field-theoretic calculations and Borel resummations of various amplitude functions in three dimensions.

Both conceptual and computational steps towards this goal have already been performed. The conceptual progress includes the demonstration that the $d=3$ field theory suggested by Parisi[21] can well be realized within the minimal subtraction scheme at $d=3$ (Refs. 7–9) by incorporating Symanzik's nonvanishing mass shift[22] and that spurious Goldstone singularities for $n>1$ below T_c can well be treated within this approach[12] by using an appropriately defined pseudocorrelation length.[9] The computational steps include the determination of the amplitude functions $F_+(u)$ and $F_-(u)$ of the specific heat in three dimensions above T_c for $n=1,2,3$ (Ref. 10) and below T_c for $n=1$,[11] respectively, up to five-loop order, and their Borel resummation. These calculations, however, were not yet complete because of an approximation regarding the additive renormalization $A(u,\epsilon)$ of the specific heat and the associated RG function $B(u)$. Due to the lack of knowledge in the literature about higher-order terms, $A(u,\epsilon)$ and $B(u)$ were approximated by their two-loop expressions. Although the good agreement between low-order $d=3$ perturbation results[7,23,24] and accurate experiments[2,25,26] provided some indication for the smallness of the effect of the higher-order terms of $B(u)$, no reliable estimate could be given for the remaining uncertainty of $F_{\pm}(u)$ which could well be of relevance at the level of accuracy anticipated in future experiments.[3] Furthermore we recall that any inaccuracy of $B(u)$ enters not only the formulas[9] for several universal amplitude ratios but also the formulas needed to determine the effective coupling $u(l)$ from the specific heat.[7,23,24]

It is the purpose of the present paper to provide the missing information on the higher-order terms of $A(u,\epsilon)$ and $B(u)$ by means of a new five-loop calculation. We shall show that the analytic calculation of $A(u,\epsilon)$ and $B(u)$ can be directly related to the previous calculations[16–19] of the four-point vertex function. This provides the crucial simplification that no new evaluations of three-, four- and five-loop integrals are necessary but that only a new determination of symmetry and $O(n)$ group factors of vacuum diagrams is sufficient.

Using the five-loop expression of $B(u)$ we are in the position to determine the correct higher-order terms of the minimally renormalized amplitude functions $F_+(u)$ for $n=1,2,3$ and $F_-(u)$ for $n=1$ in three dimensions on the basis of previous work[27–30] where a different renormalization scheme was used. The new coefficients of the higher-order terms of $F_+(u)$ turn out to differ considerably from the previous approximate coefficients[10] whereas the coefficients of $F_-(u)$ are only weakly affected by the new higher-order terms of $B(u)$.

We also perform new Borel resummations of $B(u)$ for $n=1,2,3$ as well as of $F_-(u)$ and of $F_-(u)-F_+(u)$ for $n=1$. It turns out that the result of the Borel resummation for $B(u)$ including the new terms up to five-loop order differs from the two-loop result $B(u)=n/2+O(u^2)$ by only less than 1% at the fixed point. For the amplitude functions, our new Borel resummation results differ from the previous ones[10,11] by about 1% for F_- and by less than 0.1% for $F_- - F_+$ at the fixed point. This is a nontrivial and important confirmation of the previous conjectures about the smallness of resummed higher-order contributions.[9–11]

As a first application, we calculate the universal ratios A^+/A^- and a_c^+/a_c^- of the leading and subleading amplitudes of the specific heat above and below T_c for $n=1$ and $d=3$. In addition we calculate Borel resummation values of A^+/A^- for $n=2$ and 3 by combining our present results for $B(u)$ and $F_+(u)$ for $n=2$ and 3 with those of a recent three-loop calculation of $F_-(u)$ for general n.[31] All of our calculations are performed at $d=3$ dimensions, without using the ϵ expansion. Our result $A^+/A^-=1.056\pm0.004$ for $n=2$ is more accurate than the previous result[32] 1.0294 ± 0.0134 obtained from the $\epsilon=4-d$ expansion up to $O(\epsilon^2)$ and agrees well with the high-precision experimental result[5] 1.054 ± 0.001 for ^{4}He near the superfluid transition obtained from a recent experiment in space.

II. ADDITIVE RENORMALIZATION OF THE SPECIFIC HEAT

The $O(n)$ symmetric ϕ^4 model is defined by the usual Landau-Ginzburg-Wilson functional

$$\mathcal{H}\{\boldsymbol{\phi}_0(\mathbf{x})\}=\int_V d^dx\left(\frac{1}{2}r_0\phi_0^2+\frac{1}{2}\sum_i(\nabla\phi_{0i})^2+u_0(\phi_0^2)^2-\mathbf{h}_0\cdot\boldsymbol{\phi}_0\right)\tag{2.1}$$

for the n-component field $\boldsymbol{\phi}_0(\mathbf{x})=[\phi_{01}(\mathbf{x}),\ldots,\phi_{0_n}(\mathbf{x})]$ where

$$r_0=r_{0c}+a_0t,\quad t=(T-T_c)/T_c,\tag{2.2}$$

and $\mathbf{h}_0=(h_0,0,\ldots,0)$. The Gibbs free energy per unit volume (divided by k_BT) is

$$F_0(r_0,u_0,h_0)=-V^{-1}\ln\int\mathcal{D}\boldsymbol{\phi}_0\exp(-\mathcal{H}).\tag{2.3}$$

We shall consider the bulk limit $V\to\infty$. We are interested in the specific heat $\mathring{C}^{\pm}$ per unit volume at vanishing external field $h_0=0$ (divided by Boltzmann's constant k_B) where $\pm$ refers to $T>T_c$ and $T<T_c$, respectively. Near T_c, $\mathring{C}^{\pm}$ is determined by[9]

$$\mathring{C}^{\pm}=C_B-T_c^2\frac{\partial^2}{\partial T^2}F_0(r_0,u_0,0)=C_B-a_0^2\frac{\partial^2}{\partial r_0^2}F_0(r_0,u_0,0),\tag{2.4}$$

where C_B is an analytic background term. Alternatively the Helmholtz free energy per unit volume $\Gamma_0(r_0,u_0,M_0)=F(r_0,u_0,h_0)+h_0M_0$ with $M_0\equiv\langle\phi_{01}\rangle$ determines $\mathring{C}^{\pm}$ in the $h_0\to0$ limit according to

$$\mathring{C}^{\pm}=C_B-a_0^2\frac{d^2}{dr_0^2}\Gamma_0(r_0,u_0,M_0(r_0,u_0)).\tag{2.5}$$

The perturbative expression for $\Gamma_0(r_0,u_0,M_0)$ is obtained from the negative sum of all one-particle irreducible vacuum diagrams.[15] The perturbative expression for $\mathring{C}^{\pm}$ is then determined by the vertex functions $\mathring{\Gamma}_{\pm}^{(2,0)}=d^2\Gamma_0/dr_0^2$ which we consider as functions of appropriately defined correlation lengths ξ_+ and ξ_- above and below T_c,[9,12]

$$\mathring{C}^{\pm}=C_B-a_0^2\mathring{\Gamma}_{\pm}^{(2,0)}(\xi_\pm,u_0,d).\tag{2.6}$$

A description of the critical behavior requires to turn to the renormalized vertex functions

$$\Gamma_{\pm}^{(2,0)}(\xi_\pm,u,\mu,d)=Z_r^2\mathring{\Gamma}_{\pm}^{(2,0)}(\xi_\pm,\mu^{\epsilon}Z_uZ_\phi^{-2}A_d^{-1}u,d)-\frac{1}{4}\mu^{-\epsilon}A_dA(u,\epsilon).\tag{2.7}$$

We work at infinite cutoff using the prescriptions of dimensional regularization and minimal subtraction at fixed dimension $2<d<4$ without employing the $\epsilon=4-d$ expansion.[7–9] The Z factors are introduced as

$$r=Z_r^{-1}(r_0-r_{0c}),\quad u=\mu^{-\epsilon}A_dZ_u^{-1}Z_\phi^2u_0,\quad \boldsymbol{\phi}=Z_\phi^{-1/2}\boldsymbol{\phi}_0,\tag{2.8}$$

where the geometric factor

$$A_d = \Gamma(3-d/2)2^{2-d}\pi^{-d/2}(d-2)^{-1} \tag{2.9}$$

becomes $A_3=(4\pi)^{-1}$ for $d=3$ and $A_4=(8\pi^2)^{-1}$ for $d=4$. These Z factors $Z_i(u,\epsilon)$ and the associated field-theoretic functions[8]

$$\zeta_r(u) = \mu\partial_\mu \ln Z_r(u,\epsilon)^{-1}|_0, \tag{2.10}$$

$$\zeta_\phi(u) = \mu\partial_\mu \ln Z_\phi(u,\epsilon)^{-1}|_0, \tag{2.11}$$

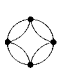
$$\beta_u(u,\epsilon) = -\epsilon u + \tilde{\beta}(u) = u[-\epsilon + \mu\partial_\mu(Z_u^{-1}Z_\phi^2)|_0], \tag{2.12}$$

are known up to five-loop order.[17–19]

The main quantity of interest in the present paper is the renormalization constant $A(u,\epsilon)$ in Eq. (2.7) which absorbs the additive poles of both $\mathring{\Gamma}_+^{(2,0)}$ and $\mathring{\Gamma}_-^{(2,0)}$. Previously[10,11] $A(u,\epsilon)$ was employed only in its two-loop form[7]

$$A(u,\epsilon) = -2n\frac{1}{\epsilon} - 8n(n+2)\frac{u}{\epsilon^2} + O(u^2). \tag{2.13}$$

