Anisotropy-isotropy transition in a Sierpinski gasket fractal

Bibhas Bhattacharyya*
Department of Physics, Scottish Church College 1 & 3, Urquhart Square, Calcutta 700 006, India

Arunava Chakrabarti†
Department of Physics, University of Kalyani, Kalyani, West Bengal 741 235, India
(Received 6 November 1997; revised manuscript received 18 February 1998)

The possibility of restoration of isotropy in a macroscopic scale has been addressed in the case of a Sierpinski gasket fractal. In recent times it has been shown that for a resistor network with an anisotropic distribution of the values of the resistances, isotropy, on the macroscopic scale is always restored in the case of a Sierpinski gasket fractal. We show that the problem of isotropy restoration is not at all trivial when one considers other problems on the same lattice. As examples, we discuss the cases of an anisotropic Ising model and a tight-binding model for noninteracting electrons on a Sierpinski gasket fractal. While in the first case, restoration of isotropy has been shown to be impossible, in the second one restoration does occur, but only in a very restrictive sense. [S0163-1829(98)04029-6]

The problem of restoration of macroscopic isotropy (homogenization) in fractals with microscopic anisotropy has received attention in recent literature.[1,2] Barlow *et al.*[1] addressed the problem in the context of finitely and infinitely ramified fractals, and have concluded that restoration of isotropy is universal. Such a phenomenon stands out to be typical for fractal systems and does not show up in regular lattices.[1] In Ref. 1 a resistor network is constructed on a Sierpinski gasket (SG) and a carpet to investigate the restoration of isotropy. Anisotropy is introduced through different values of the resistances connecting the nodal points. For an SG, restoration has been shown to occur for all initial values of the anisotropy parameter $H=R_y/R_x$, R_y and R_x being the two different values of the resistances used. The rates of restoration of isotropy have been calculated and different scaling regimes have been identified. Very recently, Lin and Goda[2] extended the ideas of Barlow *et al.*[1] to a variation of the SG (the so-called three-simplex lattice) in which the resistances are distributed in a hierarchical pattern. They report that the system undergoes an isotropy-anisotropy transition on tuning the value of the hierarchical parameter. In both the works referred to above the problem has been investigated in terms of a resistor network. While the restoration seems almost inevitable in the case of resistive networks, it is not known whether the same will occur for other problems as well. We therefore find it highly interesting to address the problem of restoration of macroscopic isotropy in fractals in the context of two different problems. As a prototype example of a finitely ramified fractal we choose an SG. We then separately study the classical Ising model with anisotropic interactions on this gasket and investigate the possibility of restoration of macroscopic isotropy. In the other problem we discuss the case of tightly bound noninteracting electrons on the same lattice. In this case we distribute the nearest-neighbor hopping integrals in an anisotropic fashion. We study both the problems within the framework of a real-space renormalization-group (RSRG) method. In the first problem, i.e., the spin model we analyze the flow of the anisotropic exchange terms to show that a restoration to the isotropic limit is impossible here. This result is in complete contrast to a recent claim by Brody and Ritz.[3] We discuss the cause of the difference in these two results. In the electronic case, however, we show that a restoration to isotropy, as far as the values of the hopping integrals are concerned, is possible in a very restrictive sense. In what follows we discuss our results.

Anisotropic Ising model on a Sierpinski gasket:

The Hamiltonian for the Ising model with ferromagnetic coupling between nearest-neighbor spins is given by

$$H=-\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j, \quad (1)$$

where the summation runs over all distinct nearest-neighbor (NN) pairs $\langle ij\rangle$. $J_{ij}=J_x$ when both i and j lie on a horizontal bond [Fig. 1(a)], while $J_{ij}=J_y$ for all other bonds. J_x and J_y are different in the anisotropic limit. We consider the parameter space $J_x, J_y\geqslant 0$ (ferromagnetic coupling).

It is well known[4] that in the isotropic limit ($J_x=J_y$) and in the decoupled one-dimensional chain limit ($J_y=0, J_x>0$) the transition temperature $T_c=0$. Therefore, it is expected that, this system with $J_x\neq J_y$ would not show a finite-temperature phase transition. However, it is interesting to see whether such an anisotropic system tends to restore isotropy

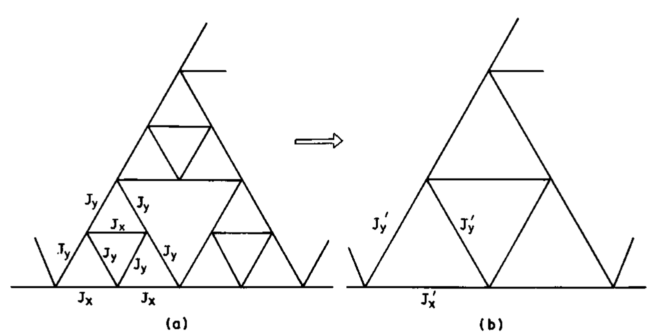

FIG. 1. (a) Part of an infinite Sierpinski gasket and (b) its renormalized version. Anisotropic coupling constants have been shown.

