Lowest-Landau-level theory of the quantum Hall effect: The Fermi-liquid-like state of bosons at filling factor one

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A theory for a Fermi-liquid-like state in a system of charged bosons at filling factor 1 is developed, working in the lowest Landau level. The approach is based on a representation of the problem as fermions with a system of constraints, introduced by Pasquier and Haldane (unpublished). This makes the system a gauge theory with gauge algebra W_∞. The low-energy theory is analyzed based on a Hartree-Fock approximation and a corresponding conserving approximation. This is shown to be equivalent to introducing a gauge field, which at long wavelengths gives an infinite-coupling U(1) gauge theory, *without* a Chern-Simons term. The system is compressible, and the Fermi-liquid properties are similar, but not identical, to those in the previous U(1) Chern-Simons fermion theory. The fermions in the theory are effectively neutral, but carry a dipole moment. The density-density response, longitudinal conductivity, and current density are considered explicitly.
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I. INTRODUCTION

The so-called composite-particle view of the liquid states of electrons (or other charged particles) in two dimensions in a high magnetic field[1] has been developed gradually over more than a decade.[2–12] Girvin[2] proposed to develop a Ginzburg-Landau theory of the fractional quantum Hall effect, with an action for a complex scalar (boson) field and containing a Chern-Simons (CS) term to enforce the condition that the quantized vortices carry a fractional charge. Girvin and MacDonald[3] introduced a singular gauge transformation, and exhibited algebraic long-range order in a bosonic field. This transformation, which attaches δ-function flux tubes to particles (via a CS term in the action of the field theory) and so in general changes the statistics of the particles as in the theory of anyons,[13] was then used in several theories, in conjunction with the mean-field approximation of replacing the gauge field strength by its expectation value, to obtain a system in a different magnetic field. Thus anyon superconductivity was discovered by mapping anyons in zero magnetic field to fermions filling Landau levels in a magnetic field;[4] the Laughlin states[14] were described by mapping fermions to bosons in zero net magnetic field and then Bose condensing them;[5] the Laughlin and hierarchy[15,16] states were reinterpeted by mapping fermions to fermions in a reduced magnetic field and then filling Landau levels;[6,7] the hierarchy states and the anyon superconductors in zero magnetic field were redescribed by hierarchical extension of the mapping to bosons, using duality methods.[8] At the same time, a lowest-Landau-level (LLL) treatment of the Ginzburg-Landau idea was developed,[9,10] without using δ-function flux tubes, by attaching vortices to electrons to convert them to bosons; in this case, the bosons condense and have true long-range order.

It has to be admitted that these ways of viewing the fractional quantum Hall effect produced little in the way of distinctive experimental predictions or explanations that were not already known by other methods, though interesting speculations concerning the phase transitions between the quantized Hall plateaus[17] may be an exception. The situation changed, however, following the discovery of an anomaly in the surface acoustic wave propagation at filling factor $\nu=\frac{1}{2}$ (and less strongly at other filling factors, such as $\frac{1}{4}$ and $\frac{3}{2}$).[18] This result speeded the development of a theory[11] (to be referred to as HLR) for a case not included in the above list, in which fermions (electrons) are mapped to fermions at zero magnetic field and form a Fermi sea. In the simplest cases, this occurs for filling factor $\nu=\frac{1}{2},\ \frac{1}{4},\ \frac{1}{6},\ \ldots$. The Fermi sea was predicted to be a compressible state that does not produce a Hall plateau, and the experimental result of a longitudinal conductivity increasing linearly with wave vector[19] was explained[11]. The Fermi surface, at which the fermions exist as genuine low-energy excitations, was observed through geometric resonance effects at ν close to $\frac{1}{2}$ in further surface acoustic wave experiments[20] (as predicted explicitly in Ref. 11), and in other experiments.[21,22] (We should point out that for other filling factors in the fermion description, the fermions are dressed to become the fractionally charged, fractional-statistics quasiparticles,[14–16,23] and so are not observed as fermions.)

In this paper, we return to the basic theory of the Fermi-liquid-like state. Recent work[24–26] has raised the possibility of changes in the way we think about the theory of the low-energy excitations near the Fermi surface. In particular, these authors find constraints not mentioned in any earlier papers known to the present author. At the same time, we may be motivated by trying to avoid the seemingly artificial CS approach, which begins with a singular gauge transformation. Ultimately, it would aid our understanding to have more intuition about what drives the formation of the Fermi-liquid-like (and other) states. There are no flux tubes attached to the particles in reality; the background magnetic field remains essentially uniform in these states of matter. The approach begun in Ref. 10 was intended to head in this direction. It uses LLL states only, so is valid in the (not entirely realistic) limit of interactions weak compared with ω_c, and binds *vor-*

tices to the electrons to lower the energy, thus forming the composite particles. Several implications of this approach were pointed out in Ref. 12 for the Fermi sea and the Bose condensate.

The approach taken in the present paper avoids the CS approach. While it is perhaps not as simple minded as one would want, it does make close contact with the work just cited.[12] Here we start from the approach of Pasquier and Haldane,[26,27] that gives an exact representation of the LLL problem in the case of charged bosons in a magnetic field at $\nu=1$, where a Fermi-liquid (FL) state is possible. Although our paper is long and fairly detailed, we can give a succinct summary of our results. *The low-energy, long-wavelength theory is a FL coupled to a gauge field* (not to be confused with the physical electromagnetic field). In contrast to the scenario arising[11] in the CS (singular gauge transformation) approach, *there is no CS term in this low-energy theory*. Consequently, the gauge field is said to be ''strongly coupled,'' and one of its effects is to enforce constraints that agree with those of Refs. 24–26. This in turn has the effect of making the fermions uncharged, but they are left with a subleading coupling to electromagnetic fields through a dipole moment. The interplay of this moment with the transverse part of the gauge field leads to a finite compressibility, in spite of the neutrality of the particles. It also leads to CS equations that relate the curl of the vector potential to the density, and a similar equation for the current, still valid in spite of the absence of a CS term in the action, in agreement with Ref. 12. In general, the good agreement with experimentally-observed phenomena achieved in the theory of HLR is not spoiled in the present theory. Nonetheless, the detailed structure of this FL-like theory is modified. While the theory is developed here for $\nu=1$ bosons, there are many indications that the results are more general. These include the derivation in Ref. 24 for general number of attached flux.

Section II contains a more detailed review of previous work, and a more detailed overview of the paper. In Sec. III, we explain the formalism due to Pasquier and Haldane that will be used in this paper. In Sec. IV, we perform explicit calculations of response functions, including those for the constraint operators, and interpret the results in terms of a strongly coupled gauge field. In Sec. V we outline the extension of the results to all orders, and provide some general discussion. Section VI is the conclusion. Appendix A discusses some details of the formalism, including the noncommutative Fourier transform, and Appendix B indicates how a Hubbard-Stratonovich transformation can be used.

II. REVIEW AND OVERVIEW

In this section, we review some of the background necessary for the discussion in this paper. We begin with the U(1) CS fermion approach developed in Ref. 11. The Fermi liquid-like state proposed in that paper is the main topic of the present work; however, we will not review the relation to experiments. In Sec. II B we review ''physical'' pictures which are based on consideration of the wave functions of the system, as opposed to field-theoretic methods. In Sec. II C, we review recent work which attempts to push the U(1) CS approach down to a low-energy effective theory in the LLL. Finally, in Sec. II D, we give a brief overview of the main results and of the layout of the remainder of the paper.

A. U(1) Chern-Simons fermion theory

In this approach the particles are represented as fermions with a δ function of flux attached, whose strength is an integral number $\tilde{\phi}$ of flux quanta Φ_0. Then the underlying particles must be bosons when $\tilde{\phi}$ is an odd integer, and fermions when $\tilde{\phi}$ is even (for noninteger $\tilde{\phi}$, the underlying particles must be anyons). We will reserve the term ''particles'' for these original particles, and refer to the transformed particles as ''fermions'' or ''quasiparticles.'' The imaginary time action (see, e.g., Ref. 11, to be referred to as HLR) is (in the gauge where $\nabla\cdot\mathbf{a}=0$)

$$S=\int d\tau\, d^2r\left[\psi^\dagger\left(\frac{\partial}{\partial\tau}-ia_0-\mu\right)\psi+\frac{1}{2m}|(-i\nabla-\mathbf{a}-\mathbf{A})\psi|^2 -\frac{i}{2\pi\tilde{\phi}}a_0\nabla\wedge\mathbf{a}\right]+\frac{1}{2}\int d\tau\, d^2r\, d^2r'\, V(\mathbf{r}-\mathbf{r}') \times\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}). \tag{2.1}$$

Here ψ is the field operator for the fermions, rather than that for the underlying particles, which could be fermions (electrons) or bosons. We will use the notation (note the use of the summation convention for repeated Greek indices)

$$\mathbf{a}\wedge\mathbf{b}=\varepsilon_{\mu\nu}a_\mu b_\nu \tag{2.2}$$

for a cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ in two dimensions, μ, $\nu,\ldots=x,y$ to label the two components, and $\varepsilon_{\mu\nu}=-\varepsilon_{\nu\mu}$, $\varepsilon_{xy}=1$ for the two-dimensional alternating tensor. We have set $\hbar=1$ and, starting with Gaussian units, we have absorbed $-e$ into the scalar potential and electric field, and $(-e/c)$ into the vector potential and magnetic field, so the charge of the particles is one and the flux quantum is 2π. The uniform background magnetic field is $\nabla\wedge\mathbf{A}=B>0$, which corresponds to the negative $\hat{z}$ direction (in the three-dimensional sense) in conventional units. We choose the unit of length so that the magnetic length $\ell_B^{-2}=B=1$. It will also be convenient to write $\wedge\mathbf{a}$ for the vector whose components are $(\wedge\mathbf{a})_\mu=\varepsilon_{\mu\nu}a_\nu$; then $\mathbf{a}\cdot\wedge\mathbf{b}=\mathbf{a}\wedge\mathbf{b}$.