Here we report on a calculation of $A(u,\epsilon)$ up to five-loop order. We would like to stress that this calculation does not require new five-loop integrations but can be performed on the basis of the previous five-loop calculation[17–19] of the four-point vertex function combined with an appropriate identification of symmetry and $O(n)$ group factors of vacuum diagrams which are shown in Fig. 1. Their negative sum determines the Helmholtz free energy Γ_0 up to five-loop order. For the present purpose of determining the pole terms at $d=4$ it suffices to consider only the case $r_0>0$ and $M_0=0$ where only four-point vertices exist. The diagrams are labeled (1) in one-loop order, (2) in two-loop order, (3), (4) in three-loop order, (5)–(8) in four-loop order and (9)-(18) in five-loop order. The analytic expression of an m-loop diagram (i) is given by the product of the coupling $(-u_0)^{m-1}$, the symmetry factor $S^{(i)}$, the $O(n)$ group factor $G^{(i)}(n)$ and the momentum integral expression $I^{(i)}(r_0,\epsilon)$. Thus the structure of the diagrammatic expression of a typical diagram, e.g., (16) is



$$= S^{(16)}(-u_0)^4 G^{(16)} I^{(16)}(r_0,\epsilon) \tag{2.14}$$

$$= 2592(-u_0)^4 \frac{n^4+8n^3+32n^2+40n}{81} \times \int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4}\int_{p_5} \times G_1\cdot G_2\cdot G_{1+2-3}\cdot G_3\cdot G_{1+2-4}\cdot G_4\cdot G_{1+2-5}\cdot G_5 \tag{2.15}$$

with $\int_p \equiv (2\pi)^{-d}\int d^dp$ and the propagators $G_{i\pm j} \equiv (r_0+|\mathbf{p}_i \pm \mathbf{p}_j|^2)^{-1}$. The symmetry and group factors are listed in Table I.

To calculate the additive renormalization constant $A(u,\epsilon)$ one needs to calculate those ultraviolet $d=4$ pole terms of the diagrams contributing to $\Gamma_{0+}^{(2,0)}$ that are left after subtraction of subdivergences. One obtains $\Gamma_{0+}^{(2,0)}$ by taking two derivatives of Γ_0 with respect to r_0. The analytic calculation of the poles of the diagrams for $\Gamma_{0+}^{(2,0)}$ is identical to that carried out previously[17–19] for the four-point vertex function $\Gamma_0^{(0,4)}$. To see this, one should take into account that in the minimal subtraction scheme[14] the ultraviolet pole terms specified above do not depend on r_0. Then by using the method of infrared rearrangement[33] one can nullify r_0 and introduce for each diagram a new fictitious external momentum to regularize infrared divergences. Then one can see that only a particular subset of those diagrams of $\Gamma_0^{(0,4)}$ are relevant in the present context, namely those where the four external legs are connected to each diagram through only two four-point vertices (rather than three four-point vertices or four four-point vertices). The details of the calculation are presented in Appendix A.

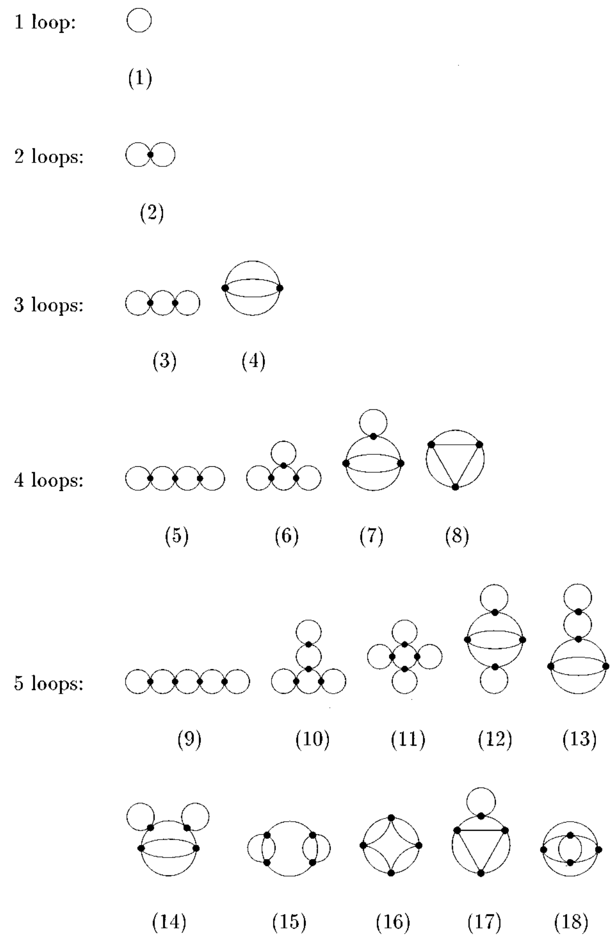


FIG. 1. Vacuum diagrams up to five-loop order determining the Helmholtz free energy Γ_0 for $M_0=0$, $r_0>0$. Diagrams (6), (10), and (11) do not contribute to $A(u,\epsilon)$. The pole terms derived from the vacuum diagrams are given in Eqs. (A1)–(A54) of Appendix A.

The result reads

$$A(u,\epsilon) = \sum_{m=1}^{5} A^{(m)}(u,\epsilon) + O(u^5), \tag{2.16}$$

where $A^{(m)}$ denotes the contribution of m-loop order,

TABLE I. Symmetry and group factors of the vacuum diagrams shown in Fig. 1. Diagrams (6), (10), and (11) do not contribute to $A(u,\epsilon)$.

Loop order	Diagram (i)	Symmetry factor $S^{(i)}$	Group factor $G^{(i)}(n)$
One loop	(1)	1	n
Two loops	(2)	3	$\frac{1}{3}(n^2+2n)$
Three loops	(3)	36	$\frac{1}{9}(n^3+4n^2+4n)$
	(4)	12	$\frac{1}{3}(n^2+2n)$
Four loops	(5)	432	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(7)	576	$\frac{1}{9}(n^3+4n^2+4n)$
	(8)	288	$\frac{1}{27}(n^3+10n^2+16n)$
Five loops	(9)	5184	$\frac{1}{81}(n^5+8n^4+24n^3+32n^2+16n)$
	(12)	10368	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(13)	6912	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(14)	6912	$\frac{1}{27}(n^4+6n^3+12n^2+8n)$
	(15)	2304	$\frac{1}{9}(n^3+4n^2+4n)$
	(16)	2592	$\frac{1}{81}(n^4+8n^3+32n^2+40n)$
	(17)	20736	$\frac{1}{81}(n^4+12n^3+36n^2+32n)$
	(18)	10368	$\frac{1}{81}(5n^3+32n^2+44n)$

$$A^{(3)}(u,\epsilon)=-\frac{4}{3}n(n+2)\left[\frac{3}{\epsilon}-\frac{40}{\epsilon^2}+\frac{24(n+4)}{\epsilon^3}\right]u^2, \tag{2.17}$$

$$A^{(4)}(u,\epsilon)=-\frac{8}{3}n(n+2)\left[\frac{(n+8)(12\zeta(3)-25)}{\epsilon}+\frac{96n+696}{\epsilon^2}-\frac{248n+1024}{\epsilon^3}+\frac{48(n+4)(n+5)}{\epsilon^4}\right]u^3, \tag{2.18}$$

$$\begin{aligned}A^{(5)}(u,\epsilon)=-\frac{2}{15}n(n+2)\Bigg[&\frac{768(n+4)(n+5)(5n+28)}{\epsilon^5}-\frac{128(293n^2+2624n+5840)}{\epsilon^4}\\&+\frac{9216\zeta(3)(5n+22)+32(519n^2+8462n+25048)}{\epsilon^3}\\&-\frac{192\zeta(3)(7n^2-28n+48)+4608\zeta(4)(5n+22)+64(31n^2+2354n+9306)}{\epsilon^2}+[48\zeta(3)(3n^2-382n-1700)\\&+288\zeta(4)(4n^2+39n+146)-3072\zeta(5)(5n+22)-3(319n^2-13968n-64864)]\frac{1}{\epsilon}\Bigg]u^4,\end{aligned} \tag{2.19}$$

where $\zeta(s)=\Sigma_{j=1}^{\infty}j^{-s}$ is the Riemann zeta function with $\zeta(3)=1.202\,056\,90$, $\zeta(4)=\pi^4/90$, and $\zeta(5)=1.036\,927\,76$. Most important is the d-independent RG function $B(u)$ which is determined by[9]

$$4B(u)=[2\zeta_r(u)-\epsilon]A(u,\epsilon)+\beta_u(u,\epsilon)\frac{\partial A(u,\epsilon)}{\partial u}. \tag{2.20}$$

Using $A(u,\epsilon)$ of Eqs. (2.16)–(2.19) and the perturbative expressions for ζ_r and β_u of Refs. 17–19, we find

$$\begin{aligned}B(u)=&\frac{n}{2}+3n(n+2)u^2-\frac{8}{3}n(n+2)(n+8)[25-12\zeta(3)]u^3+\frac{1}{2}n(n+2)[16\zeta(3)(3n^2-382n-1700)-1024\zeta(5)(5n+22)\\&+96\zeta(4)(4n^2+39n+146)-319n^2+13\,968n+64\,864]u^4+O(u^5).\end{aligned} \tag{2.21}$$

The terms of $O(u^2)$ and $O(u^3)$ agree with those of Ref. 34. In Table II the coefficients c_{Bm} of the power series

$$B(u)=\sum_{m=0}^{\infty}c_{Bm}u^m \tag{2.22}$$

are given for $n=0,1,2,3$ up to $m=4$ corresponding to five-loop order. Table II also contains the coefficients $f_i^{(m)}$ of the power series of the functions

$$f_i(u)=\sum_{m=1}^{\infty}f_i^{(m)}u^m, \tag{2.23}$$

TABLE II. Coefficients $f_i^{(m)}$ of the functions $\tilde{\beta}(u)$, $\zeta_r(u)$, and $\zeta_\phi(u)$ for $i=1,2,3$, respectively, and coefficients c_{Bm} of $B(u)$, compare Eqs. (2.22) and (2.23), for $n=0$, 1, 2, 3 up to five-loop order ($m=6$ for $\tilde{\beta}$, $m=5$ for ζ_r and ζ_ϕ, $m=4$ for B). For $f_i^{(m)}$ compare Table I of Ref. 8.