on a larger length scale. To investigate this, we rescale the lattice using the standard RSRG technique.[4] The scaled lattice is shown in Fig. 1(b). The renormalized coupling constants are given by

$$K_x' = \frac{1}{4}\ln[A(K_x,K_y)B(K_x,K_y)/C^2(K_x,K_y)], \quad (2)$$

$$K_y' = \frac{1}{4}\ln[A(K_x,K_y)/B(K_x,K_y)], \quad (3)$$

where

$$A(K_x,K_y) = e^{3K_x}\cosh 6K_y + e^{-3K_x} + e^{-K_x}\cosh 2K_y + e^{K_x}, \quad (4)$$

$$B(K_x,K_y) = e^{3K_x}\cosh 2K_y + e^{-3K_x} + e^{-K_x}\cosh 2K_y + e^{K_x}, \quad (5)$$

$$C(K_x,K_y) = e^{K_x}\cosh 4K_y + e^{K_x} + 2e^{-K_x}\cosh 2K_y, \quad (6)$$

where $K_i = J_i/kT$, $i = x$ or y, T is the temperature, and k is the Boltzmann constant. Primed quantities refer to the renormalized values. It is easy to check that the above recursion relations reproduce those of Gefen *et al.*[4] in the isotropic limit $J_x = J_y$. Additionally, it is expected that, if one starts with the initial value of $J_y = 0$ with any arbitrary nonzero value of J_x, then J_y should continue to be zero under successive renormalization, since the gasket now consists of a set of decoupled chains. On the other hand, even if we set $J_x = 0$ and J_y nonzero at the beginning, J_x should grow nonzero values under iteration, as different sites on the ''horizontal'' axis will be connected via the ''angular'' bonds at different scales of length. These two essential features are correctly reproduced from the set of recursion relations derived above. Analysis of the recursion relations (2) and (3) reveals that there are two fixed points (FP's): a stable one at (0,0) and an unstable one at (∞,∞). This shows that indeed $T_c = 0$ for such a system and any such system with finite J_x and J_y flows down to the FP at (0,0). However, when one addresses the question of restoration of isotropy, one should look at the flow of the parameters K_x and K_y or, at the ratio K_x/K_y under successive renormalization. It is extremely important to appreciate that both K_x and K_y approach zero under iteration as (0,0) is a stable fixed point. One can now work out a little bit of algebra to discover that, when both K_x and K_y become very very small (we call it the ''weak-coupling limit''), then $K_x' \simeq K_x^2$ and $K_y' \simeq K_y^2$, so that the ratio K_x'/K_y' either blows up or flows down to zero, depending on the initial choice of values. This is shown schematically in Fig. 2 where we have divided the K_x-K_y parameter space into two zones by the ''isotropy line '' $K_x = K_y$. Now if we start with a point above the ''isotropy line,'' for several initial stages of iteration the parameters approach the isotropy line, but always from above. As we keep on renormalizing, at one stage both K_x and K_y become very small, i.e., they enter the ''weak-coupling regime.'' But, as the ratio K_x/K_y still remains less than unity and as $K_x'/K_y' \simeq (K_x/K_y)^2$ in this limit, the ratio keeps on decreasing and ultimately flows to the value zero. This is why in Fig. 2 we find an initial bending of flow towards the isotropy line and then its recession from the same to the FP (0,0). A similar reasoning shows

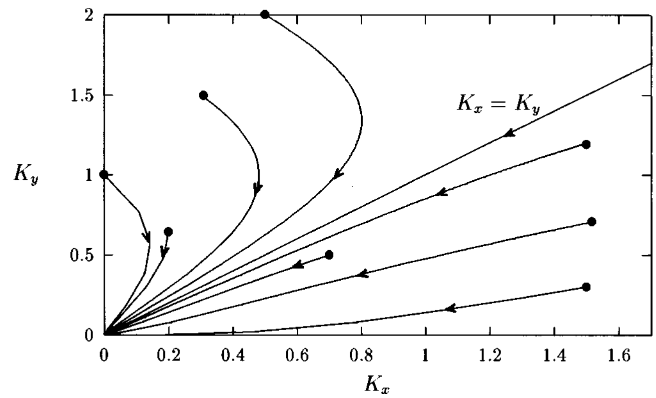

FIG. 2. Flow diagram in the K_x-K_y space. Initial points are marked by solid circles.

that for any starting point in the parameter space that is below the isotropy line, the RG flow initially approaches the isotropy line from below and finally reaches the (0,0) FP.