Varying a_0 in the action leads to

$$\nabla\wedge\mathbf{a}=-2\pi\tilde{\phi}\rho, \tag{2.3}$$

where $\rho(\mathbf{r})=\psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ is the number density both of the Chern-Simons fermions and of the underlying particles. When the filling factor $\nu=2\pi\bar{\rho}/B$ is $1/\tilde{\phi}$ (where $\bar{\rho}$ is the average density), there is no net field for the fermions, and, within a mean-field approximation, a Fermi sea ground state is possible.

The leading approximation for the linear-response functions is the random phase approximation (RPA), in both the gauge field a_0, $\mathbf{a}$, and the Coulomb (or other) interaction $V(\mathbf{r})$. In Fourier space the full density-density response function is then,[11] before any approximation,

$$\chi_{\rho\rho}=\frac{\chi_{\rho\rho}^{\rm irr}}{1+V(\mathbf{q})\chi_{\rho\rho}^{\rm irr}}, \tag{2.4}$$

and in the RPA $\chi^{\rm irr}_{\rho\rho}=\chi^{\rm irr}_0$, where

$$\chi_0^{\rm irr}=\frac{\chi_0}{1-(2\pi\tilde{\phi})^2\chi_0\chi_0^{\perp}/q^2}. \tag{2.5}$$

Here $\chi^{\rm irr}$ is the response function which is irreducible with respect to the interaction V *only* (i.e., diagrammatically, it does not become disconnected when a single interaction line is cut), while χ_0 is the density-density response for the noninteracting sea of fermions of mass m (the bare or band mass), and $\chi_0^{\perp}$ is the transverse current-current response, of the same Fermi sea, including the constant ''diamagnetic current'' term. In the limit where first the frequency ω and then the wave vector q tend to zero, we have

$$\chi_0=m/2\pi, \tag{2.6}$$

$$\chi_0^{\perp}\sim -q^2/12\pi m, \tag{2.7}$$

and hence

$$\frac{\partial n}{\partial\mu}\equiv\lim_{q\to 0}\chi^{\rm irr}_{\rho\rho}(q,0)=\frac{m/2\pi}{1+\tilde{\phi}^2/6}. \tag{2.8}$$

(For a long-range potential, i.e. one that is divergent as $q\to 0$, this is the appropriate definition of the compressibility $\partial n/\partial\mu$. For a short-range interaction, one would use $\chi_{\rho\rho}$ in place of $\chi^{\rm irr}_{\rho\rho}$.) Thus the theory predicts that the system is *compressible*. Note, however, that the approach describes the properties that the system has if it is in the phase described. For a highly correlated system such as particles in the lowest Landau level, it is difficult to find any approach that can accurately predict, for a given Hamiltonian, in which phase the system will be. For example, an alternative phase that is possible at the same filling factors as the Fermi liquid is the Pfaffian state,[28] which is believed to be incompressible.[29] Nevertheless, the question of the properties of the Fermi-liquid state—which has a Fermi surface in the excitation spectrum for the fermions—is well defined.

For the conductivity, the general statement[30] is that the resistivity tensors add,

$$\rho=\rho_{\rm CS}+\rho_{\psi}, \tag{2.9}$$

where $\rho_{{\rm CS}\mu\nu}=2\pi\tilde{\phi}\varepsilon_{\mu\nu}$, coincides with the Hall resistivity at $\nu=1/\tilde{\phi}$, and $\rho_{\psi\mu\nu}$ is the resistivity tensor of the fermions, the inverse of the conductivity tensor which is related to the current-current response function that is irreducible with respect to both the interaction and the gauge field. In the RPA, using the Drude approximation to include impurities, one has, taking $q\to 0$, then $\omega\to 0$, $\rho_{\psi\mu\nu}=\delta_{\mu\nu}/\sigma_{\psi xx}$, where $\sigma_{\psi xx}$ is the usual Drude result for the Fermi sea in zero magnetic field with impurity scattering. There is also an unusual scattering mechanism[11,31] in which the fermions scatter off the static vector potential $\delta\mathbf{a}$ induced in the Chern-Simons gauge field by a variation in the density of particles produced by the impurity potential, since $\nabla\wedge\delta\mathbf{a}=-2\pi\tilde{\phi}\delta\rho$.

The effects of interactions and gauge field fluctuations beyond the RPA would be expected to have a variety of effects. By analogy with the Landau-Silin treatment of fermions with a long-range interaction, one would expect that when both the long-range parts of the interaction (if any) and of the Chern-Simons gauge field are extracted, by considering responses irreducible with respect to both the interaction and the gauge field as above, the remaining effects can be handled to all orders by renormalizing parameters, and the leading long-wavelength effects expressed in terms of Landau interaction parameters F_{ℓ} and an effective mass m^*. Since the system is translationally and Galilean invariant (in the absence of impurities), the latter mass must satisfy the usual relation[11,32]

$$m^*/m=1+F_1 \tag{2.10}$$

(details of our two-dimensional normalization of the Landau parameters such as F_1 are given later). In addition, in the limit where the cyclotron energy $\omega_c=1/m$ is large compared with the typical interaction strength between particles, $V(\overline{\rho}^{-1/2})$, (e.g., as $m\to 0$), the dynamics should be governed entirely by the interactions, and so $1/m^*$ should scale with the interaction strength, and be of order the typical interaction strength up to numerical factors.

This expectation that the theory would be a renormalized Fermi liquid, coupled to the long-range interaction and the gauge field, turned out to be too naive, however. The fluctuations of the gauge field have singular effects that appear to cause a partial breakdown of the Fermi-liquid picture.[11] The effects of such fluctuations were evaluated in leading order in the RPA gauge field propagator in HLR (the small parameter being $\tilde{\phi}$, with the background magnetic field being adjusted such that the net field seen by the fermions on average was zero for any value of $\tilde{\phi}$, i.e., the filling factor was always $1/\tilde{\phi}$; recall that for generic values of $\tilde{\phi}$ the particles are anyons). The main effects were, first, that the propagator itself shows the appearance of a mode at the cyclotron frequency $1/m$, which carries all of the f-sum rule spectral weight to order q^2. Thus this mode is the physical cyclotron mode. The virtual excitation of this mode, which is the longitudinal part of the gauge field, led, in first order, to a contribution to the fermion self-energy that was logarithmically infrared divergent. The effect could plausibly be exponentiated to give, for the quasiparticle residue Z_F of a fermion at the Fermi wave vector k_F,

$$Z_F\sim L^{-\tilde{\phi}/2}, \tag{2.11}$$

where L is the system size (or, presumably, $|k-k_F|^{\tilde{\phi}/2}$ as k approaches k_F for infinite L). This would correspond to the Girvin-MacDonald (GM) power law,[3] generalized to the fermion case; in particular, the exponent should be exact. This is supported by further analysis of these fluctuations which, similarly to the boson case[33], lead to a factor $\Pi_{i<j}|z_i-z_j|^{\tilde{\phi}}$, times a Gaussian, in the ground-state wave function of the fermions (the result for the fermion case is widely known, but does not appear to have been explicitly published). This in turn leads to the GM power $r^{-\tilde{\phi}/2}$ as a factor in the equal-time Green's function of the fermion,[34]

$$\langle\psi(\mathbf{r})\psi^{\dagger}(\mathbf{0})\rangle\sim r^{-(3/2+\tilde{\phi}/2)}\sin(k_F r-\pi/4), \tag{2.12}$$

and correspondingly to the above result for Z_F (see also Ref. 35). (The GM power law in the composite boson case has also been recovered field theoretically in Ref. 36.) Related effects were also found in the work of Shankar and Murthy,[24] to which we shall turn shortly. In the work of HLR and others, it was assumed that the vanishing quasiparticle residue for the original CS fermions was of little significance, since as with many similar effects in field theory, in particular the nonsingular quasiparticle residue in an ordinary Fermi liquid, it is canceled in physical response functions that measure quasiparticle properties. However, the recent results to be reviewed below, and those of the present paper, suggest that things are not quite so simple, and rather than just ignoring these effects on the assumption that they cancel, the longitudinal mode should be integrated out ''exactly'' to obtain an effective field theory, before proceeding to the effects of the other lower-energy fluctuations, such as the transverse fluctuations.

The fluctuations in the transverse part of the gauge field have received more attention (due to the CS term, there are also cross-terms that mix the longitudinal and transverse fluctuations; however, these are assumed to have some intermediate significance). The first-order self-energy contains power-law infrared-divergent terms for the case of a short-range interaction, which are weakened by the presence of a long-range interaction because the latter suppresses density fluctuations which correspond to fluctuations of the transverse CS vector potential **a**. For the $1/r$ Coulomb interaction, the effects become logarithmic, and for an interaction which is longer range than $1/r$ they become finite. In the Coulomb case, the structure of the effects is similar to those in an electron gas coupled to the transverse part of the ordinary electromagnetic field (since there is no CS term in this case, these effects are not weakened by the Coulomb interaction, but are always logarithmic—however, they are extremely weak in practice).[37,38] In both of these systems, it can be argued by treating the self-energy self-consistently[37,11,39] that the effects lead to an effective mass diverging as $m^*\sim-\ln|k-k_F|$, a quasiparticle scattering rate $\sim-|\varepsilon_k^*-\mu|/\ln|\varepsilon_k^*-\mu|$ (where ε_k^* is the dispersion relation that corresponds to the stated behavior of the effective mass near k_F), and a quasiparticle residue $Z_F\sim-1/\ln|k-k_F|$ (the latter would be in addition to the effect of the longitudinal fluctuations described above). These results suggest that while the effective mass diverges at k_F, the quasiparticles remain just marginally well defined due to the reciprocal logarithm in the decay rate, and thus the system is a ''marginal Fermi liquid.'' For longer-range interaction, there is no such breakdown of Landau Fermi-liquid theory, and for the extreme case of $V(r)\sim\ln r$, the scattering rate recovers its usual form $\sim(\varepsilon_k-\mu)^2$ (all these results are for zero temperature).

There are many other studies of this,[40–46] often with conflicting results. We believe that the correct results are those that agree with the above scenario of HLR for the behavior of the effective mass, etc.