	$\tilde{\beta}_u$	ζ_r	ζ_φ	B
$n=0$	0	8	0	0
	32	−80	−16	0
	−672	3552	128	0
	43989.9534	−223152.607	−8000	0
	−4166409.19	18836823.8	500639.112	0
	498653403.0			
$n=1$	0	12	0	1/2
	36	−120	−24	0
	−816	6048	216	9
	56245.8519	−413813.942	−14040	−761.422836
	−5632017.54	37512804.7	958294.321	44244.7100
	708814936.0			
$n=2$	0	16	0	1
	40	−160	−32	0
	−960	9024	320	24
	69029.7505	−660870.017	−21120	−2256.06766
	−7268274.40	63662497.1	1566676.69	141294.329
	956636505.0			
$n=3$	0	20	0	3/2
	44	−200	−40	0
	−1104	12480	440	45
	82341.6490	−967074.371	−29000	−4653.13955
	−9075019.76	98265069.9	2333667.84	310944.846
	1243816220.0			

where $f_i(u)$ denotes the functions $\tilde{\beta}(u)$, $\zeta_r(u)$, and $\zeta_\phi(u)$ for $i=1,2,3$, respectively. These coefficients are taken from Refs. 17–19. Up to four-loop order they agree with those in Table I of Ref. 8. (Note that $f_i^{(k)}$ in the table caption of Ref. 8 should read $f_i^{(k)}\times 10^{-4}$.) The five-loop coefficients $f_1^{(6)}$, $f_2^{(5)}$, and $f_3^{(5)}$ differ from those in Table I of Ref. 8 according to the corrections in five-loop order in Ref. 19.

In Fig. 2 the partial sums of $B(u)$ from two- to five-loop order are shown for the example $n=2$. As expected, the contributions for $m\geqslant 2$ have alternating signs and increase considerably in magnitude. Clearly a resummation of $B(u)$ is necessary similar to that for $\zeta_r(u)$, $\zeta_\phi(u)$, and $\tilde{\beta}(u)$ performed previously.[8]

First we reexamine the fixed-point values u^*, $\beta_u(u^*,1)=0$, for $n=1,2,3$ obtained in Refs. 8 and 24 by means of Borel resummation on the basis of previous five-loop results[17,18] and in Ref. 16 on the basis of four-loop results. Here we employ the corrected five-loop coefficients for the ϵ expansion of the fixed-point value which we have derived from Eq. (8) of Ref. 19. Employing the standard Borel resummation method[8,35] we have obtained the fixed-point values in three dimensions

$$u^*=0.0404\pm 0.0003 \quad \text{for } n=1, \tag{2.24}$$

$$u^*=0.0362\pm 0.0002 \quad \text{for } n=2, \tag{2.25}$$

$$u^*=0.0327\pm 0.0001 \quad \text{for } n=3. \tag{2.26}$$

The corresponding resummation parameters α and $b=5.5+n/2$ (Ref. 35) are

$$2.22\leqslant\alpha\leqslant 3.41, \quad b=6.0 \quad \text{for } n=1, \tag{2.27}$$

$$2.45\leqslant\alpha\leqslant 3.43, \quad b=6.5 \quad \text{for } n=2, \tag{2.28}$$

$$2.71\leqslant\alpha\leqslant 3.43, \quad b=7.0 \quad \text{for } n=3. \tag{2.29}$$

The previous fixed-point values[8,16,24] are consistent with Eqs. (2.24)–(2.26) within the previous error bars. The present error bars are smaller than the previous ones.[8,16,24] (The range of α determines our error bars, as described further below.)

We have performed Borel resummations of $B(u)$ at the fixed point u^* for the cases $n=1,2,3$. In addition, for the important case $n=2$ (superfluid ^{4}He), we have determined the Borel resummed function $B(u)$ at various values of u. The results are given in Eq. (2.34)–(2.36) and in Figs. 2 and 3.

A description of the Borel resummation method[35] for the present purpose has been given in Sec. V of Ref. 8. [In Eq. (5.10) of Ref. 8 a_{jk} should read $a_{j-m,k}$.] In the present work, however, we use a different way of determining the parameters α and b of the summations. This implies a different determination of the error bars.

For $B(u)$ the value of the parameter b is not known from an analysis of the large-order behavior [see Eq. (5.6) of Ref. 8, and references therein]. Here we fix both b and α by requiring fastest convergence of the series of the partial

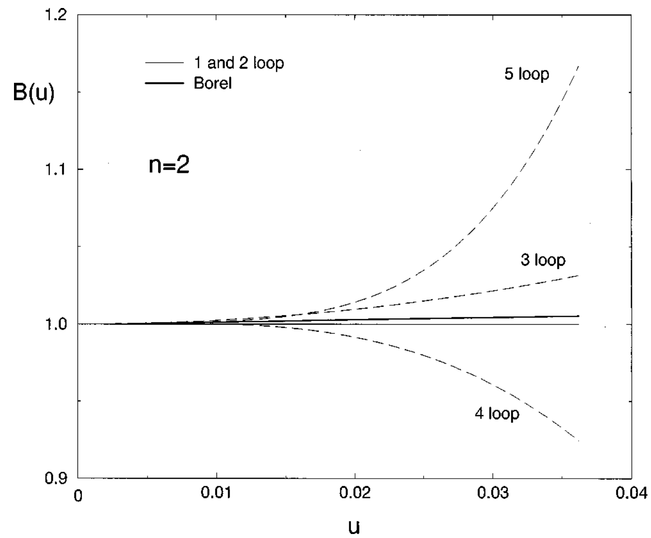


FIG. 2. Partial sums $B_M(u)=\Sigma_{m=0}^{M} c_{Bm} u^m$ of $B(u)$, Eq. (2.22), as a function of u for $n=2$ from $M=1$ (two-loop order) to $M=4$ (five-loop order). Also shown is the Borel resummed result (solid line) which deviates from the two-loop result $B_1=1$ by only 0.5% at the fixed point $u^\star=0.0362$.

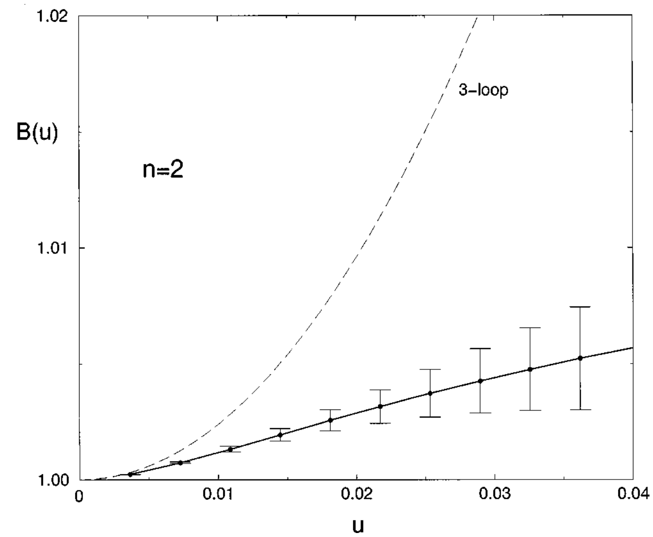


FIG. 3. Borel resummation result for the function $B(u)$, Eq. (2.21), for $n=2$ (solid line) obtained by interpolation between the resummed values of $B(u)$ at $u_k=ku^\star/10$, $k=1,\ldots,10$ of the renormalized coupling u in the range $0<u\leqslant u^\star=0.0362$, with error bars. Also shown is the three-loop result $B_2(u)=1+24u^2$ (dashed line), compare Fig. 2. The two-loop result is $B_1=1$. The Borel values $B(u_k)$ are 1.000 233, 1.000 73, 1.0013, 1.0019, 1.0026, 1.0032, 1.0037, 1.0043, 1.0048, 1.0053 for $k=1,\ldots,10$, respectively.

Borel sums $S^{(L)}=S^{(L)}(u,\alpha,b)$ for $B(u)$ defined in Eq. (5.12) of Ref. 8 [here L corresponds to $(L+1)$-loop order]. To do so we look for the minima of $\Delta^{(4)}$ and $\Delta^{(3)}$ with regard to variations of both b and α where

$$\Delta^{(L)}(u,\alpha,b)=\left|\frac{S^{(L)}-S^{(L-1)}}{S^{(L-1)}}\right|. \tag{2.30}$$

This yields five-loop values of the parameters α, b for each u. In order to define an error bar, we apply the same method to the four-loop result of $B(u)$, i.e., to $\Delta^{(3)}$ and $\Delta^{(2)}$. The four-loop values of α, b together with the five-loop values provide the ranges of the best values of α and b as a result of the combined four- and five-loop analysis. Then we define the error bar of the five-loop result for $B(u)$ by the maximum and minimum of the resummed four- and five-loop values for $B(u)$ over the ranges of the best values of α, b. At the fixed point u^*, Eqs. (2.24)–(2.26), we find the ranges

$$0.95\leqslant\alpha\leqslant 1.08,\quad 5.7\leqslant b\leqslant 7.75\quad \text{for}\ n=1, \tag{2.31}$$

$$0.94\leqslant\alpha\leqslant 1.04,\quad 7.0\leqslant b\leqslant 8.59\quad \text{for}\ n=2, \tag{2.32}$$

$$0.94\leqslant\alpha\leqslant 1.02,\quad 8.18\leqslant b\leqslant 9.76\quad \text{for}\ n=3. \tag{2.33}$$

The corresponding Borel resummed results for $B(u^*)$ are

$$B(u^*)=0.5024\pm 0.0011\quad \text{for}\ n=1, \tag{2.34}$$

$$B(u^*)=1.0053\pm 0.0022\quad \text{for}\ n=2, \tag{2.35}$$

$$B(u^*)=1.5080\pm 0.0034\quad \text{for}\ n=3. \tag{2.36}$$

We have also determined the function $B(u)$ for $n=2$ (superfluid ^{4}He) in the range $0\leqslant u\leqslant 0.04$ as shown in Fig. 3.