In both the situations above, the bending of the flow line indicates that the parameters have entered the weak-coupling regime. However, it should be mentioned that, if we start with $K_y = 0$, the RG flow is unidirectional to the (0,0) FP along the K_x axis. This essentially shows that starting with a set of decoupled chains one should not restore the coupling in the transverse direction at any iteration. If one starts very close to the K_x axis (the gasket now behaves as a set of chains very weakly coupled to each other in the transverse direction), the RG flow bends towards the K_x axis at the very outset and then continues to flow towards the (0,0) FP.

In Fig. 3 we plot the isotropy parameter $R = K_x/K_y = J_x/J_y$ against the number of iterations n. It is seen that for all initial values of R, it ultimately flows away from unity, indicating an absence of restoration. The bending of the flow lines in Fig. 2 are reflected in Fig. 3 also. It may be noted that in Fig. 3 we have taken $J_y = 1$ for all starting values of R. However, one can choose J_y arbitrarily (for the same values of R as in Fig. 3) to obtain a qualitatively similar set of diagrams which always confirm that there will be no restoration of isotropy for the above spin model.

At this point, it is essential that we discuss the cause that makes our result totally different from that in Ref. 3. In their work, Brody and Ritz[3] have rescaled both the ''horizontal'' and the ''angular'' coupling constants (K_x and K_y in our

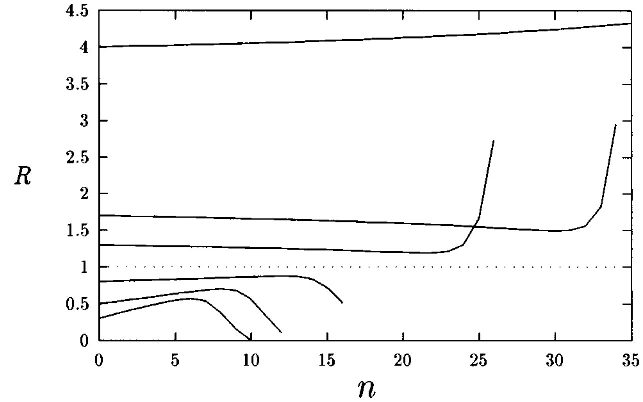

FIG. 3. Variation of the ratio J_x/J_y under iteration for different starting values. The dotted line refers to $J_x = J_y$.

case) by an arbitrary factor at each step of renormalization. The scale factor has been chosen to be $1/K_x(n)$ at the nth stage of iteration. They have rescaled by this factor in order to keep the value of the other coupling constant equal to unity at all stages of renormalization. As a result, the coupling constants are *never allowed* to enter the ''weak-coupling regime'' and hence there is an apparent restoration of isotropy.

To end this section we note that a similar decimation study can easily be done with the antiferromagnetic case as well.

Tightly bound electron with anisotropic hopping amplitudes on a Sierpinski gasket:

We start by describing a single electron on an infinite SG lattice by the usual tight-binding Hamiltonian in the Wannier representation,

$$H=\sum_i \epsilon_i|i\rangle\langle i|+\sum_{\langle ij\rangle} t_{ij}|i\rangle\langle j|, \tag{7}$$

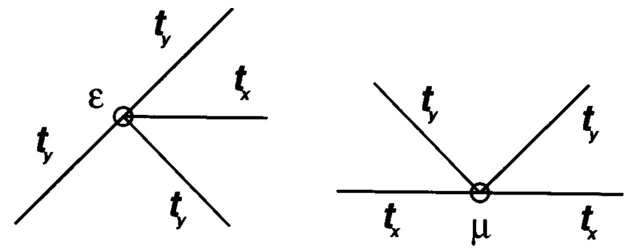

FIG. 4. The two distinct types of vertices with site energies ϵ and μ in the electronic case.

where summation extends over NN pairs $\langle ij\rangle$. Hopping integral $t_{ij}=t_x$ for i and j lying on a horizontal bond and $t_{ij}=t_y$ otherwise. ϵ_n is the on-site potential at the nth atomic site. In particular, $\epsilon_n=\epsilon$ for the site n joining one t_x bond with three t_y bonds while, $\epsilon_n=\mu$ for the site n joining two t_x bonds with two t_y bonds (Fig. 4).