If we are not too concerned about the latter effects of transverse gauge field fluctuations, for example if we consider an interaction longer range than the Coulomb interaction, or in the Coulomb case neglecting the logarithmic effects in view of how slowly they diverge at k_F, then we are led to a physical picture of what to expect from the system to all orders in the fluctuations. It is essentially the Landau theory with due regard to the long-range effects, as described above, and thus retains the CS structure present in the RPA. For the density-density response, the responses χ_0 and $\chi_0^\perp$ that appeared in the RPA will therefore be replaced by renormalized versions, and, according to this scenario, we then expect that, in the limit that gives (for example) the compressibility, χ_0 and $\chi_0^\perp$ that appeared in the RPA will be replaced by renormalized versions $m^*/[2\pi(1+F_0)]$ and $q^2\chi_d^*$, respectively, where χ_d^* is a renormalized long-wavelength Landau diamagnetic susceptibility, which is a non-Fermi-liquid quantity as it involves derivatives at the Fermi surface. Explicitly, we expect

$$\frac{\partial n}{\partial\mu}=\frac{m^*}{2\pi(1+F_0)-(2\pi\tilde{\phi})^2\chi_d^*m^*}. \tag{2.13}$$

(We expect that F_0 diverges the same way as m^*, so that the renormalized version of χ_0 remains finite.[44,45]) Thus the system remains compressible in this scenario.

B. Physical pictures

In this subsection, we review aspects discussed in Ref. 12, which was in part an elucidation of Ref. 10 (see also Ref. 47). The approach is based on the wave functions of the particles, which are assumed from the beginning to be in the lowest Landau level. To lower the repulsive interaction energy, each particle would like to bind to $\tilde{\phi}$ vortices, which at $\nu=1/\tilde{\phi}$ leaves no vortices left. (Note that in the LLL, the number of zeros in the wave function of each particle is equal to the number of flux quanta threading the relevant area, and that a vortex means a simultaneous zero in the wave function of every particle other than the one under consideration.) For the same choices of statistics of the particles and filling factor as before, the bound states behave as fermions in a zero net magnetic field (this statement again involves the mean-field assumption that the average density of particles is uniform, as we will see).

To make the idea concrete, we may consider trial wave functions in which the fermionic bound states occupy a Slater determinant of plane waves, or spherical harmonics on the sphere[48] (these resemble Jain's trial wave functions,[6] except that the fermions are in zero effective magnetic field)

$$\tilde{\Psi}(z_1,\ldots,z_N)=\mathcal{P}_{\rm LLL}\det M_{ij}\prod_{i<j}(z_i-z_j)^{\tilde{\phi}}. \tag{2.14}$$

Here we write the wave function on the sphere[15] with $z_i=2Rv_i/u_i$, the complex coordinate of particle i in stereographic projection to the plane. Only the polynomial part of the wave function is shown, as indicated by the tilde on Ψ. The full wave function is recovered by multiplication by $\Pi_i(1+|z_i|^2/2R^2)^{-(N_\phi+2)/2}$, and this must be done before integration of the z_i coordinates over the complex plane to give the correct integration measure, in particular when applying the LLL projection operator $\mathcal{P}_{\rm LLL}$. In the limit where the radius R and the number N_ϕ of flux quanta through the surface of the sphere go to infinity with the field strength fixed, the nonpolynomial factor approaches the usual $e^{-(1/4)\Sigma_i|z_i|^2}$. M_{ij} are the spherical harmonics of angular momentum L_i, M_i for the jth particle, or can be replaced by plane waves $e^{i\mathbf{k}_i\cdot\mathbf{r}_j}$ in the plane. L_i, M_i (or $\mathbf{k}_i$) can be chosen to fill the Fermi sea to obtain a trial ground state.[48] Different choices of the sets of L_i, M_i do *not* give orthogonal states in general, except when the total angular momenta differ. Note that apart from the projection to the LLL, the wave functions have the form that would be expected from the CS approach, on including the fluctuations at the RPA level that produce the amplitude of the Laughlin-Jastrow (LJ) factor in the wave function, as noted above.

The fermionic bound states or ''quasiparticles'' described here are created by operators of the form $\psi_e^\dagger U^{\tilde{\phi}}$, where $\psi_e^\dagger$ creates a particle in the LLL, and $U(z)=\Pi_i(z_i-z)$ is Laughlin's quasihole operator,[14] which creates a vortex.[10] As for the wave functions; this differs from the CS fermion operator $\psi^\dagger$ by including the amplitude of the quasihole operator, and not just the phase (like the wave functions, it should also include a nonpolynomial factor in z, which we have suppressed here). Consequently, like the corresponding boson operator,[10] its equal-time Green's function is not expected to include the GM power-law factor $r^{-\tilde{\phi}/2}$; this has been confirmed by calculation.[34] Since at $\nu=1/\tilde{\phi}$ the $\tilde{\phi}$ vortices induce a hole in the density of the other particles that contains a deficiency in the particle number of exactly unity, there has always been a temptation to say that the bound states formed this way are neutral objects. This should be contrasted with the CS fermions and bosons, which carry particle number unity.

The plane-wave factors, in the flat space limit, can be rewritten using[49,50] (see also Appendix A)

$$\mathcal{P}_{\mathrm{LLL}} e^{i\mathbf{k}\cdot\mathbf{r}_i}\mathcal{P}_{\mathrm{LLL}}=e^{i\mathbf{k}\cdot\mathbf{R}_i}e^{-(1/4)|k|^2}, \tag{2.15}$$

where $\mathbf{R}_i$ is the guiding-center coordinate of particle i, which has no matrix elements between states in different Landau levels. The operator $\mathbf{K}_i=-\wedge\mathbf{R}_i$ is the pseudomomentum that generates magnetic translations of particle i. Thus the plane-wave factors in the Slater determinant can be replaced by $e^{i\mathbf{k}\cdot\mathbf{R}_i}$ and each such factor displaces the ith particle by $\wedge\mathbf{k}$ (in units where the magnetic length is 1) from its vortices. This picture of particles bound to vortices but displaced by $\wedge\mathbf{k}$ from their center has several consequences.[12]

The first consequence is that, if we consider the interaction of the particle with the vortices (or correlation hole) to which it is bound (neglecting the exchange effects due to the latter being constructed from other particles, indistinguishable from the first), then, for $\mathbf{k}=\mathbf{0}$, the particle is precisely on the vortices as in the Laughlin states, and for $\mathbf{k}\neq\mathbf{0}$ it is displaced by $\wedge\mathbf{k}$. Consequently, the energy should increase, and the interaction between the particle and its vortices becomes an effective kinetic (i.e., $\mathbf{k}$-dependent) energy for the fermion, which is the origin of the effective mass at the Fermi wave vector, and scales inversely with V. A formula for this energy can be found for the analogous boson case in Ref. 10. Notice that the displacements in the Fermi sea ground state are bounded above by $k_F=\sqrt{2/\tilde{\phi}}$, which is much less than the typical distance between neighboring particles which is of order $\sim\sqrt{\tilde{\phi}}$. Thus for $\tilde{\phi}>$ order 1, which is the case of interest when the particles are bosons or fermions, not anyons, the displacements do not unduly perturb the bound states.

Second, if we accept that the fermions are neutral, then their leading coupling to the electric potential is through a dipole moment $\wedge\mathbf{k}$. It is important to realize that the wave vectors of the fermions contribute to the total momentum of the system, which is a conserved quantity. One might imagine that the dipole moment could be renormalized by effects not yet included, or that the vortices might not all be at the same point as we have implicitly assumed. Indeed, when the underlying particles are fermions, the wave function must have one vortex exactly on every particle, because of antisymmetry. This will not affect the dipole moment, because the plane-wave factors must produce the displacement shown, and, when the particles are fermions, this is accomplished by displacing the other vortices further to compensate for the one that is not displaced at all. Also, if the vortices are viewed as point objects, then their relative displacements can only produce multipole moments of even order, and not a contribution to the dipole moment, which is determined by the displacement of the particle from the center of mass of the vortices. Thus the dipole moment is not renormalized. A more rigorous version of this argument will be given later in this paper.

Third, when the $\tilde{\phi}$ vortices are dragged around adiabatically, they pick up a Berry phase factor[23] which can be interpreted as a vector and scalar potential governed by the particle number and number drift-current densities ρ and $\mathbf{j}$.[10,12,47] This means that the fermionic bound states, in addition to the electromagnetic $\mathbf{A}$ and A_0, experience $\mathbf{a}$ and a_0 given by

$$\nabla\wedge\mathbf{a}=-2\pi\tilde{\phi}\rho, \tag{2.16}$$

$$-\dot{\mathbf{a}}-\nabla a_0=2\pi\tilde{\phi}\wedge\mathbf{j}. \tag{2.17}$$

These have the form of the equations in the CS fermion approach, but it is important to emphasize that they have been obtained[10,12,47] without the use of δ-function fluxes attached to the particles, and that they still involve the physical density and current, which cannot be identified with the density and current of the fermions because the latter are (or may be) neutral.

In Ref. 12, these were used as an alternative approach that was stated to be equivalent to the CS approach, and the neutrality of the quasiparticles was not invoked. It was felt that, although the fermions and bosons appear neutral, the situation might be like that in the usual electron gas problem with a Coulomb interaction, where at low energies the quasiparticles are neutral in their couplings to external longitudinal electric fields because of screening; however, in the Fermi-liquid viewpoint, one nonetheless views the fermions as having charge unity, and the low-energy behavior of the Fermi liquid itself produces the screening effects in the limit $\omega/q\to 0$ in the response functions. In the opposite limit $\omega/q\to\infty$, the charge of the quasiparticles does show up in the conductivity (and also in the transverse response in both regimes). However, recent work to be discussed in Sec. II C, and the work in the present paper, suggests that in the quantum Hall effect context, we can in fact obtain a consistent picture in which the quasiparticles have only dipolar couplings to external fields. The obvious question is then whether the Fermi liquid is still compressible. We will answer this question in the affirmative.