Most remarkable is the smallness of the deviation of the resummed function $B(u)$ for $n=1,2,3$ from its two-loop approximation $n/2$. This confirms previous conjectures[9–11] and justifies earlier analyses.[7,23,24]

III. AMPLITUDE FUNCTIONS $F_\pm$ IN THREE DIMENSIONS

A. Definition of $F_\pm$

By means of the renormalized vertex functions in Eq. (2.7) we define the dimensionless amplitude functions

$$F_\pm(\mu\xi_\pm,u,d)=-4\mu^\epsilon A_d^{-1}\Gamma_\pm^{(2,0)}(\xi_\pm,u,\mu,d). \tag{3.1}$$

They enter the critical behavior of the specific heat in three dimensions in the form of the functions

$$F_\pm(1,u,3)\equiv F_\pm(u) \tag{3.2}$$

according to[9]

$$\mathring{C}^\pm=C_B+\frac{1}{4}\,a^2\mu^{-1}A_3K_\pm(u(l_\pm))$$
$$\times\exp\int_u^{u(l_\pm)}\frac{2\zeta_r(u')-1}{\beta_u(u',1)}\,du', \tag{3.3}$$

where

$$K_\pm(u)=F_\pm(u)-A(u,1) \tag{3.4}$$

and $a=Z_r(u,1)^{-1}a_0$. In Eq. (3.3), $u(l)$ is the effective coupling satisfying

$$l\,\frac{du(l)}{dl}=\beta_u(u(l),1) \tag{3.5}$$

with $u(1)=u$. The flow parameters l_+ and l_- are chosen as $l_+=(\mu\xi_+)^{-1}$ and $l_-=(\mu\xi_-)^{-1}$ above and below T_c.

B. Power series of $F_\pm$

The amplitude functions are expandable in integer powers of u (Ref. 9) and have the power series[10]

$$F_+(u)=\sum_{m=0}^{\infty} c_{Fm}^{+}u^m \tag{3.6}$$

and[11]

$$F_-(u)=\frac{1}{u}\sum_{m=0}^{\infty} c_{Fm}^{-}u^m. \tag{3.7}$$

We have determined c_{Fm}^{+} up to five-loop order (i.e., up to $m=4$) for $n=1,2,3$ and c_{Fm}^{-} up to five-loop order (i.e., up to $m=5$) for $n=1$ in two different ways.

(i) The coefficients c_{Fm}^{+} and c_{Fm}^{-} can be calculated from Eqs. (3.1), (2.7) in three dimensions according to

$$F_\pm(u)=-16\pi Z_r^2\xi_\pm^{-1}\mathring{\Gamma}_\pm^{(2,0)}(\xi_\pm,4\pi\xi_\pm^{-1}Z_uZ_\phi^{-2}u,3)+A(u,1), \tag{3.8}$$

where the Z factors have the arguments $Z_i(u,1)$. The perturbative expression for $\mathring{\Gamma}_+^{(2,0)}$ can be obtained for $n=1,2,3$ from

$$\mathring{\Gamma}_+^{(2,0)}(\xi_+,u_0,3)=\frac{1}{4u_0}Z_5^{-1}(\lambda), \tag{3.9}$$

where the renormalization factor $Z_5(\lambda)$ and its relation to the specific heat have been presented in numerical form by Bervillier and Godrèche[28] and by Bagnuls and Bervillier,[29,36] see also Ref. 10. For $d=3$ their renormalized coupling λ is related to our u_0 via the renormalization factor $Z_3(\lambda)$ according to $u_0\xi_+=-2\pi\lambda Z_3(\lambda)^{-1/2}$ as noted in Ref. 10. The perturbative expression of $\mathring{\Gamma}_-^{(2,0)}$ for $n=1$ can be determined according to

$$\begin{aligned}\mathring{\Gamma}_-^{(2,0)}(\xi_-,u_0,3)&=\frac{\partial^2}{\partial r_0'^2}\mathring{\Gamma}_-(\xi_-,u_0,3)\\&=\left(\frac{\partial r_0'}{\partial\xi_-}\right)^{-1}\frac{\partial}{\partial\xi_-}\\&\quad\times\left[\left(\frac{\partial r_0'}{\partial\xi_-}\right)^{-1}\frac{\partial}{\partial\xi_-}\mathring{\Gamma}_-(\xi_-,u_0,3)\right],\end{aligned} \tag{3.10}$$

where $r_0'(\xi_-,u_0)$ is given by Eq. (3.8) of Ref. 11. The Helmholtz free energy $\mathring{\Gamma}_-(\xi_-,u_0,3)$ is given in numerical form by Eq. (3.15) of Ref. 11 where our $\mathring{\Gamma}_-(\xi_-,u_0,3)$ is denoted by $\tilde{\Gamma}_{-0}(\xi_-,u_0)$. Our numerical results for $c_{Fm}^{\pm}$ up to nine digits are presented in Table III.

(ii) Alternatively the coefficients $c_{Fm}^{\pm}$ can be determined via the relation

$$\begin{aligned}8A_3^{-1}P_\pm(u)f_\pm^{(3,0)}(u)=&[1-2\zeta_r(u)]F_\pm(u)+4B(u)\\&-\beta_u(u)\partial F_\pm(u)/\partial u\end{aligned} \tag{3.11}$$

as done previously.[10,11] For the definition of $P_\pm$ and $f_\pm^{(3,0)}$ and for a derivation of Eq. (3.11) we refer to Refs. 8 and 9.

TABLE III. Coefficients $c_{Fm}^{\pm}$ of $F_+(u)$ and $F_-(u)$ for $n=1,2,3$ defined in Eqs. (3.6) and (3.7), respectively. For c_{Fm}^{+}, m refers to u^m corresponding to $(m+1)$-loop order whereas for c_{Fm}^{-}, m refers to u^{m-1} corresponding to m-loop order. The coefficients c_{F2}^{+} and c_{F3}^{-} (three-loop order) are taken from Ref. 31.

	m	c_{Fm}^{+}	c_{Fm}^{-}
$n=1$	0	-1	1/2
	1	-6	-4
	2	-22.6976284	72
	3	-722.742498	-5189.75474
	4	34775.5861	433582.586
	5		-47754702.5
$n=2$	0	-2	1/2
	1	-16	-4
	2	-92.5270090	64
	3	-2430.86460	-5918.07320
	4	102469.659	
$n=3$	0	-3	1/2
	1	-30	-4
	2	-233.488142	56
	3	-5742.02976	-6607.95641
	4	204463.777	

In the present context we need the contributions to $P_\pm$ and $f_\pm^{(3,0)}$ only up to $O(u^4)$ as given in Table IV of Ref. 10 and Table III of Ref. 11, since $B(u)$ is known only up to $O(u^4)$ as well. (We recall that the coefficients of P_- are determined by those of P_+ according to $P_-(u)=-\frac{1}{2}\{1+2[1-P_+(u)]-\frac{3}{2}\zeta_r(u)\}$.[9]) This calculation via Eq. (3.11) yields coefficients $c_{Fm}^{\pm}$ that agree with those obtained via Eq. (3.8) up to eight digits for c_{Fm}^{-} and up to seven digits for c_{Fm}^{+}. The slight differences between the results of the calculations (i) and (ii) are due to the fact that $Z_3(\lambda)$, $Z_5(\lambda)$, and $\mathring{\Gamma}_-$ are available up to five-loop order only in numerical form. We consider the calculation (i) via Eq. (3.8) as slightly more reliable since fewer numerical operations are required than in calculation (ii) using Eq. (3.11).

Since here we have used the perturbative contributions of $B(u)$ up to five-loop order, the resulting higher-order coefficients $c_{Fm}^{\pm}$ given in Table III differ from those determined previously (see Table IV of Ref. 10 and Table III of Ref. 11) where the approximation $B(u)=n/2+O(u^2)$ was used. Only our low-order coefficients c_{F0}^{+}, c_{F1}^{+}, c_{F0}^{-}, c_{F1}^{-}, and c_{F2}^{-} agree with the previous ones.[10,11] The coefficients c_{Fm}^{+} with $m>1$ differ considerably from the previous ones whereas the coefficients c_{Fm}^{-} with $m>2$ differ only by 0.2% $(m=3)$, 0.1% $(m=4)$, and 2% $(m=5)$.

Very recently the coefficients c_{Fm}^{+} and c_{Fm}^{-} have been determined analytically for general n up to three-loop order.[31] The corresponding three-loop values c_{F2}^{+} for $n=1,2,3$ as well as c_{F3}^{-} for $n=1,2,3$ are taken from Ref. 31 and are included in Table III up to nine digits. No results for c_{F3}^{-} were available in the previous literature for $n>1$. The new information on c_{F3}^{-} for $n>1$ enables us to perform the first Borel resummations for $F_-(u)$ and $F_-(u)-F_+(u)$ for $n=2$ and 3 in Sec. III D below.

C. Borel resummation of five-loop results for $n=1$

In order to study the effect of the new higher-order terms we have performed Borel resummations of the series for $uF_-(u)$ and for $u[F_-(u)-F_+(u)]$ at the fixed point u^* for the case $n=1$. The method employed is the same as for $B(u)$ in Sec. II. The parameter ranges turn out to be

$$1.6\leqslant\alpha\leqslant1.7,\quad 7.48\leqslant b\leqslant 8.70 \tag{3.12}$$

for $u^*F_-(u^*)$, and

$$1.4\leqslant\alpha\leqslant1.7,\quad 6.0\leqslant b\leqslant 11.7 \tag{3.13}$$

for $u^*[F_-(u^*)-F_+(u^*)]$.