Here also we use the RSRG scheme to rescale the lattice and this yields the following recursion relations:

$$X'=\frac{(Z^2-1)^2-4Z^2X-X^2-2Z^2X^2-3Z^2XY-2Z^2Y-XY+X^3Y}{(X+Z^2)^2+(2+Y)Z^2-1}, \tag{8}$$

$$Y'=\frac{X^2Y^2-2X^2-4Z^2XY-Y^2-2Z^2Y-4Z^2X-4Z^2+2Z^4+2}{(X+Z^2)^2+(2+Y)Z^2-1}, \tag{9}$$

$$Z'=\frac{Z^2(Y+2)(X+1)}{(X+Z^2)^2+(2+Y)Z^2-1}, \tag{10}$$

where we have defined three dimensionless parameters X, Y and Z as follows:

$$X=\frac{(E-\epsilon)}{t_x}, \tag{11}$$

$$Y=\frac{(E-\mu)}{t_x}, \tag{12}$$

$$Z=\frac{t_y}{t_x}. \tag{13}$$

Here, E is the energy of the electron and the parameter Z really measures the degree of anisotropy in hopping. The primed quantities in the recursion relations (8)–(10) refer to the renormalized ones. We want to find out whether it is possible to restore the isotropic limit $Z=1$ at a macroscopic scale under RSRG flow by starting from a microscopic anisotropy, i.e., $Z\neq 1$; and even if it is possible, what are the restrictions imposed on the flow.

To this end we analyze the recursion relations (8)–(10) to help us find the FP's. These are (0,0,1), (4,4,1), (∞,∞,1), (-2,0,-1) out of which the nontrivial stable FP's are (0,0,1) and (4,4,1) while (-2,0,-1) is an unstable FP.

Of the two stable FP's the first one at (0,0,1) is stable in the sense that if we start out with small deviations from zero in the values of X and Y, while maintaining $Z=1$, then the system flows back to (0,0,1). This is, however, not a case of restoration of isotropy as $Z=t_y/t_x$ is always one. Linearizing Eqs. (8)–(10) around the stable FP (4,4,1), we get an eigenvalue 4/5 which corresponds to an eigenvector $(\delta X,\delta Y,\delta Z)=(3,2,1)$. This implies that (4,4,1) is a stable FP only if we start with initial values of the quantities X, Y, and Z that lie on the eigenvector (3,2,1) terminating at (4,4,1) in the parameter space. For any such starting set of values of X, Y, and Z, the recursion relations flow down to (4,4,1) even if we start with $Z\neq 1$. That is, starting with microscopic anisotropy we can restore isotropy in the $n\to\infty$ limit (n being the number of iterations) provided we are on the line given by $\delta X:\delta Y:\delta Z=3:2:1$. The restoration is, therefore, highly *restrictive*. We have been able to find two distinct scaling regimes for the isotropy parameter Z on the line $\delta X:\delta Y:\delta Z=3:2:1$ around the FP (4,4,1). These are discussed below:

(i) $Z\gtrsim 1$ and $Z\lesssim 1$: If we are close to the isotropic zone either from above or below the value of unity, we can find the rate of restoration of isotropy by noting that

$$Z^{(n+1)}-1\simeq\frac{4}{5}(Z^{(n)}-1)$$

in the limit $n \to \infty$. The superscript refers to the RG iteration.

(ii) $Z \gg 1$: In this highly anisotropic region, one can analyze the recursion relation (10) for Z', keeping in mind that X and Y will always be related to Z in the manner, $X = 3Z + 1$ and $Y = 2Z + 2$, so that (X,Y,Z) is a point on the abovementioned eigenvector. The analysis then yields, for $Z \gg 1$,

$$Z^{(n+1)} = 6 - \frac{32}{Z^{(n)}} + \mathcal{O}\left(\frac{1}{Z^{(n)2}}\right)$$

in the limit $n \to \infty$.

In conclusion, we note that a restoration to macroscopic isotropy, starting from microscopic anisotropy, though almost trivially possible in the case of resistance networks on a fractal system, it is not so for other models on the same fractal lattice. For simple Ising spins on a Sierpinski gasket such a restoration has been shown to be impossible in contrast to a recent demand, whereas, for a tight-binding model for noninteracting electrons isotropy in the large scale is restored only for a very specific model.

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*Electronic address: bibhas@hp1.saha.ernet.in
†Electronic address: rkm@cmp.saha.ernet.in and arunava@klyuniv.ernet.in

[1] M. T. Barlow, K. Hattori, T. Hattori, and H. Watanabe, Phys. Rev. Lett. **75**, 3042 (1995).
[2] Z. Lin and M. Goda, J. Phys. A **29**, L217 (1996).
[3] D. Brody and A. Ritz, Phys. Lett. A **233**, 430 (1997).
[4] Y. Gefen, B. B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. **45**, 855 (1980).