C. Recent approaches to the LLL

Several recent works have taken up the outstanding issues discussed in the previous subsections. They are concerned with obtaining results for the Fermi-liquid state, including the effects of all the particles being in the lowest Landau level, or, as would seem to be at least roughly equivalent, including the effects of the amplitude of the correlation factors produced by the zero-point fluctuations of the cyclotron-frequency longitudinal modes of the CS gauge field. The aim of such work is, of course, to test the validity of the results of HLR. Different approaches have been used. Shankar and Murthy (SM) (Ref. 24) based their work on the U(1) CS fermion field theory approach; however, they worked in a Hamiltonian formalism, and aimed to eliminate the cyclotron variables by canonical transformation, rather than by resummation of perturbation theory. The cyclotron modes are represented as oscillators whose zero-point motion produces the amplitude of the LJ factor in the ground-state wave function. However, when fermions are excited to different **k** states, the oscillators must adjust to a displaced ground state, and this seems to reproduce many of the effects of the correlation hole discussed in Sec. II B, as well as other effects connected with the cyclotron mode and the projection to the lowest Landau level. Lee[25] used duality methods, which are good for representing vortices. In his approach, the particles are fermions at $\nu=\frac{1}{2}$, but, in view of the single vortex exactly on each particle because of Fermi statistics (for LLL wave functions), they can be represented as bosons at $\nu=1$. In these two works, only the leading long-wavelength effects can be treated. Pasquier and Haldane (PH) (Ref. 26) used a method that is valid only for $\tilde{\phi}=1$ (that is, the particles are bosons at $\nu=1$), and represents the LLL problem exactly, through equations valid for all wavelengths. A version of their method will be described in Sec. III and used extensively in this paper.

All these groups arrive at the following points in common. The LLL physics is described by Fermi fields c and $c^\dagger$ with canonical anticommutation relations, and the physical states must obey the operator constraints for each wave vector **q**:

$$\int \frac{d^2k}{(2\pi)^2} c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}} \left(1-\frac{1}{2}i\mathbf{k}\wedge\mathbf{q}\right)+O(q^2) -\bar{\rho}(2\pi)^2\delta(\mathbf{q})=0. \tag{2.18}$$

In SM and Lee, the $O(q^2)$ terms are unknown, and in SM the constraints are further restricted to apply only for q less than a cutoff Q whch is chosen to equal k_F. In PH, the terms higher order in q are known. The physical particle number-density operator reduces to the form

$$\rho(\mathbf{q})=\bar{\rho}(2\pi)^2\delta(\mathbf{q})+\int \frac{d^2k}{(2\pi)^2} i\mathbf{k}\wedge\mathbf{q}\, c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}}, \tag{2.19}$$

again to leading order in q on using the constraints. Note that this is the Fourier transform of a dipolar or polarization expression for the density, $\rho=\bar{\rho}-\nabla\cdot\mathbf{P}$, where the polarization **P** is that due to a dipole moment of $\wedge\mathbf{k}$ on a fermion of wave vector **k** (this semiclassical way of describing it will be quite useful; compare the discussion of fermions with a fairly well-defined wave vector and position in Fermi-liquid theory, which can be better described formally by the Wigner distribution function).

A result for the effective mass was obtained as follows. Beginning from the interaction Hamiltonian that is all that is left when the kinetic energy of the particles has been quenched,

$$H_{\rm int}=\frac{1}{2}\int \frac{d^2q}{(2\pi)^2} V(q) {:}\rho(\mathbf{q})\rho(-\mathbf{q}){:}, \tag{2.20}$$

where colons $:\cdots:$ represent normal ordering; the normal ordering is then dropped as it produces only a constant proportional to the number of particles. The density is then replaced by the form in Eq. (2.19). When this is written in first quantization it becomes

$$H_{\rm int}=\frac{1}{2}\sum_{ij}\int \frac{d^2q}{(2\pi)^2} V(q)\mathbf{q}\wedge\mathbf{k}_i\mathbf{q}\wedge\mathbf{k}_j. \tag{2.21}$$

On taking the $i=j$ term of this expression, they obtain an effective kinetic energy due to interactions,

$$\sum_i \mathbf{k}_i^2/(2m^*), \tag{2.22}$$

where the effective mass is given by

$$1/m^*=\frac{1}{2}\int \frac{d^2q}{(2\pi)^2} V(q)q^2, \tag{2.23}$$

which has the form of the dipole moment squared term in the self-interaction energy of a dipole; if the q integral is cut off as in SM, the density profile is smeared as it would be in the correlation hole. It is therefore similar to the proposal of Refs. 10 and 12.

For the density-density response function, these authors found, using the dipolar form of the density,

$$\chi_{\rho\rho}(\mathbf{q},0)=\langle\rho(\mathbf{q})\rho(-\mathbf{q})\rangle=q^2\langle PP\rangle=q^2\bar{\rho}m^*+O(q^4). \tag{2.24}$$

In the last step, the transverse momentum-momentum response function of the Fermi gas with effective mass m^* was used. In these calculations, constraints (2.18) were either ignored,[26] or were handled by introducing functional-integral representations of δ functions of the constraints, which were then treated in the RPA (Ref. 24); the results take the form stated in either case.

If this last result is taken seriously, it implies that the system is incompressible. However, SM stated some reservations about the calculation, because of the way the constraint was handled. They suggested that the symmetry of the Hamiltonian under translations of the wave vectors of all the particles could lead to cancellations and to factors of $1/q^2$ that could restore a finite compressibility to the system. This proposal is very close to the results that will be obtained in the present paper by a systematic treatment of the constraints. While this paper was being completed, a short comment[51] and a revised version of Ref. 25 appeared which used the same symmetry just mentioned, and obtained results very similar to some of ours below, including the fact that the system is compressible. We will comment further on the relation of the symmetries being used in Sec. V.

D. Overview of the results of the present paper

Here we describe results of the present paper. First we give a simple discussion of our central result for the density-density response function. With the benefit of hindsight, using arguments that are justifed by the more detailed and formal calculations below, the results can in fact be obtained from the results of Sec. II B. Then we describe the results of this paper.

The dipolar form of the density in Sec. II B can be expressed as

$$\rho(\mathbf{r})=\overline{\rho}-\nabla\wedge\mathbf{g}, \tag{2.25}$$

where $\mathbf{g}(\mathbf{r})$ is the momentum density of the fermions, since $\mathbf{P}=\wedge\mathbf{g}$. On the other hand, we also have

$$\rho(\mathbf{r})=\overline{\rho}-\overline{\rho}\nabla\wedge(\mathbf{a}+\mathbf{A}). \tag{2.26}$$

This suggests that we write

$$\mathbf{a}+\mathbf{A}=\mathbf{g}/\overline{\rho} \tag{2.27}$$

in general, even though the above argument only implies this for the transverse part of $\mathbf{a}$. This equation suggests there is a gauge-invariant current $\mathbf{j}^R$, which is not the physical current, such that (for excitations near the Fermi surface),

$$\begin{aligned}\mathbf{j}^R&=\{-i\tfrac{1}{2}[c^\dagger\nabla c-(\nabla c^\dagger)c]-(\mathbf{a}+\mathbf{A})c^\dagger c\}/m^*\\&=[\mathbf{g}(\mathbf{r})-(\mathbf{a}+\mathbf{A})\rho^R]/m^*,\end{aligned} \tag{2.28}$$

which is required to vanish, $\mathbf{j}^R=\mathbf{0}$. Assuming that the ''density'' $\rho^R=c^\dagger c$ is just $\overline{\rho}$, this is equivalent to Eq. (2.27). Indeed, a vanishing current would be consistent with such a constraint, $\rho^R=\overline{\rho}$, if they together obey a continuity equation

$$\partial\rho^R/\partial t+\nabla\cdot\mathbf{j}^R=0. \tag{2.29}$$

This involves the longitudinal part of the current, so we have an argument for both parts of Eq. (2.27). The condition $\rho^R=\overline{\rho}$ should of course be viewed as the long-wavelength version of the constraint found by SM, Lee, and PH.

The gauge-invariant form of the ''current'' $\mathbf{j}^R$ encourages us to consider an effective Hamiltonian

$$H_{\rm eff}=\frac{1}{2m^*}\int d^2r\,|(-i\nabla-\mathbf{a}-\mathbf{A})c|^2+\cdots, \tag{2.30}$$

which, apart from higher covariant derivatives of c and $c^\dagger$, contains no other terms in $\mathbf{a}$, not even a CS term. We view $H_{\rm eff}$ as a Hamiltonian for c and c^+, but as a Lagrangian for a. Thus $\mathbf{a}$ is a strongly coupled gauge field, and varying $H_{\rm eff}$ with respect to $\mathbf{a}$ yields $\mathbf{j}^R=0$. Then, neglecting other terms in $H_{\rm eff}$, we can use the RPA, or the following self-consistent-field argument, to find the density-density response function. From the form of the density, an external scalar potential couples to $\nabla\wedge\mathbf{g}$. The irreducible density response contains two parts. The first part is from the transverse momentum-momentum response function of the gas with mass m^*; it is the part found by SM, Lee, and PH (Refs. 24–26) (Lee has since revised this result[25]). The second is the response of the same gas to the induced vector potential $\mathbf{a}$. (In both responses, the constant ''diamagnetic current'' term is absent.) Thus

$$\chi^{\rm irr}_{\rho\rho}=(\overline{\rho}+m^*\chi_0^\perp)(q^2m^*+iq\,\delta a_\perp), \tag{2.31}$$

where in the last factor the two terms arise from the two parts just described, and $\delta a_\perp$ is the response in the transverse vector potential to the perturbation, and is therefore given by

$$iq\,\delta a_\perp=\chi^{\rm irr}_{\rho\rho}/\overline{\rho}. \tag{2.32}$$

From these self-consistent equations we find

$$\chi^{\rm irr}_{\rho\rho}=-\overline{\rho}(\overline{\rho}+m^*\chi_0^\perp)q^2/\chi_0^\perp, \tag{2.33}$$

which is exactly the result we obtain in this paper. This yields for the compressibility $dn/d\mu=-\overline{\rho}^2/\chi^*_{\rm d}>0$, where $\chi^*_{\rm d}$ is the diamagnetic susceptibility for this fermion gas. This result differs from that in the scenario based on the U(1) CS approach, described at the end of Sec. II A. Several other observables are similarly in close, but not always exact, agreement with the scenario based on HLR, described above.