We have found that our present method does not yield a reliable estimate of the parameters α and b for $F_+(u^*)$ separately; this may be related to the fact that, unlike c_{Fm}^-, the coefficients c_{Fm}^+ (see Table III) do not have alternating signs for $m\leqslant 3$ (this alternation is predicted for the asymptotic large-order behavior[8,35]). $F_+(u)$ will be further studied elsewhere. In the application to amplitude ratios given below we shall not need $F_+(u^*)$ separately.

The resummation results are

$$u^*F_-(u^*)=0.3687\pm0.0040 \tag{3.14}$$

and

$$u^*[F_-(u^*)-F_+(u^*)]=0.4170\pm0.0036. \tag{3.15}$$

The previous approximate resummation results[10,11] for $n=1$ were $u^*F_-(u^*)=0.3648$ and $u^*[F_-(u^*)-F_+(u^*)]=0.4170$ with an error bar of about 1%. Thus our resummation results differ from the previous ones only by about 1% for F_- and by less than 0.1% for $F_- - F_+$, confirming previous conjectures.[10,11] The parameter d_F in the effective representation[11] $F_-(u)=(2u)^{-1}-4(1+d_F u)$ now becomes $d_F=-4.64$ (compared to -4.04 in Ref. 11). This leaves the solid line in Fig. 4 of Ref. 11 essentially unchanged.

D. Borel resummation of three-loop results for $n=2$ and 3

While the previous two-loop result[9] for $F_-(u)$ did not yet provide sufficient information for a controlled resummation procedure for $n>1$ and thus did not yet lead to an error estimate, the new three-loop coefficients c_{F3}^- have significantly improved the situation. On the basis of these three-loop results we have performed Borel resummations of the series for $uF_-(u)$ and for $u[F_-(u)-F_+(u)]$ for the cases $n=2$ and 3. The method employed is the same as in Sec. II. The parameter ranges turn out to be

$$-1.70\leqslant\alpha\leqslant1.10,\quad 1.5\leqslant b\leqslant4.3,\quad n=2, \tag{3.16}$$

$$-1.90\leqslant\alpha\leqslant1.15,\quad 1.5\leqslant b\leqslant4.7,\quad n=3 \tag{3.17}$$

for $u^*F_-(u^*)$, and

$$-0.80\leqslant\alpha\leqslant0.95,\quad 0.65\leqslant b\leqslant8.3,\quad n=2, \tag{3.18}$$

$$-0.30\leqslant\alpha\leqslant0.55,\quad 1.10\leqslant b\leqslant7.5,\quad n=3 \tag{3.19}$$

for $u^*[F_-(u^*)-F_+(u^*)]$. The resummation results are

$$u^*F_-(u^*)=0.384\pm0.025,\quad n=2, \tag{3.20}$$

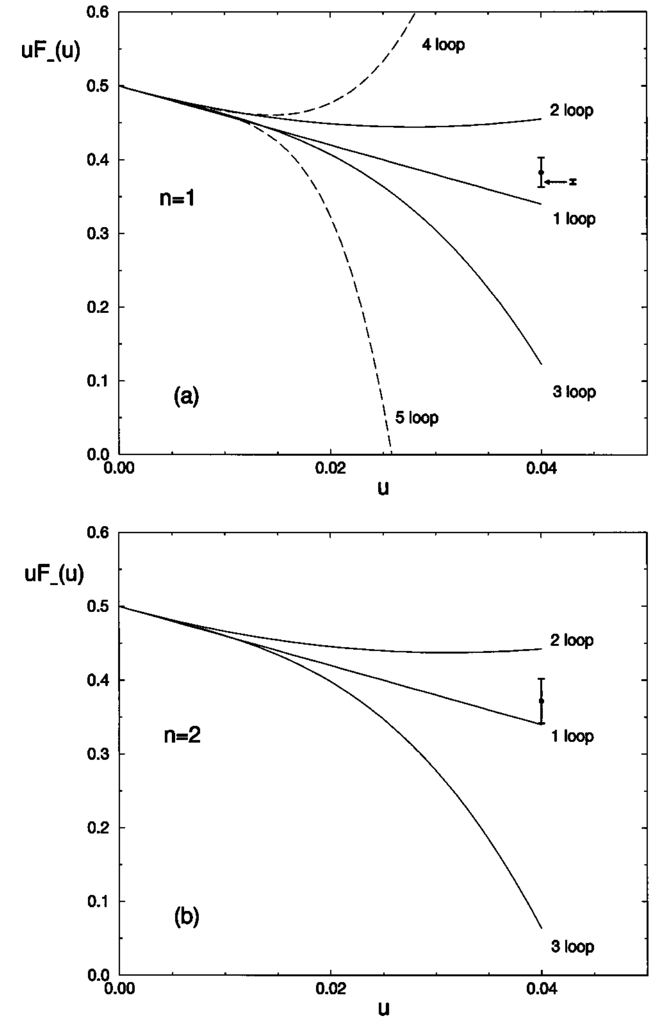


FIG. 4. Amplitude function $F_-(u)$, multiplied by the renormalized coupling u, in three dimensions as a function of u in one-, two-, and three-loop order (solid lines) for $n=1$ (a) and $n=2$ (b) and in four- and five-loop order for $n=1$ (a) (dashed lines). The dot with error bars at $u=0.04$ indicates the result of the Borel resummation on the basis of the three-loop series for $uF_-(u)$. The small error bar (a) indicates the Borel resummation result on the basis of the five-loop series for $uF_-(u)$ at $u=0.04$ for $n=1$, the arrow indicates the corresponding central value at $u=0.04$.

$$u^*F_-(u^*)=0.387\pm0.026,\quad n=3, \tag{3.21}$$

and

$$u^*[F_-(u^*)-F_+(u^*)]=0.461\pm0.019,\quad n=2, \tag{3.22}$$

$$u^*[F_-(u^*)-F_+(u^*)]=0.498\pm0.011,\quad n=3. \tag{3.23}$$

In order to substantiate the reliability of these resummations of three-loop results we have repeated our resummation for $n=1$ but now on the basis of three-loop results. In Fig. 4(a) we show the Borel resummation value for $uF_-(u)$ at $u=0.04$ for $n=1$ together with the corresponding error bar. For comparison, the smaller error bar resulting from the Borel summation on the basis of the five-loop results is also shown. We see that the central value of the three-loop Borel resummation is not far from the central value of the five-loop

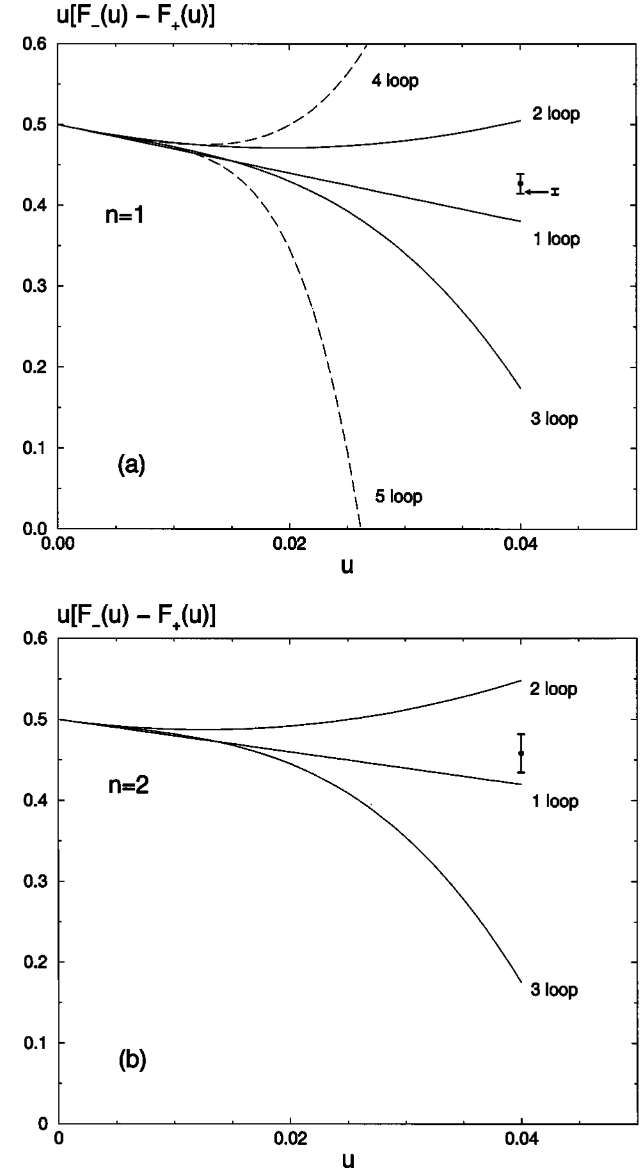


FIG. 5. Amplitude function $F_{-}(u)-F_{+}(u)$, multiplied by the renormalized coupling u, in three dimensions as a function of u in one-, two- and three-loop order (solid lines) for $n=1$ (a) and $n=2$ (b) and in four- and five-loop order for $n=1$ (a) (dashed lines). The dot with error bars at $u=0.04$ indicates the result of the Borel resummation on the basis of the three-loop series for $u[F_{-}(u)-F_{+}(u)]$. The small error bar (a) indicates the Borel resummation result on the basis of the five-loop series for $u[F_{-}(u)-F_{+}(u)]$ at $u=0.04$ for $n=1$, the arrow indicates the corresponding central value at $u=0.04$.

resummation [arrow in Fig. 4(a)]. Analogous results are shown in Fig. 5(a) for $u[F_{-}(u)-F_{+}(u)]$ at $u=0.04$ for $n=1$. These results provide confidence in our resummation procedure based on three-loop results. In particular they demonstrate that Borel resummations of three-loop results yield reliable results with error bars that significantly reduce the uncertainties of one- and two-loop calculations. Since the n dependence beyond three-loop order is expected to be smooth and weak in the range $1\leqslant n\leqslant 3$ there is no reason to expect that the reliability of our resummation procedure is significantly different for $n=2$ and 3. In Figs. 4(b) and 5(b) the corresponding resummation results (with error bars) on the basis of three-loop perturbation series are shown for $n=2$ at the same value $u=0.04$ as for $n=1$ in Figs. 4(a) and 5(a). They clearly demonstrate the significant improvement over the previous situation at two-loop order[9,12] where no error bar could be determined in a convincing fashion.

