Penetration depth and the conductivity sum rule for a model with incoherent *c*-axis coupling

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The conductivity sum rule for a one-band hopping model relates the integrated spectral weight of the real part of the conductivity to the average kinetic energy. For such a model, the superconducting penetration depth is therefore dependent upon both the change in the conductivity spectral weight and the change in kinetic energy between the normal and superconducting states. Here we examine the consequences of this for the c-axis penetration depth of a layered system in which the charge transfer perpendicular to the layers (along the c axis) is mediated by interlayer impurity scattering.
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The nature of the frequency- and temperature-dependent c-axis conductivity, $\sigma_{1c}(\omega,T)$, in the cuprate superconductors remains controversial, but for a number of these materials it appears to be weak and incoherent.[1] Recently, a simple model[2–4] consisting of layers with BCS quasiparticles which have a $d_{x^2-y^2}$ gap and an interlayer coupling mediated by impurity scattering was used to calculate $\sigma_{1c}(\omega,T)$. For this model, the conductivity sum rule relates the integrated spectral weight under $\sigma_{1c}(\omega,T)$ to the average kinetic energy per unit cell in the c direction.[5] For such a model, the superconducting penetration depth is dependent upon both the change in the conductivity spectral weight and the change in the kinetic energy. Here we examine this and discuss its consequences.

We consider a Hamiltonian of the form

$$H=H_{ab}+H_c\,, \tag{1}$$

where H_{ab} describes the intralayer dynamics and H_c is the interlayer coupling

$$H_c=\sum_{l,s}\; V_l(c^\dagger_{l+z,s}c_{l,s}+c^\dagger_{l,s}c_{l+z,s}). \tag{2}$$

Here V_l is a random potential due to impurity scattering between layers. We assume that H_{ab} describes quasiparticles with energy ε_p in the normal state and BCS quasiparticles with dispersion $E_p=\sqrt{\varepsilon_p^2+\Delta_p^2}$ in the superconducting state with $\Delta_p=\Delta_0\cos 2\phi_p$, a $d_{x^2-y^2}$ gap.

For this model, the c-axis conductivity sum rule has the form[5]

$$\frac{2}{\pi e^2 d^2}\int_0^\infty \sigma_{1c}(\omega)d\omega=-\langle K_c\rangle, \tag{3}$$

where d is the interlayer spacing, and $\langle K_c\rangle$ is the c-axis kinetic energy per unit volume

$$\langle K_c\rangle=\frac{\langle H_c\rangle}{V}. \tag{4}$$

If the change in $\langle K_c\rangle$ between the normal and superconducting states is negligible, one has the usual relationship between the loss in the $(\omega>0)$ spectral weight of the conductivity in the superconducting state relative to the normal state and the c-axis penetration depth, λ_c (Refs. 6 and 7)

$$\frac{c^2}{4\pi\lambda_c^2}=\frac{2}{\pi}\int_{0^+}^\infty d\omega[\sigma_{1c}^N(\omega)-\sigma_{1c}^S(\omega)]. \tag{5}$$

Here σ_{1c}^N and σ_{1c}^S are the normal and superconducting c-axis conductivities, respectively. However, when the c-axis tunneling process is incoherent and the gap has a strong momentum dependence, the change in $\langle K_c\rangle$ between the superconducting and normal states becomes important. Then Eq. (5) is modified to

$$\frac{c^2}{4\pi\lambda_c^2}=\frac{2}{\pi}\int_{0^+}^\infty d\omega[\sigma_{1c}^N(\omega)-\sigma_{1c}^S(\omega)] -e^2d^2(\langle K_c\rangle^S-\langle K_c\rangle^N). \tag{6}$$

For the case of a $d_{x^2-y^2}$ superconductor, if the tunneling process is diffuse, the Josephson coupling between the layers vanishes[2–4] and λ_c is infinite. In this case, $\sigma_{1c}(\omega)$ is still suppressed when the gap is opened (see Fig. 2 of Ref. 4) but the change in the kinetic energy in Eq. (6) cancels the change in the spectral weight, leading to an infinite λ_c. If the incoherent tunneling process is anisotropic, there will only be a partial cancellation, leading to a larger λ_c than one would find using Eq. (5). Here, we examine this effect for an impurity scattering model of the interlayer transport.

Taking V_l to be weak, the first nonvanishing contribution to $\langle K_c\rangle$, after averaging over impurities,[4,8] is

$$\frac{4n^c_{\rm imp}}{N_{ab}^2}\sum_{k,p}\overline{|V_{pk}|^2}T\sum_n\frac{(i\omega_n+\epsilon_p)(i\omega_n+\epsilon_k)}{[(i\omega_n)^2-E_p^2][(i\omega_n)^2-E_k^2]} -\frac{4n^c_{\rm imp}}{N_{ab}^2}\sum_{k,p}\overline{|V_{pk}|^2}T\sum_n\frac{\Delta_k\Delta_p}{[(i\omega_n)^2-E_p^2][(i\omega_n)^2-E_k^2]}, \tag{7}$$

where $n^c_{\rm imp}$ is the impurity concentration which causes c-axis transport, N_{ab} is the number of sites in the ab plane, $\omega_n=(2n+1)\pi T$, and we will take the impurity potential to have the separable form

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$$\overline{|V_{pk}|^2} = |V_0|^2 + |V_1|^2 \cos 2\phi_k \cos 2\phi_p . \tag{8}$$

Physically, the first term in Eq. (7) is due to quasiparticle fluctuations between the layers, while the second term is due to superconducting pair fluctuations.