In this argument, we neglected the Landau parameters. These can be included without significantly changing the results. However, the Landau parameter F_1 should not be added, since it is already included in the gauge field effects. *The strongly coupled gauge field in the Fermi liquid is equivalent to a Landau parameter* $F_1=-1$, provided $m^*>0$.[52] Thus we are led to a scenario in which the Fermi-liquid-like state has many FL properties in common with the theory of HLR, including a finite compressibility, yet differs in that there is no CS term for the gauge field, while the physical density is dipolar or (using an equation of motion) is $-\overline{\rho}\nabla\wedge\mathbf{a}$.

In the rest of the paper, we follow a different argument from that just presented. We give a detailed microscopic derivation, in which the relationship $\rho(\mathbf{r})=-\overline{\rho}\nabla\wedge\mathbf{a}$ appears only at the end; thus we do not rely on the Berry phase argument. The starting point is an approach of Pasquier and Haldane, described in Sec. III below. In this approach, which works for $\tilde{\phi}=1$ only, that is bosons at $\nu=1$, each fermion is described by two coordinates, which we term ''left'' and ''right,'' but the available states are those of a particle in zero magnetic field, because the wave functions are complex analytic in the left and antianalytic in the right coordinates. The left coordinate is that of the underlying particle contained in the fermion, while the right coordinate represents an attached vortex, as in the pictures in Sec. II B. The system must obey a constraint of fixed density $\rho^R=\overline{\rho}$ in the right coordinates. Since the separation of the left from the right coordinate is $\wedge\mathbf{k}$ when the fermion is in a plane-wave state of wave vector $\mathbf{k}$, the physical density is dipolar. In order to maintain the constraint, the longitudinal part of the current $\mathbf{j}^R$ of the vortices (right coordinates) must vanish, as argued above. In Sec. IV, we consider a conserving approximation for observable response functions. We show that the constraints are satisfied in this method. We calculate the density-

density response, its spectral density, the longitudinal conductivity, the scattering of the fermions by a potential, and the current-density operator. From the results we deduce that the system can be described in terms of the strongly coupled gauge field mentioned above. The gauge invariance is a manifestation of the constraint. The gauge fields obey the CS *equations*, even though there is no CS *term* in the action. In Sec. V, we indicate the form we expect for the exact results to all orders in the interactions, and give arguments that these are correct. We conjecture that a certain sum rule for the spectral density is exact. While at present this approach works for $\tilde{\phi}=1$—that is, for bosons at $\nu=1$—we expect that the conclusions are more general, as the results and arguments of the previous subsections and the beginning of this one are.

III. PASQUIER-HALDANE APPROACH FOR $\tilde{\phi}=1$

In this section we review (with a few variations of our own) the method of PH,[53] which works only for $\tilde{\phi}=1$, though the filling factor does not necessarily have to be 1. A similar method works for fermions with one vortex attached, mapping them to composite bosons. Since the formalism has not appeared elsewhere in the form in which we will use it, it will be presented in self-contained fashion.

We begin abstractly, labeling arbitrary single-particle states with indices. Hopefully the later development in coordinate space, though less general, will seem less abstract and give more physical insight, and clearly show the connection with composites particles and the LLL. We take fermion operators which are matrices with two indices, c_{mn} and $c^{\dagger}_{nm}$, with canonical anticommutation relations

$$\{c_{mn},c^{\dagger}_{n'm'}\}=\delta_{mm'}\delta_{nn'} \tag{3.1}$$

(and others vanish) where m, m', n, and n' run from 1 to N (this case of square matrices is convenient for the $\nu=1$ boson problem, while rectangular matrices would be used for $\nu\neq 1$). The anticommutation relations are invariant under independent unitary transformations on the left and right indices, under which

$$c\mapsto U_L c U_R,$$
$$c^{\dagger}\mapsto U_R^{\dagger}c^{\dagger}U_L^{\dagger}, \tag{3.2}$$

where U_L and U_R are unitary $N\times N$ matrices. These transformations are generated by the operators

$$\rho^R_{nn'}=\sum_m c^{\dagger}_{nm}c_{mn'}, \tag{3.3}$$

$$\rho^L_{mm'}=\sum_n c^{\dagger}_{nm'}c_{mn}. \tag{3.4}$$

The right generators ρ^R generate the group $\mathrm{U}(N)_R$ of unitary matrices. These are used to specify a set of N^2 constraints on the system

$$(\rho^R_{nn'}-\delta_{nn'})|\Psi_{\rm phys}\rangle=0, \tag{3.5}$$

which defines a subspace of states that will be identified with the physical Hilbert space. By taking the trace, we see that these imply that the U(1) generator or fermion number operator [which is common to $\mathrm{U}(N)_R$ and $\mathrm{U}(N)_L$],

$$\hat{N}=\sum_{mn} c^{\dagger}_{nm}c_{mn}, \tag{3.6}$$

must have an eigenvalue equal to N. Thus, in the allowed subspace, N is both the range of the indices, and the number of fermions. The remaining right generators generate $\mathrm{SU}(N)_R$, and physical states must be singlets under the action of this group. The other group $\mathrm{SU}(N)_L$ is not used for constraints, and will be broken by the Hamiltonian to a subgroup that represents translations and/or rotations on the two-dimensional manifold (say, the sphere, torus, or infinite plane) on which the physical particles move. At the same time, the generators $\rho^L_{mm'}$ will represent the physical density on this manifold.

The physical states that satisfy the constraints can be written as linear combinations of

$$|\Psi^{m_1\cdots m_N}_{\rm phys}\rangle=\sum_{n_1,\ldots,n_N}\varepsilon^{n_1,\ldots,n_N}c^{\dagger}_{n_1m_1}c^{\dagger}_{n_2m_2}\cdots c^{\dagger}_{n_Nm_N}|0\rangle, \tag{3.7}$$

where $|0\rangle$ is the vacuum containing no fermions. These states contain N fermions and are clearly singlets under $\mathrm{SU}(N)_R$ since they are antisymmetric in the n (right) indices. On the other hand, the anticommutation of the $c^{\dagger}$'s implies that they are symmetric in the remaining m (left) indices. Thus these states can be viewed as basis states for a system of N bosons, each of which can be in any one of N single-particle states. Such a boson system could be described by basis states

$$a^{\dagger}_{m_1}\cdots a^{\dagger}_{m_N}|0\rangle, \tag{3.8}$$

where $[a_m,a^{\dagger}_{m'}]=\delta_{mm'}$, and others vanish. Each such state is obtained in this way, which proves that the fermion system of $c's$ with the constraints is equivalent to the unconstrained boson system. If we define a filling factor as the particle number divided by the number of available orbitals, as $N\to\infty$, then in our case we clearly have bosons at filling factor $\nu=1$.

We note that in the larger Hilbert space without the constraints, which is just the Fock space of the c's, each fermion can be in any of N^2 states, so there are

$$\binom{N^2}{N} \tag{3.9}$$

linearly independent states for N fermions. The states satisfying the constraints form the Fock space of the bosons a, which contains only

$$\binom{2N-1}{N} \tag{3.10}$$

linearly independent states.

The left indices m can run over any range, and this can be used to represent any filling factor ν. The constrained system can also be set up using canonical *commutation* relations for the c's, and a similar argument then shows that the physical states represent *fermions*, (e.g., electrons) at $\nu \leqslant 1$.

So far we have only a way of representing bosons by fermions (or vice versa), and the technique is reminiscent of the methods used for quantum spin systems, in the case of a single quantum spin (see e.g., Ref. 54). If we now view m and n as indices for lowest Landau-level states, say on a sphere where there are $N_\phi+1$ such states for N_ϕ flux quanta through the sphere,[15] then for the case where both indices range from 1 to N, we have $N=N_\phi+1$, and the filling factor ν agrees with that defined as N/N_ϕ as $N\to\infty$. We can introduce coordinate space wave functions for the left index m, which are just those of the physical bosons. We do the same for the right indices n, except that they are complex conjugated so that the field strength (or the charge) is effectively reversed. Using orthonormal single-particle LLL basis states $u_m(z)$, we write in analogy with the usual field operators,

$$c(z,\bar{w})=\sum_{mn} u_m(z)\,\overline{u_n(w)c_{mn}},$$

$$c^\dagger(w,\bar{z})=\sum_{mn} u_n(w)\,\overline{u_m(z)c^\dagger_{nm}}, \qquad (3.11)$$

which are adjoints of each other. Note that we use z's for ''left'' indices, corresponding to m's (which however appear on the right in $c^\dagger$) and w's for ''right'' indices, corresponding to n's. The appearance of two coordinates on c and $c^\dagger$ means that they behave like operators on the LLL single-particle Hilbert space, just like the matrix structure they had in index notation. A formalism for handling such operators as integral kernels is given in Appendix A. For the sphere, we can write $\tilde{u}_m(z)\propto z^m$, for $m=0,\ \ldots\ ,N_\phi=N-1$, and the factor $(1+|z|^2/4R^2)^{-(N_\phi+2)/2}$ must be attached before integration. Following this convention we will write only the polynomial part in the following wave functions.

In the z, w variables, the densities become

$$\rho^R(w,\bar{w}')=\int d^2z\, c^\dagger(w,\bar{z})c(z,\bar{w}'), \qquad (3.12)$$

$$\rho^L(z,\bar{z}')=\int d^2w\, c^\dagger(w,\bar{z}')c(z,\bar{w}). \qquad (3.13)$$

Matrix multiplication has been replaced by integration, so that all operators in the single-particle Hilbert space of LLL functions of z and $\bar{w}$ become integral kernels (see Appendix A). One can see that $\rho^L(z,\bar{z})$ is the LLL-projected density operator denoted $\bar{\rho}$ by Girvin, MacDonald, and Platzman (GMP),[50] and ρ^R is analogous.