IV. APPLICATIONS

We apply our results to the specific heat in the asymptotic critical region where it can be represented as[1,9]

$$\mathring{C}^{\pm}=\frac{A^{\pm}}{\alpha}\,|t|^{-\alpha}(1+a_c^{\pm}|t|^{\Delta}+\cdots)+B \tag{4.1}$$

with the Wegner exponent[38] Δ. We consider the universal amplitude ratios[1] A^{+}/A^{-} and a_c^{+}/a_c^{-}. The former can be expressed in terms of $B^{*}\equiv B(u^{*})$ and $F_{\pm}^{*}\equiv F_{\pm}(u^{*})$ in three dimensions as[9]

$$\frac{A^{+}}{A^{-}}=\left(\frac{b_{+}}{b_{-}}\right)^{\alpha}\left[1-\alpha\,\frac{F_{-}^{*}-F_{+}^{*}}{4\nu B^{*}+\alpha F_{-}^{*}}\right], \tag{4.2}$$

where

$$\frac{b_{+}}{b_{-}}=\frac{2\nu P_{+}^{*}}{(3/2)-2\nu P_{+}^{*}} \tag{4.3}$$

with $P_{+}^{*}\equiv P_{+}(u^{*})$. This expression for A^{+}/A^{-} has several independent sources of inaccuracies: (i) The values for the critical exponents α and ν, (ii) the values for P_{+}^{*}, (iii) the values for B^{*}, (iv) the values for $F_{-}^{*}-F_{+}^{*}$, and F_{-}^{*}. In evaluating this expression for A^{+}/A^{-} for $n=1,2,3$ we shall take (i) the most reliable values for α and ν that are presently available, (ii) the Borel resummed values for P_{+}^{*} based on five-loop results for $n=1,2,3$, (iii) the Borel resummed values for B^{*} for $n=1,2,3$ as given by Eqs. (2.34)–(2.36), and (iv) the Borel resummed values for $F_{\pm}^{*}$ as given in Eqs. (3.14), (3.15), and (3.20)–(3.23). For the critical exponents α and ν we take

$$\alpha=0.1070\pm 0.0045,\quad \nu=0.6310\pm 0.0015,\quad n=1, \tag{4.4}$$

$$\alpha=-0.01285\pm 0.000\,38,\quad \nu=0.67095\pm 0.000\,13,\quad n=2, \tag{4.5}$$

$$\alpha=-0.1150\pm 0.0090,\quad \nu=0.7050\pm 0.0030,\quad n=3, \tag{4.6}$$

according to Refs. 13, 5, and 37, respectively. Borel summation values for P_{+}^{*} have been calculated previously.[10] These calculations, however, were based on five-loop coefficients that were derived from the five-loop results of Refs. 17 and 18 which were corrected by Ref. 19. Taking into account the latter corrections we have derived corrected five-loop coefficients c_{P5} of the power series $P_{+}(u)=\Sigma_0^{\infty}c_{Pm}u^{m}$. Our new values for c_{P5} are (compare Table IV of Ref. 10)

$$c_{p5}=-1\,026\,631.56,\quad n=1, \tag{4.7}$$

$$c_{p5}=-1\,763\,840.20,\quad n=2, \tag{4.8}$$

$$c_{p5}=-2\,767\,879.03,\quad n=3. \tag{4.9}$$

TABLE IV. Universal amplitude ratio A^+/A^- of the specific heat, Eqs. (4.1)–(4.3), for $n=1,2,3$ in three dimensions.

	Field theory		Lattice series	
n	Present work	Previous work	expansions	Experiment
1	0.540 ± 0.011	0.524 ± 0.010[a] 0.541 ± 0.014[b]	0.523 ± 0.009[c]	0.56–0.63[d],0.53[e]
2	1.056 ± 0.004	1.0294 ± 0.0134[a] 1.05[f]	1.08[g]	1.054 ± 0.001[h] 1.058 ± 0.004[i] 1.067 ± 0.003[j] 1.088 ± 0.007[k]
3	$1.51\ \pm0.04$	1.521 ± 0.022[a] 1.58[f]	1.52[g]	1.40[l]

[a]Reference 32.
[b]Reference 30.
[c]Reference 39.
[d]Reference 40.
[e]Reference 41.
[f]Reference 9.
[g]Reference 42.
[h]Reference 5.
[i]Reference 43.
[j]Reference 44.
[k]Reference 45.
[l]Reference 46.

For the corresponding Borel resummed values of P_+^* we have obtained

$$P_+^*=0.7568\pm0.0044, \quad n=1, \tag{4.10}$$

$$P_+^*=0.7091\pm0.0045, \quad n=2, \tag{4.11}$$

$$P_+^*=0.6709\pm0.0039, \quad n=3. \tag{4.12}$$

The values P_+^* obtained from Eq. (3.2) and Table I of Ref. 10 are by about 0.7% smaller than the values in Eqs. (4.10)–(4.12).

If we use the central values of the critical exponents given by Eqs. (4.4)–(4.6) and collect the results of Eqs. (2.24)–(2.26), (2.34)–(2.36), (3.14), (3.15), (3.20)–(3.23), and (4.10)–(4.12) we arrive at the values for A^+/A^- according to Eqs. (4.2) and (4.3) as given in Table IV for $n=1,2,3$. For comparison this Table also contains the results of previous calculations as well as experimental or numerical results. We see that there is good agreement between the previous and our results for $n=1$ and $n=3$. For $n=2$, however, our result is significantly more accurate and agrees well with the high-precision experimental result for ^{4}He near the superfluid transition obtained from a recent experiment in space.[5]

The smallness of our error bar for A^+/A^- for $n=2$ is due to the small value of α for $n=2$ which, according to the structure of Eq. (4.2), suppresses the error of $F_\pm^*$. Exploiting this structure, i.e., separating exponents from amplitude functions, is a particular advantage of our $d=3$ formulation of field theory. This structure was not explicitly taken into account in the previous ϵ-expansion analysis for $n=2$.[32] The previous result obtained from the ϵ expansion up to $O(\epsilon^2)$ (Ref. 32) does not agree with the experimental result for $n=2$ within the error bars. The results 1.05 and 1.58 of Ref. 9 for $n=2$ and $n=3$ are partly based on the one-loop form of $F_-(u)$ for which no error bar was available in Ref. 9.

The expression of a_c^+/a_c^- is more complicated and depends on the derivatives of the functions $F_-(u)$, $F_-(u)-F_+(u)$, $P_+(u)$, $B(u)$, and $\zeta_r(u)$ at the fixed point, as specified in Eqs. (4.24), (5.19) and (5.21)–(5.24) of Ref. 9. We have performed Borel resummations for these quantities on the basis of our new five-loop results for $n=1$. For a_c^+/a_c^- we obtain the value 1.0 ± 0.1. This is consistent with the previous result[30] 0.96 ± 0.25 for $n=1$. Since for $n>1$ only three-loop results are available (which would still yield larger error bars for the derivatives of the various functions mentioned above) we postpone corresponding calculations of a_c^+/a_c^- for $n>1$ until four-loop results become available.

Finally we note that not only the fixed-point values B^* and $F_\pm^*$ but also the entire functions $B(u)$ and $F_\pm(u)$ are of physical relevance. They are needed (i) for determining the effective renormalized coupling from experimental data of the specific heat,[6,7] and (ii) for representing the specific heat in the entire nonasymptotic critical region well away from T_c according to Eqs. (3.3)–(3.5). We also note that B^* and $F_\pm^*$ enter various other important universal ratios, e.g., R_ξ^T related to the superfluid density ρ_s of ^{4}He.[9]

Note added. After completion of the present work we learned of Ref. 47, where the perturbative terms of a function equivalent to $B(u)$ have been calculated up to five-loop order. These terms agree with ours in Eq. (2.21). At the end of this work it is asserted that in our paper we use the ϵ expansion. Our work does not use the ϵ expansion to calculate the amplitude functions $F_\pm(u)$ and the amplitude ratios A^+/A^- and a_c^+/a_c^-. Only the fixed-point value u^* has been determined via a Borel resummation of the ϵ expansion series for u^*.

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APPENDIX A: DERIVATION OF THE ADDITIVE RENORMALIZATION $A(u,\epsilon)$

The following equations (A1)–(A54) give the ultraviolet $d=4$ poles of the diagrams shown in Fig. 1 defined by

$KR'(\partial^2 I^{(i)}/\partial r_0^2)$ according to the standard notations, see e.g., Ref. 18. Here R' denotes the incomplete ultraviolet R operation which subtracts subdivergences, and K denotes the operation of taking pole parts. We use subscripts $(a.b)$ for the pole terms on the right-hand sides of Eqs. (A1)–(A8) that are identical with the numbers associated with the three- and four-loop diagrams of Ref. 16 (the first number a in the brackets indicates the number of loops and the second number b indicates the consecutive number of a diagram in Table I of Ref. 16). The subscripts on the right-hand sides of Eqs. (A9)–(A18) correspond to the numbers of the five-loop diagrams of Ref. 18. We have multiplied the left-hand sides by factors $(16\pi^2)^m$ in m-loop order because of the definition of the bare four-point coupling $16\pi^2 g_0/24$ in Refs. 16–19.