Setting $\Delta_k = 0$ in Eq. (7) gives us $\langle K_c \rangle^N$. Taking $\Delta_k = \Delta_0 \cos 2\phi_k$ gives $\langle K_c \rangle^S$ for a $d_{x^2-y^2}$ superconductor. Thus we find that

$$\begin{aligned}\langle K_c \rangle^S_{d_{x^2-y^2}} - \langle K_c \rangle^N = & -16 n^c_{\mathrm{imp}} N^2(0) T \sum_n |V_0|^2 \\ & \times \left[\frac{\omega_n^2}{\Delta_0^2 + \omega_n^2} \mathbf{K}^2\left(\frac{\Delta_0}{\sqrt{\Delta_0^2 + \omega_n^2}} \right) - \left(\frac{\pi}{2} \right)^2 \right] \\ & - 16 n^c_{\mathrm{imp}} N^2(0) T \sum_n \left\{ \frac{|V_1|^2}{\Delta_0^2 (\Delta_0^2 + \omega_n^2)} \right. \\ & \times \left[\omega_n^2 \mathbf{K}\left(\frac{\Delta_0}{\sqrt{\Delta_0^2 + \omega_n^2}} \right) \right. \\ & \left. \left. - (\Delta_0^2 + \omega_n^2) \mathbf{E}\left(\frac{\Delta_0}{\sqrt{\Delta_0^2 + \omega_n^2}} \right) \right]^2 \right\}, \end{aligned} \tag{9}$$

where $N(0)$ is the bare single-particle density of states, and **K** and **E** are complete elliptic integrals of the first and second kinds, respectively.[9] For $T \ll \Delta_0$,

$$\begin{aligned}\langle K_c \rangle^S_{d_{x^2-y^2}} - \langle K_c \rangle^N = & \frac{8 n^c_{\mathrm{imp}} N^2(0)}{\pi} \Delta_0 (5.12 |V_0|^2 - 2.37 |V_1|^2) \\ & + \mathcal{O}\left[\left(\frac{T}{\Delta_0} \right)^3 \ln^2 \left(\frac{T}{\Delta_0} \right) \right]. \end{aligned} \tag{10}$$

Then from Eq. (6) we have

$$\begin{aligned}\frac{c^2}{4\pi\lambda_c^2} = & \frac{2}{\pi} \int_{0^+}^{\infty} d\omega [\sigma^N_{1c}(\omega) - \sigma^S_{1c}(\omega)] \\ & - \frac{8 n^c_{\mathrm{imp}} N^2(0)}{\pi} e^2 d^2 \Delta_0 (5.12 |V_0|^2 - 2.37 |V_1|^2). \end{aligned} \tag{11}$$

However, we know that when $|V_1|^2 = 0$ there is no pair transport and λ_c becomes infinite. In this case, the $|V_0|^2$ term gives the difference between the area under $\sigma^N_{1c}(\omega,T)$ and $\sigma^S_{1c}(\omega,T)$ for $\omega > 0$ and there is no δ-function contribution at $\omega = 0$. For $|V_1|^2$ small but nonvanishing, λ_c becomes finite but *larger* than one would estimate from the missing spectral area $\sigma^N_{1c}(\omega,T) - \sigma^S_{1c}(\omega,T)$ for $\omega > 0$. If $|V_1|^2$ increases sufficiently so that $|V_1|^2 = 2.16 |V_0|^2$ then there is no change between $\langle K_c \rangle^S$ and $\langle K_c \rangle^N$ and the correct λ_c is obtained from the familiar sum rule, Eq. (5).

Equation (18) of Ref. 4 gives a prediction for the c-axis penetration depth for the model we considered here. It is

$$\frac{c^2}{4\pi\lambda_c^2} \simeq 4\pi e^2 d^2 n^c_{\mathrm{imp}} N^2(0) |V_1|^2 \Delta_0 (0.48). \tag{12}$$

This is the result one would obtain if a direct magnetic measurement of the penetration depth were made. Our Eq. (11) also gives a prediction for the c-axis penetration depth. However, Eq. (11) is the penetration depth inferred from a measurement of the conductivity. Our results show that for a momentum-dependent gap, the conductivity sum rule must be applied with care to determine the penetration depth. Our results show that for a momentum-dependent gap, there is a change in the c-axis kinetic energy between the normal and superconducting states; this change in kinetic energy must be taken into account in order to correctly obtain the penetration depth from the conductivity sum rule. A naive application of the conductivity sum rule [Eq. (5)] would imply a penetration depth which is smaller or larger than what would be measured. From a correct application of the sum rule [Eq. (6)], the correct value of the penetration depth could be inferred. Equations (11) and (12) give the same value for the penetration depth. However, Eq. (11) is what one would use to infer the penetration depth from a measurement of the conductivity.

One can ask what has happened to the conductivity spectral weight. As Hirsch discussed,[10] spectral weight can be transferred to or from higher bands which are not included in our simple interlayer hopping model. Note however, for $|V_1|^2 < 2.16 |V_0|^2$, we have the opposite effect to that discussed by Hirsch for his model of hole superconductivity. That is, for the impurity model we have considered here, when the system goes into the superconducting state, if $|V_1|^2 < 2.16 |V_0|^2$, spectral weight is transferred to higher bands and the true λ_c is larger than one would obtain by simply determining the missing spectral weight according to Eq. (5). Conversely, if $|V_1|^2 > 2.16 |V_0|^2$ spectral weight is transferred down from higher bands and the true λ_c is actually smaller than that given by Eq. (5).

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