Passing to the thermodynamic limit at fixed field strength and density equal to $\bar{\rho}$, the radius of the sphere goes to infinity, the system becomes flat locally, and we may use Fourier transforms. The version of the Fourier transform required is defined in Appendix A. To avoid discussion of global issues, which would distinguish this thermodynamic limit from that of a torus, we will view the use of Fourier transforms as a technique for handling local calculations, in which we could include damping factors which tend to unity at the end. Alternatively, every calculation could, with only a little extra difficulty, be done in coordinate space. A third alternative would be to use the analog of the Fourier transform, involving spherical harmonics, on the finite-size sphere. This is more tedious. Introducing the Fourier transform in the plane, then we notice that the pair of coordinates z and $\bar{w}$ for each particle or field operator c is replaced by a single ordinary two-dimensional wave vector $\mathbf{k}$. This makes sense because, by choosing equal and opposite field strengths for the basis functions in these coordinates, the particles effectively ''see'' zero magnetic field for our filling factor $\nu=1/\tilde{\phi}=1$. Note that, because the functions are analytic in z, $\bar{w}$ (the LLL restriction), we do not effectively have four real variables per particle, as we would if the basis states had not been restricted to the LLL. The transformation of the matrix $c(z,\bar{w})$ into a plane-wave operator is similar to that for the density operator, say ρ^L, which can clearly be traded for its Fourier components (see, e.g., GMP).

In terms of $c_{\mathbf{k}}$ and $c^\dagger_{\mathbf{k}}$, which are defined in Appendix A, and which satisfy

$$\{c_{\mathbf{k}},c^\dagger_{\mathbf{k}'}\}=(2\pi)^2\delta(\mathbf{k}-\mathbf{k}'), \qquad (3.14)$$

we have

$$\rho^R(\mathbf{q})=\int \frac{d^2k}{(2\pi)^2}e^{-(1/2)i\mathbf{k}\wedge\mathbf{q}}\,c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}}c_{\mathbf{k}+(1/2)\mathbf{q}}, \qquad (3.15)$$

$$\rho^L(\mathbf{q})=\int \frac{d^2k}{(2\pi)^2}e^{(1/2)i\mathbf{k}\wedge\mathbf{q}}\,c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}}c_{\mathbf{k}+(1/2)\mathbf{q}}, \qquad (3.16)$$

and we can show that

$$[\rho^R(\mathbf{q}),\rho^L(\mathbf{q}')]=0, \qquad (3.17)$$

$$[\rho^R(\mathbf{q}),\rho^R(\mathbf{q}')]=-2i\sin\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}'\,\rho^R(\mathbf{q}+\mathbf{q}'), \qquad (3.18)$$

$$[\rho^L(\mathbf{q}),\rho^L(\mathbf{q}')]=2i\sin\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}'\,\rho^L(\mathbf{q}+\mathbf{q}'). \qquad (3.19)$$

The Lie algebra commutation relations defined by Eq. (3.19) appeared in GMP and in Ref. 55, and the algebra so defined has become known as W_∞ [the defining relations are often given in a different basis of the Lie algebra, essentially the expansion of our $\rho^L(z,\bar{z}')$ in angular momentum eigenstates z^m and $\bar{z}'^m$]. In the notation of GMP, our $\rho^L(\mathbf{q})=e^{(1/4)|q|^2}\bar{\rho}(\mathbf{q})$. (The following algebraic comments will not be used in the following.) From our point of view, W_∞ is just a certain limit of SU(N) as $N\to\infty$. It is also helpful to note that if the $2\sin\frac{1}{2}\mathbf{q}\wedge\mathbf{q}'$ is replaced by $\mathbf{q}\wedge\mathbf{q}'$ in the commutation relations (for example, because $\mathbf{q}$ and $\mathbf{q}'$ or the magnetic length are small), then the resulting algebra is that of ''area-preserving diffeomorphisms,'' or equivalently (for the corresponding Poisson bracket relations) Fourier components of functions on classical phase space. W_∞ can then be viewed as a quantum deformation of the latter, thus as ''diffeomorphisms of the quantum analogue of phase space,'' a fairly familiar view of the LLL. The connection of W_∞ with the quantum Hall effect has often been remarked.[56] Our interest here is in the isomorphic algebra generated by the ρ^R's, which are the constraints of our problem.

The constraints become

$$[\rho^R(\mathbf{q})-\overline{\rho}(2\pi)^2\delta(\mathbf{q})]|\Psi_{\text{phys}}\rangle=0. \tag{3.20}$$

Thus states can be built up in the ''large'' Hilbert space as combinations of

$$\prod_{\{\mathbf{k}_i\}} c^\dagger_{\mathbf{k}}|0\rangle \tag{3.21}$$

(where the product is indexed by $\mathbf{k}$'s in a set of N wave vectors $\mathbf{k}_i$), and then projected to satisfy the constraints. The effect of projection can be more easily appreciated in terms of wave functions in coordinate space, by returning to the finite-size system.

In coordinate space, the constraints require that the $\overline{w}$ dependence of wave functions be that of a full LLL,

$$\tilde{\Psi}_{\text{phys}}(z_1,\overline{w}_1,\ldots,z_N,\overline{w}_N)=f(z_1,\ldots,z_N)\prod_{i<j}(\overline{w}_i-\overline{w}_j), \tag{3.22}$$

because the LJ factor in the $\overline{w}$'s is the unique totally antisymmetric function annihilated by the ρ^R's, since the full LLL has no density fluctuations. Hence f is a symmetric polynomial in the z_i's, as appropriate for bosons. Projection of the wave function of any state in the ''large'' Hilbert space to this physical subspace, where states can be characterized just by f, is accomplished by multiplying by $\Pi(w_i-w_j)$ and integrating over the w_i's with the appropriate measure, leaving a symmetric function f in the z_i's (possibly zero). If as a family of examples we take states (3.21), or their analogs on the sphere, in first quantization they become Slater determinants $\det[Y_{L_iM_i}(z_j,\overline{w}_j)]$, where the $Y_{LM}(z,\overline{w})$ are spherical harmonics projected to the LLL, which correspond to the plane waves $\tau_{\mathbf{k}}$ in the plane, defined in Appendix A. Then the projection gives

$$\begin{aligned} f&=\int\prod_k d^2w_k\prod_{i,j}(w_i-w_j)\det Y_{L_iM_i}(z_j,\overline{w}_j)\\ &=\mathcal{P}_{\text{LLL}}\det Y_{L_iM_i}(\Omega_j)\prod(z_i-z_j), \end{aligned}\tag{3.23}$$

that is, the projection to the LLL of ordinary spherical harmonics in a Slater determinant times the LJ factor. These are just the trial wave functions described in Sec. II B. Thus the formalism not only describes bosons at $\nu=1$, but the fermions are closely related to those in the ''physical'' approach, where the amplitude of the LJ factor is automatically included in the trial wave functions. Contrast this with the CS approach, where the trial wave functions satisfying the CS constraint of one flux attached to each particle consist of the Slater determinant times only the phase of the LJ factor, and no LLL projection. Note also that while the projection into a strictly smaller subspace implies that states described by distinct sets of $\mathbf{k}_i$ before projection may not be orthogonal after projection, they do not usually vanish, except in some exceptional cases noted in Ref. 48.

Since the right coordinates $\overline{w}$ of the fermions become, in the trial wave functions after projection, the locations of the vortices, it seems natural to refer to them as such even before projection. Thus we can say that *each fermion consists of a particle (boson) at the left coordinate z, and a vortex at the right coordinate* $\overline{w}$, *and so as a whole is effectively neutral*. The constraints demand that the density ρ^R of vortex coordinates is fixed, as an operator statement. This seems natural if the vortices are thought of as forming a two-dimensional plasma (in view of the LJ factor and Laughlin's plasma mapping[14]), since the plasma is in a screening phase and suppresses long-wavelength density fluctuations; indeed, in this case of $\nu=1$, there are no fluctuations in the LLL density at all in the Laughlin state (the full LLL or Vandemonde determinant). In retrospect, this condition on the vortices seems to be the main effect that was left out in Refs. 10 and 12.

Now we finally specify the Hamiltonian appropriate to bosons in the LLL at $\nu=1$. In terms of the boson operators a introduced earlier, we have, assuming a potential interaction between the bosons,

$$H=\tfrac{1}{2}\sum_{m_1,\ldots,m_4}V_{m_1m_2;m_3m_4}a^\dagger_{m_1}a^\dagger_{m_2}a_{m_4}a_{m_3}, \tag{3.24}$$

where the matrix elements of the interaction in the LLL are[57]

$$\begin{aligned} V_{m_1m_2;m_3m_4}=\int d^2r_1\,d^2r_2\,&\overline{u_{m_1}(z_1)u_{m_2}(z_2)V(\mathbf{r}_1}-\mathbf{r}_2)\\ &\times u_{m_3}(z_1)u_{m_4}(z_2). \end{aligned}\tag{3.25}$$

The corresponding operator in the large Hilbert space, where it commutes with the constraints ρ^R, and so projects to H in Eq. (3.24), is

$$H=\tfrac{1}{2}\sum_{\substack{m_1,\ldots,m_4\\ n_1,n_2}}V_{m_1m_2;m_3m_4}c^\dagger_{n_1m_1}c^\dagger_{n_2m_2}c_{m_4n_2}c_{m_3n_1}. \tag{3.26}$$

Then using the definition of $c(z,\overline{w})$, we obtain

$$H=\tfrac{1}{2}\int d^2r_1d^2r_2V(\mathbf{r}_1-\mathbf{r}_2):\rho^L(z_1,\overline{z}_1)\rho^L(z_2,\overline{z}_2):, \tag{3.27}$$

where the normal ordering is with respect to the vacuum of the c's, $|0\rangle$. Thus this is simply a potential interaction written in terms of the LLL-projected density ρ^L. In Fourier space this becomes

$$H=\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\tilde{V}(\mathbf{q}):\rho^L(\mathbf{q})\rho^L(-\mathbf{q}):, \tag{3.28}$$

where $\tilde{V}(\mathbf{q})=e^{-(1/2)|q|^2}V(\mathbf{q})$ absorbs a factor left from the definition of the Fourier transform of ρ^L, and $V(\mathbf{q})$ is the usual Fourier transform

$$V(\mathbf{q})=\int d^2r\,e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r}). \tag{3.29}$$

The interaction Hamiltonian breaks the symmetry group from $SU(N)_L$ (in the absence of interaction) to SU(2) (for the sphere) or to magnetic translations and rotations in the case of the plane. It still commutes with the ''constraint operators''

$$G(\mathbf{q})\equiv\rho^R(\mathbf{q})-\overline{\rho}(2\pi)^2\delta(\mathbf{q}). \tag{3.30}$$

Our expression for the Hamiltonian differs somewhat from that in the paper of PH. They work on the torus, which is a relatively unimportant difference, and write the Hamiltonian using the constraints to make the ansatz explained in Sec. II C, which results in a one-body term that gives the fermions an effective kinetic energy coming from the interaction. In our approach we do not wish to make such a substitution since the commutator of H with the $G(\mathbf{q})$ would not vanish identically, but only on using the conditions $G(\mathbf{q})=0$. The reason for our insistence on retaining $[H,G(\mathbf{q})]=0$ will be discussed in Sec. IV. Of course, if everything is done correctly, the results should be the same, in the end, since the starting point is the same.