The one- and two-loop expressions read

$$16\pi^2 KR'\left(\frac{\partial^2}{\partial r_0^2}\left[-\frac{1}{2}\int_p \ln(r_0+p^2)\right]\right)=\frac{1}{2}I_{(1.1)}\equiv J^{(1)}, \tag{A1}$$

$$(16\pi^2)^2 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(2)}(r_0,\epsilon)\right)=2I_{(2.2)}\equiv J^{(2)}, \tag{A2}$$

the three-loop expressions read

$$(16\pi^2)^3 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(3)}(r_0,\epsilon)\right)=2I_{(3.2)}\equiv J^{(3)}, \tag{A3}$$

$$(16\pi^2)^3 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(4)}(r_0,\epsilon)\right)=8I_{(3.4)}+12I_{(3.9)}\equiv J^{(4)}, \tag{A4}$$

the four-loop expressions read

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(5)}(r_0,\epsilon)\right)=2I_{(4.5)}\equiv J^{(5)}, \tag{A5}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(6)}(r_0,\epsilon)\right)=0\equiv J^{(6)}, \tag{A6}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(7)}(r_0,\epsilon)\right)=4I_{(4.9)}+6I_{(4.11)}\equiv J^{(7)}, \tag{A7}$$

$$(16\pi^2)^4 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(8)}(r_0,\epsilon)\right)=24I_{(4.12)}+6I_{(4.18)}+12I_{(4.19)}\equiv J^{(8)}, \tag{A8}$$

and the five-loop expressions read

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(9)}(r_0,\epsilon)\right)=2I_{117}\equiv J^{(9)}, \tag{A9}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(10)}(r_0,\epsilon)\right)=0\equiv J^{(10)}, \tag{A10}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(11)}(r_0,\epsilon)\right)=0\equiv J^{(11)}, \tag{A11}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(12)}(r_0,\epsilon)\right)=2I_{121}\equiv J^{(12)}, \tag{A12}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(13)}(r_0,\epsilon)\right)=4I_7+6I_{120}\equiv J^{(13)}, \tag{A13}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(14)}(r_0,\epsilon)\right)=2I_8\equiv J^{(14)}, \tag{A14}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(15)}(r_0,\epsilon)\right)=2I_{14}+12I_{15}+12I_{19}+24I_{21}+4I_{23}+18I_{106}\equiv J^{(15)}, \tag{A15}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(16)}(r_0,\epsilon)\right)=8I_{79}+16I_{88}+32I_{95}+16I_{100}\equiv J^{(16)}, \tag{A16}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(17)}(r_0,\epsilon)\right)=2I_{81}+8I_{84}+4I_{99}\equiv J^{(17)}, \tag{A17}$$

$$(16\pi^2)^5 KR'\left(\frac{\partial^2}{\partial r_0^2}I^{(18)}(r_0,\epsilon)\right)=8I_{32}+32I_{47}+4I_{73}+8I_{93}+8I_{98}+4I_{109}+8I_{116}\equiv J^{(18)}. \tag{A18}$$

The pole terms up to four loops are[16]

$$I_{(1.1)}=\frac{2}{\epsilon}, \tag{A19}$$

$$I_{(2.2)}=-\frac{4}{\epsilon^2}, \tag{A20}$$

$$I_{(3.2)}=\frac{8}{\epsilon^3}, \tag{A21}$$

$$I_{(3.4)}=\frac{2}{3\epsilon^2}-\frac{3}{4\epsilon}, \tag{A22}$$

$$I_{(3.9)}=\frac{8}{3\epsilon^3}-\frac{8}{3\epsilon^2}+\frac{2}{3\epsilon}, \tag{A23}$$

$$I_{(4.5)}=-\frac{16}{\epsilon^4}, \tag{A24}$$

$$I_{(4.9)}=-\frac{4}{3\epsilon^3}+\frac{3}{2\epsilon^2}, \tag{A25}$$

$$I_{(4.11)}=-\frac{16}{3\epsilon^4}+\frac{16}{3\epsilon^3}-\frac{4}{3\epsilon^2}, \tag{A26}$$

$$I_{(4.12)}=-\frac{4}{3\epsilon^4}+\frac{10}{3\epsilon^3}-\frac{13}{3\epsilon^2}+\frac{11-6\zeta(3)}{6\epsilon}, \tag{A27}$$

$$I_{(4.18)}=-\frac{8}{3\epsilon^4}+\frac{8}{3\epsilon^3}+\frac{4}{3\epsilon^2}+\frac{2\zeta(3)-2}{\epsilon}, \qquad (A28)$$

$$I_{(4.19)}=-\frac{2}{3\epsilon^3}+\frac{1}{\epsilon^2}-\frac{7}{12\epsilon}, \qquad (A29)$$

and the five-loop pole terms are[18]

$$I_7=\frac{8}{3\epsilon^4}-\frac{3}{\epsilon^3}, \qquad (A30)$$

$$I_8=\frac{8}{3\epsilon^4}-\frac{3}{\epsilon^3}, \qquad (A31)$$

$$I_{14}=\frac{4}{15\epsilon^3}-\frac{3}{10\epsilon^2}-\frac{5}{96\epsilon}, \qquad (A32)$$

$$I_{15}=\frac{2}{15\epsilon^3}-\frac{13}{20\epsilon^2}+\frac{857}{960\epsilon}, \qquad (A33)$$

$$I_{19}=\frac{8}{15\epsilon^4}-\frac{7}{3\epsilon^3}+\frac{25}{6\epsilon^2}-\frac{215}{96\epsilon}, \qquad (A34)$$

$$I_{21}=\frac{16}{15\epsilon^4}-\frac{7}{5\epsilon^3}-\frac{11}{60\epsilon^2}+\frac{157}{320\epsilon}, \qquad (A35)$$

$$I_{23}=\frac{4}{15\epsilon^3}-\frac{3}{10\epsilon^2}-\frac{5}{96\epsilon}, \qquad (A36)$$

$$I_{32}=\frac{48\zeta(3)}{5\epsilon^3}-\frac{48\zeta(3)+24\zeta(4)}{5\epsilon^2}+\frac{14\zeta(3)+9\zeta(4)-16\zeta(5)}{5\epsilon}, \qquad (A37)$$

$$I_{47}=\frac{8}{15\epsilon^5}-\frac{12}{5\epsilon^4}+\frac{6}{\epsilon^3}+\frac{18\zeta(3)-45}{5\epsilon^2}+\frac{146-90\zeta(3)-9\zeta(4)}{30\epsilon}, \qquad (A38)$$

$$I_{73}=\frac{16}{15\epsilon^5}-\frac{8}{3\epsilon^4}+\frac{28}{15\epsilon^3}+\frac{6+4\zeta(3)}{5\epsilon^2}-\frac{32-12\zeta(3)-18\zeta(4)}{30\epsilon}, \qquad (A39)$$

$$I_{79}=\frac{16}{5\epsilon^5}-\frac{16}{5\epsilon^4}-\frac{8}{5\epsilon^3}+\frac{4+4\zeta(3)}{5\epsilon^2}+\frac{7+6\zeta(3)-12\zeta(4)}{5\epsilon}, \qquad (A40)$$

$$I_{81}=\frac{16}{3\epsilon^5}-\frac{16}{3\epsilon^4}-\frac{8}{3\epsilon^3}+\frac{4-4\zeta(3)}{\epsilon^2}, \qquad (A41)$$

$$I_{84}=\frac{8}{3\epsilon^5}-\frac{20}{3\epsilon^4}+\frac{26}{3\epsilon^3}-\frac{44-24\zeta(3)}{12\epsilon^2}, \qquad (A42)$$

$$I_{88}=\frac{16}{15\epsilon^5}-\frac{8}{3\epsilon^4}+\frac{28}{15\epsilon^3}+\frac{6-12\zeta(3)}{5\epsilon^2}-\frac{16-18\zeta(4)}{15\epsilon}, \qquad (A43)$$

$$I_{93}=\frac{8}{5\epsilon^5}-\frac{52}{15\epsilon^4}+\frac{34}{15\epsilon^3}+\frac{116-24\zeta(3)}{60\epsilon^2}-\frac{56-44\zeta(3)+6\zeta(4)}{20\epsilon}, \qquad (A44)$$

$$I_{95}=\frac{16}{15\epsilon^5}-\frac{8}{3\epsilon^4}+\frac{28}{15\epsilon^3}+\frac{6-12\zeta(3)}{5\epsilon^2}-\frac{16-18\zeta(4)}{15\epsilon}, \qquad (A45)$$

$$I_{98}=\frac{4}{15\epsilon^4}-\frac{14}{15\epsilon^3}+\frac{19}{15\epsilon^2}-\frac{386+768\zeta(3)}{960\epsilon}, \qquad (A46)$$

$$I_{99}=\frac{4}{3\epsilon^4}-\frac{2}{\epsilon^3}+\frac{7}{6\epsilon^2}, \qquad (A47)$$

$$I_{100}=\frac{4}{5\epsilon^4}-\frac{6}{5\epsilon^3}+\frac{1}{5\epsilon^2}+\frac{81-48\zeta(3)}{160\epsilon}, \qquad (A48)$$

$$I_{106}=\frac{64}{15\epsilon^5}-\frac{32}{5\epsilon^4}+\frac{8}{5\epsilon^3}+\frac{16}{15\epsilon^2}-\frac{2}{15\epsilon}, \qquad (A49)$$

$$I_{109}=\frac{16}{15\epsilon^5}-\frac{16}{5\epsilon^4}+\frac{16}{5\epsilon^3}+\frac{52-108\zeta(3)}{15\epsilon^2}-\frac{202-168\zeta(3)-18\zeta(4)}{30\epsilon}, \qquad (A50)$$

$$I_{116}=\frac{8}{15\epsilon^4}-\frac{4}{3\epsilon^3}+\frac{32}{15\epsilon^2}-\frac{250-96\zeta(3)}{120\epsilon}, \qquad (A51)$$

$$I_{117}=\frac{32}{\epsilon^5}, \qquad (A52)$$

$$I_{120}=\frac{32}{3\epsilon^5}-\frac{32}{3\epsilon^4}+\frac{8}{3\epsilon^3}, \qquad (A53)$$

$$I_{121}=\frac{32}{3\epsilon^5}-\frac{32}{3\epsilon^4}+\frac{8}{3\epsilon^3}. \qquad (A54)$$

In Eqs. (A37) and (A51) the corrections found in Ref. 19 have been taken into account. Note that in Eqs. (A19)–(A54) we have used $\epsilon=4-d$ whereas in Refs. 16, 18 ϵ denotes $(4-d)/2$.

Equations (A1)–(A54) determine the additive renormalization according to Eq. (2.7) as

$$A(u,\epsilon)=-2\Bigg[\frac{n}{\epsilon}-\frac{8n(n+2)}{\epsilon^2}(-u/2)+\sum_{i=3}^{4}S^{(i)}G^{(i)}J^{(i)}(-u/2)^2+\sum_{i=5}^{8}S^{(i)}G^{(i)}J^{(i)}(-u/2)^3+\sum_{i=9}^{18}S^{(i)}G^{(i)}J^{(i)}(-u/2)^4\Bigg]. \tag{A55}$$

The overall factor of 2 in Eq. (A55) arises from the $d=4$ value of the factor $(A_d/4)^{-1}/16\pi^2$ which is needed to obtain $A(u,\epsilon)$ from Eqs. (A1)–(A54) according to Eq. (2.7). The renormalized coupling u enters Eq. (A55) in the form $u/2$; the factor 1/2 takes into account that, near $d=4$, $u_0=A_d^{-1}u+O(u^2)=8\pi^2u+O(u^2)=1/2[16\pi^2u+O(u^2)]$ [see Eq. (2.8)].

APPENDIX B: Z FACTORS

In deriving the coefficients of the perturbation series of the quantities $F_\pm(u)$, $P_\pm(u)$, and $f_\pm^{(3,0)}(u)$ we needed the Z factors Z_r, Z_ϕ, and Z_u calculated previously[17–19] up to five-loop order. Since their explicit form is not available in the literature we present them here explicitly. They read

$$Z_r(u,\epsilon)=1+\sum_{m=1}^{5}Z_r^{(m)}(\epsilon)u^m+O(u^6), \tag{B1}$$

$$Z_u(u,\epsilon)=1+\sum_{m=1}^{5}Z_u^{(m)}(\epsilon)u^m+O(u^6), \tag{B2}$$

$$Z_\phi(u,\epsilon)=1+\sum_{m=1}^{5}Z_\phi^{(m)}(\epsilon)u^m+O(u^6), \tag{B3}$$

with the following coefficients in m-loop order.

Coefficients of $Z_r(u,\epsilon)$:

$$Z_r^{(1)}(\epsilon)=\frac{4(n+2)}{\epsilon}, \tag{B4}$$

$$Z_r^{(2)}(\epsilon)=4(n+2)\left[\frac{4(n+5)}{\epsilon^2}-\frac{5}{\epsilon}\right], \tag{B5}$$

$$Z_r^{(3)}(\epsilon)=\frac{16}{3}(n+2)\left[\frac{15n+111}{\epsilon}+\frac{-278-61n}{\epsilon^2}+\frac{12n^2+132n+360}{\epsilon^3}\right], \tag{B6}$$

$$\begin{aligned}Z_r^{(4)}(\epsilon)=-\frac{2}{3}(n+2)\Bigg[&\frac{288\zeta(4)(5n+22)+48\zeta(3)(3n^2+10n+68)+31\,060-n^2+7578n}{\epsilon}\\&-\frac{1152\zeta(3)(22+5n)+1236n^2+23\,580n+74\,616}{\epsilon^2}+\frac{16(245n^2+2498n+6284)}{\epsilon^3}\\&-192\frac{(n+5)(2n+13)(n+6)}{\epsilon^4}\Bigg],\end{aligned} \tag{B7}$$

$$\begin{aligned}Z_r^{(5)}(\epsilon)=\frac{4}{15}(n+2)\Bigg[&[9600\zeta(6)(55n+2n^2+186)+768\zeta(5)(-5n^2+72+14n)+288\zeta(4)(29n^2+2668-3n^3+816n)\\&+768\zeta(3)^2(-2n^2-145n-582)+48\zeta(3)(8208n+17n^3+940n^2+31\,848)+21n^3+45\,254n^2+1\,077\,120n\\&+3\,166\,528]\frac{1}{\epsilon}-[30\,720\zeta(5)(2n^2+186+55n]+576\zeta(4)(5n+22)(n-22)+96\zeta(3)(27n^3+1224n^2+14\,456n\\&+45\,448)-98n^3+277\,280n^2+3\,073\,376n+7\,449\,712)\frac{1}{\epsilon^2}+[2304\zeta(3)(13n+74)(5n+22)+21\,576n^3\\&+685\,192n^2+5\,017\,312n+10\,459\,360]\frac{1}{\epsilon^3}-\frac{32(307\,976+31\,752n^2+172\,176n+1933n^3}{\epsilon^4}\\&+\frac{384(5n+34)(n+6)(n+5)(2n+13)}{\epsilon^5}\Bigg].\end{aligned} \tag{B8}$$

Coefficients of $Z_\phi(u,\epsilon)$:

$$Z_{\phi}^{(1)}=0, \tag{B9}$$

$$Z_{\phi}^{(2)}=-\frac{4(n+2)}{\epsilon}, \tag{B10}$$

$$Z_{\phi}^{(3)}=\frac{8}{3}\,(n+2)(n+8)\left[\frac{1}{\epsilon}-\frac{4}{\epsilon^2}\right], \tag{B11}$$

$$Z_{\phi}^{(4)}=2(n+2)\left[\frac{5(n^2-18n-100)}{\epsilon}+\frac{4(n^2+234+53n)}{\epsilon^2}-\frac{16(n+8)^2}{\epsilon^3}\right], \tag{B12}$$

$$\begin{aligned}Z_{\phi}^{(5)}=&-\frac{8}{15}\,(n+2)\Bigg[[-1152\zeta(4)(5n+22)+48\zeta(3)(-6n^2+184+n^3+64n)-22\,752n-39n^3-296n^2-77\,056]\,\frac{1}{\epsilon}\\&+\frac{2304\zeta(3)(5n+22)-60n^3+135\,488+42\,440n+1844n^2}{\epsilon^2}-\frac{16(n+8)(3n^2+269n+1210)}{\epsilon^3}+\frac{192(n+8)^3}{\epsilon^4}\Bigg].\end{aligned} \tag{B13}$$

Coefficients of $Z_u(u,\epsilon)$:

$$Z_u^{(1)}=\frac{4(n+8)}{\epsilon}, \tag{B14}$$

$$Z_u^{(2)}=16\left[\frac{(n+8)^2}{\epsilon^2}-\frac{5n+22}{\epsilon}\right], \tag{B15}$$

$$Z_u^{(3)}=\frac{8}{3}\left[\frac{96\zeta(3)(5n+22)+942n+2992+35n^2}{\epsilon}-\frac{16(n+8)(17n+76)}{\epsilon^2}+\frac{24(n+8)^3}{\epsilon^3}\right], \tag{B16}$$

$$\begin{aligned}Z_u^{(4)}=&-\frac{16}{3}\Bigg[[480\zeta(5)(2n^2+55n+186)-72\zeta(4)(n+8)(5n+22)+24\zeta(3)(63n^2+764n+2332)+20\,624n+1640n^2-5n^3\\&+49\,912]\,\frac{1}{\epsilon}-\frac{480\zeta(3)(5n+22)(n+8)+67\,424n+153\,088+7736n^2+172n^3}{\epsilon^2}+\frac{16(55n+248)(n+8)^2}{\epsilon^3}-\frac{48(n+8)^4}{\epsilon^4}\Bigg],\end{aligned} \tag{B17}$$

$$\begin{aligned}Z_u^{(5)}=&\frac{4}{15}\Bigg[[6912\zeta(7)(25\,774+9261n+686n^2)-28\,800\zeta(6)(n+8)(2n^2+55n+186)+768\zeta(5)(165\,084+7466n^2+305n^3\\&+66\,986n)-288\zeta(4)(62\,656+4084n^2+28\,084n+189n^3)+2304\zeta(3)^2(3264-59n^2-6n^3+446n)+48\zeta(3)(1264n^3\\&-13n^4+1\,312\,864+551\,032n+67\,432n^2)+20\,429\,248n+2\,518\,864n^2+195n^4+40\,148\,480+39\,230n^3]\,\frac{1}{\epsilon}\\&-[99\,840\zeta(5)(n+8)(2n^2+55n+186)-14\,976\zeta(4)(5n+22)(n+8)^2+3456\zeta(3)(91n^3+15\,436n+34\,144+2196n^2)\\&+63\,219\,712n-800n^4+420\,800n^3+117\,768\,192+9\,811\,712n^2]\,\frac{1}{\epsilon^2}\\&+\frac{66\,048\zeta(3)(5n+22)(n+8)^2+32(n+8)(733n^3+40\,186n^2+353\,392n+803\,328)}{\epsilon^3}-\frac{512(193n+875)(n+8)^3}{\epsilon^4}\\&+\frac{3840(n+8)^5}{\epsilon^5}\Bigg].\end{aligned} \tag{B18}$$

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