IV. HARTREE-FOCK AND CONSERVING APPROXIMATIONS

In this section, which is the central one of the paper, we develop an approximate solution for our system that descibes the FL state. We begin in Sec. IV A with the Hartree-Fock (HF) approximation, which yields a dispersion relation for the fermions. Then in Sec. IV B we explain how the constraints can be included. We choose a gauge such that, for nonzero frequencies, they must be satisfied without any assistance from integration over auxiliary fields that impose them explicitly. This is achieved in Sec. IV C by use of conserving approximations, a familiar method of many-body and quantum-field theory. In the present case, such an approximation consistent with the HF approximation is the generalized or time-dependent HF approximation, which sums ring and ladder diagrams. We show explicitly that the constraints are obeyed in our approximation. In Sec. IV D we investigate the asymptotics of the ladder series that appears in Sec. IV C, for use in the following calculations. In Sec. IV E we apply the approach to the physical response functions, beginning with the density-density response. We show that the system is compressible and that the longitudinal conductivity relevant for the surface acoustic wave experiments, which is a certain limit of this response, is given by exactly the same expression as in HLR. We also exhibit a sum-rule-like relation for the high-frequency response, or for the first moment of the spectral density, which we will later argue is exact. We consider the scattering of a fermion by a scalar potential perturbation, and interpret the result in terms of a vector potential related to the density by the CS relation discussed in Sec. II. We calculate the longitudinal conductivity due to impurity scattering. Finally, we consider the physical current density, which we relate to the stress or momentum flux tensor of the fermions, and so recover the other CS relation.

A. Hartree-Fock approximation

In this subsection, we use the HF approximation, which is quick and is the simplest one that gives an effective kinetic energy and is consistent with a stable Fermi sea as the ground state. The treatment of the constraints will be extensively discussed in Sec. IV B, and the formalization of the exchange part of the self-energy as the saddle point approximation to a functional integral, valid in some sense in a large-M limit (in a generalization of the model to M component fermions), is left to Appendix B.

The problem for $\tilde{\phi}=1$ using the PH approach is described by the Hamiltonian (3.28), which can be written

$$H=\frac{1}{2}\int\frac{d^2k_1d^2k_2d^2q}{(2\pi)^6}\tilde{V}(q)e^{(1/2)i\mathbf{k}_1\wedge\mathbf{q}-(1/2)i\mathbf{k}_2\wedge\mathbf{q}}\times c^\dagger_{\mathbf{k}_1-(1/2)\mathbf{q}}c^\dagger_{\mathbf{k}_2+(1/2)\mathbf{q}}c_{\mathbf{k}_2-(1/2)\mathbf{q}}c_{\mathbf{k}_1+(1/2)\mathbf{q}}, \tag{4.1}$$

subject to the constraints $G(\mathbf{q})\equiv\rho^R(\mathbf{q})-\overline{\rho}(2\pi)^2\delta(\mathbf{q})=0$, that is $\hat{N}=N$, and

$$\int\frac{d^2k}{(2\pi)^2}e^{-(1/2)i\mathbf{k}\wedge\mathbf{q}}c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}}c_{\mathbf{k}+(1/2)\mathbf{q}}=0 \tag{4.2}$$

for $\mathbf{q}\neq 0$. Notice that when the phase factor containing $\mathbf{k}\wedge\mathbf{q}$ is expanded in a Taylor series, to $O(q^2)$ it takes the same form as the constraint found by SM and Lee,[24,25] as mentioned in Sec. II C.

The HF approximation for a translationally invariant system takes the energy eigenstates to be Slater determinants of plane waves, that is plane-wave-occupation-number eigenstates in the second-quantized formalism, and the energy of such a state is taken to be the expectation value of H. As is well known, for the excitation spectrum, this is equivalent to replacing H by an effective one-body Hamiltonian with an effective energy $\varepsilon_\mathbf{k}$ for each plane-wave state $\mathbf{k}$, where $\varepsilon_\mathbf{k}$ depends self-consistently on the occupation numbers $n_\mathbf{k}$. In the present case, we must also include the constraints by the use of Lagrange multipliers $\overline{\lambda_\mathbf{q}}$ and minimize

$$H-\mu N-\int\frac{d^2q}{(2\pi)^2}\overline{\lambda_\mathbf{q}}G(-\mathbf{q}) \tag{4.3}$$

with respect to $\overline{\lambda_\mathbf{q}}$ to find the ground state. When almost all particles are in the Fermi sea, $\overline{\lambda_\mathbf{q}}$ are zero by translational symmetry, except at $\mathbf{q}=\mathbf{0}$, where $\overline{\lambda_\mathbf{0}}$ absorbs the chemical potential μ, consistent with the fact that the constraints fix the particle number and hence we are actually in the canonical, not grand canonical, ensemble. Consequently one has $\overline{\lambda_\mathbf{q}}=(2\pi)^2\overline{\lambda}\,\delta(\mathbf{q})$, and $\overline{\lambda}+\mu$ is determined by the condition on the total particle number. One arrives therefore at the total-energy expectation value

$$E=\frac{1}{2L^2}\sum_{\mathbf{kk}'}f_{\mathbf{kk}'}n_\mathbf{k}n_{\mathbf{k}'} \tag{4.4}$$

(in which we have used the conventional notation for a finite system in a square box of side L, with discrete $\mathbf{k}$ values, and $n_\mathbf{k}$ are the expectation values of the occupation numbers for the corresponding states), where

$$f_{\mathbf{kk}'}=\tilde{V}(\mathbf{0})-\tilde{V}(\mathbf{k}-\mathbf{k}'). \tag{4.5}$$

The function $f_{\mathbf{kk}'}$ plays the role of the Landau interaction function when $\mathbf{k}$ and $\mathbf{k}'$ are restricted to the Fermi surface. The effective single-particle Hamiltonian $K=H-(\mu+\overline{\lambda})N$ is

$$K_{\rm eff}=\sum_{\mathbf{k}} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}, \tag{4.6}$$

where $\xi_{\mathbf{k}}=\varepsilon_{\mathbf{k}}-\mu-\overline{\lambda}$ and

$$\varepsilon_{\mathbf{k}}=\tilde{V}(\mathbf{0})\int \frac{d^2k'}{(2\pi)^2} n^0_{\mathbf{k}'}-\int \frac{d^2k'}{(2\pi)^2}\tilde{V}(\mathbf{k}-\mathbf{k}')n^0_{\mathbf{k}'}, \tag{4.7}$$

in which the first term is the direct or Hartree term, equal to $\tilde{V}(\mathbf{0})\overline{\rho}$, and the second is the exchange or Fock term, which is responsible for the $\mathbf{k}$ dependence of $\xi_{\mathbf{k}}$. Also, in the ground state at zero temperature, $n^0_{\mathbf{k}}=\theta(k_F-k)$ and $k_F=\sqrt{2}$ in our units, and $\mu+\overline{\lambda}$ is chosen so that $\xi_{k_F}=0$. Notice that the phase factors in the Hamiltonian H have turned out to be unity in the HF expressions, which are identical to those of the usual Fermi gas, except that the bare kinetic energy is zero, and that $\tilde{V}(\mathbf{q})$ replaces $V(\mathbf{q})$ for reasons connected with the LLL. This formula for $\varepsilon_{\mathbf{k}}$ differs from that of other authors, discussed in Sec. II C, in that it depends explicitly on the occupation numbers of the other $\mathbf{k}$ states, and does not reduce to the self-interaction of a dipole even for small $\mathbf{q}=\mathbf{k}-\mathbf{k}'$ in the integral in the exchange term. Our $\xi_{\mathbf{k}}$ obtains its $\mathbf{k}$ dependence from the exchange effect, while the interaction of the particle with the correlation hole that surrounds it (due to the vortices) is a ''Hartree-like'' term (and not simple Hartree) (see Ref. 10, where exchange effects were explicitly neglected). Thus the exchange effect found here in the simplest approximation seems to be complementary to the interaction with the correlation hole, and probably both terms would be present in a better approximation. As for the dipolar form of density, we will see that the density does take on this form, and this could be included in the exchange self-energy, but this would necessitate a complicated self-consistent calculation which could not be done analytically. In any case, the dipolar effect changes the form of the interaction at small $\mathbf{q}$, while intermediate $\mathbf{q}$ values are important in the exchange self-energy. Thus the expression here is a convenient starting point, and not badly wrong physically, at least in some cases, as we will see shortly.

The zero-temperature HF dispersion relation can be studied in detail. Apparently, no difficulties are caused by the absence of a bare $\varepsilon_{\mathbf{k}}$ term. For any repulsive interaction $\tilde{V}(\mathbf{q})=e^{-(1/2)|\mathbf{q}|^2}V(\mathbf{q})>0$, $\varepsilon_{\mathbf{k}}$ increases monotonically with $|\mathbf{k}|$ for all $\mathbf{k}$. At $|\mathbf{k}|=k_F$,

$$\frac{k_F}{m^*}\equiv\frac{\partial\xi_{\mathbf{k}}}{\partial|\mathbf{k}|}=-\int_{|\mathbf{k}'|<k_F}\frac{d^2k'}{(2\pi)^2}\frac{\partial\tilde{V}}{\partial|\mathbf{k}|}(\mathbf{k}'-\mathbf{k})$$
$$=\frac{k_F}{2\pi}\int\frac{d\theta_{\mathbf{kk}'}}{2\pi}\tilde{V}(\mathbf{k}'-\mathbf{k})\cos\,\theta_{\mathbf{kk}'} \tag{4.8}$$

(note that $\theta_{\mathbf{kk}'}$ parametrizes the angle between $\mathbf{k}'$ and $\mathbf{k}$ which are both on the Fermi surface). For a δ-function (short-range) potential $V(\mathbf{q})=V(\mathbf{0})$, $1/m^*$ is positive and finite. Thus the system is stable against single-particle excitations. For a Coulomb interaction $V(\mathbf{q})=2\pi e^2/|\mathbf{q}|$, there is a logarithmic singularity at $|\mathbf{k}|=k_F$:

$$\frac{\partial\xi_{\mathbf{k}}}{\partial|\mathbf{k}|}\sim-\ln|k-k_F|. \tag{4.9}$$

This is very similar to that for the Coulomb interaction in the three-dimensional electron gas at zero magnetic field treated in the HF approximation. In that case, the divergence is unphysical and is removed by replacing the bare Coulomb interaction in the exchange term by the screened one, which leaves a finite effective mass and heat capacity $C_V\sim\gamma T\sim m^*k_FT$. This conclusion of course depends on the presence of screening due to the nonzero compressiblity of the electron gas. In the present problem, the existence of such a compressiblity is one of the points we wish to study, so we must return to this later. Note, however, that replacing the unscreened interaction by the dipolar interaction also cuts off the divergence in the present problem. As mentioned already, this will also be left for later discussion. For the time being, we may consider an interaction of shorter range (decaying as a faster power) than the Coulomb interaction, and the effective mass is then finite within the HF approximation.

The question may be raised of whether a charge-density-wave (CDW) instability could take place due to the absence of a bare kinetic energy. However, the constraints $\rho^R(\mathbf{q})=0$, though not the same as $\int d^2k\, c^{\dagger}_{\mathbf{k}-(1/2)\mathbf{q}}c_{\mathbf{k}+(1/2)\mathbf{q}}=0$, may have a similar effect in maintaining the uniform density of the fluid within the HF approximation (a CDW in the underlying particles cannot be ruled out at some filling factors, especially $\nu\ll1$, but may not be describable within the HF approximation for the fermions). Another possible instability is to pairing as in BCS theory. This was argued by PH,[27] who found numerically that bosons at $\nu=1$ tend to form a ground state with high overlap with the Pfaffian state, a paired state which is presumably incompressible. However, for some interactions, such pairing may either not occur, or be very weak so that it occurs only at very low energies, and then the present results for the ''normal'' Fermi-liquid-like state will still apply at higher energies, temperatures, or wave vectors. For the state of electrons at $\nu=\frac{1}{2}$, experimental and numerical results both indicate that pairing must be either extremely weak or absent, so there would seem to be a regime to which the theory would apply, assuming that it can be extended to $\tilde{\phi}>1$. We return to the issue of pairing in Sec. V.

B. Constraints

In this subsection we begin a fuller and more systematic analysis which begins from the HF approximation but entails a careful study of the role of the constraints. In the present subsection, we explain a functional integral method for handling the constraints exactly. Approximation methods are discussed beginning in Sec. IV C, where the starting point is once again the HF approximation. The present subsection could be skipped on a first reading, but does explain why many statements later in the paper are restricted to nonzero frequencies.

The constraint operators $G(\mathbf{q})$ obey

$$[G(\mathbf{q}),G(\mathbf{q}')]=-G(\mathbf{q}+\mathbf{q}')2i\,\sin\tfrac{1}{2}\mathbf{q}\wedge\mathbf{q}', \tag{4.10}$$

$$[H,G(\mathbf{q})]=0. \tag{4.11}$$

These relations have the property that if all $G(\mathbf{q})$ are replaced by zero throughout, as stipulated by the constraint, then they are still true. Constraints with this property are termed first class, while others are termed second class.[58] Second-class constraints lead to modified commutation relations given by ''Dirac brackets'' in the constrained subspace, and are generally more awkward to handle. An example is the constraint of being in the LLL, applied to one or more charged particles in a magnetic field, which when imposed in the obvious way is second class, and consequently the coordinates x and y of the particle(s) end up not commuting when projected into the LLL. By contrast, systems with only first-class constraints can be viewed as gauge theories, and there are very well-developed methods by which they can be handled.[58] The advantage of the PH approach is that, while the fields are in the LLL from the beginning, the only constraints involved are first class.

The importance of the first-class property of the constraints is that $G(\mathbf{q})$ form a Lie algebra, SU(N) or W_∞, and are constants of the motion $dG(\mathbf{q})/dt=0$ for all $\mathbf{q}$. Thus, before considering them as constraints, $G(\mathbf{q})$ can be viewed as generators of a symmetry algebra of the Hamiltonian. As constants of the motion, the conditions $G(\mathbf{q})=0$, if imposed at the initial time, would hold for all other times. Our procedure, which is a version of the Faddev-Popov functional integral method, will differ somewhat from this, however. To find thermodynamic properties and correlation functions, we begin with the partition function

$$Z=\mathrm{Tr}_{G=0}\, e^{-\beta(H-\mu\hat{N})}, \tag{4.12}$$

where the trace is restricted to states satisfying the constraints. This can be written formally as

$$Z=\mathrm{Tr}\, e^{-\beta(H-\mu\hat{N})}\delta_{G,0}, \tag{4.13}$$

where the trace is taken in the Hilbert space, the Fock space of the fermions c, with no restriction on the fermion number $\hat{N}$. (The $\mu\hat{N}$ term is included to make this look conventional, even though the constraints fix $\hat{N}=N$, so the constrained ensemble is canonical, not grand canonical.) The δ function, which imposes all the constraints, can be given a Fourier representation which essentially, for a non-Abelian group, means integration over the group manifold. Here we return to the U(N) notation that we had for finite N

$$\delta_{G,0}=\int [U^{-1}dU]\, U\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta(\hat{N}-N)}, \tag{4.14}$$

where the first integration is over SU(N) with the invariant (Haar) normalized measure $[U^{-1}dU]$, and the second is over U(1) and imposes $\hat{N}=N$. We can write $U=e^{-\Sigma_a i\beta\lambda_a G_a}$ [where $a=1,\ldots,N^2-1$ runs over a basis of the SU(N) Lie algebra] and convert the unrestricted Tr to a functional integral in the standard way to obtain

$$Z=\int \mathcal{D}[c,c^\dagger][U^{-1}dU]\beta\frac{d\lambda_0}{2\pi}\exp\biggl[-\int_0^\beta d\tau\biggl\{\mathrm{Tr}\, c^\dagger\frac{d}{d\tau}c+H - \mu\hat{N}-i\sum_a \lambda_a G_a - i\lambda_0(\hat{N}-N)\biggr\}\biggr], \tag{4.15}$$

where H, $\hat{N}$, and G_a are given by the standard forms in terms of the Grassman variables $c_{mn}(\tau)$, $c^\dagger_{nm}(\tau)$, and the trace in the exponent is on the U(N) indices. The commutation properties (4.11) were used in obtaining this expression. The λ_a's and $\lambda_0=\theta/\beta$ now play the role of time-independent scalar potentials in the sense of gauge theory. The functional integral results from gauge fixing a manifestly gauge-invariant version,

$$Z=\int \mathcal{D}[c,c^\dagger]\mathcal{D}[\phi]\exp\biggl[-\int_0^\beta d\tau\biggl\{-\mathrm{Tr}\biggl(\frac{d}{d\tau}+i\phi\biggr)c^\dagger c+H - \mu\hat{N}+i\lambda_0 N\biggr\}\biggr], \tag{4.16}$$

in which ϕ stands for all the λ's in $N\times N$ matrix form, is τ dependent, and is functionally integrated over the U(N) Lie algebra. Under a U(N) gauge transformation U, $\phi\mapsto U^{-1}\phi U+U^{-1}dU/d\tau$. This reduces to the previous integral (4.15) by imposing the condition $d\phi/d\tau=0$ inside the functional integral (we are neglecting Faddeev-Popov determinants). This condition is not the same as $\phi=0$ (which is often used instead), which cannot be reached by a gauge transformation from an arbitrary ϕ, since gauge transformations must be periodic in τ with period β. Thus $\int d\tau\,\phi$ cannot be gauged away to zero. The holonomy $Pe^{i\int d\tau\,\phi}$ (P denotes that the integral is path ordered), which is an element of the group U(N), remains. This holonomy is the combination $Ue^{i\theta}$ of the earlier integration variables. Under a τ-independent gauge transformation it is not invariant,

$$Pe^{i\int d\tau\,\phi}\mapsto U^{-1}Pe^{i\int d\tau\,\phi}U, \tag{4.17}$$

and so only the set of eigenvalues of this matrix is gauge invariant. (Note that there are gauge transformations that permute the eigenvalues.) The integral in Eq. (4.15) is over the holonomy, but can be further gauge fixed to leave integration over the eigenvalues only:

$$\int [U^{-1}dU]e^{i\int d\tau\,\Sigma_a\lambda_a G_a} \to\frac{1}{N!}\int_0^{2\pi/\beta}\prod_{\alpha=1}^{N}\frac{d\lambda_\alpha}{2\pi/\beta}\prod_{\gamma<\delta}|e^{i\beta\lambda_\gamma}-e^{i\beta\lambda_\delta}|^2 \times e^{i\int d\tau\,\Sigma_\epsilon\lambda_\epsilon G_{\epsilon\epsilon}}, \tag{4.18}$$

with the measure well known in, for example, random matrix theory (which here has no connection with the similar-looking LJ factors).

The reduction of the constraint integrals to only zero-frequency fields shows that at low temperatures, the integration over these fields is relatively unimportant, since zero-frequency is of zero measure in integrals over frequency that appear in a diagrammatic treatment, as will be used in the following. The non-zero frequency part of the constraints

$G(\mathbf{q},\omega)=0$ will have to come out automatically without help from an integration over a field that enforces it directly [as in the totally gauge unfixed version, Eq. (4.16)]. In Sec. IV C, it will be demonstrated that this occurs.

Finally we note that when developing the HF approximation as in Sec. IV A (or when taking the saddle point of the functional integral as in Appendix B), the Lagrange multiplier $\overline{\lambda}$ is the saddle point value of $i\lambda_0$, so the saddle point value of λ_0 is imaginary. This phenomenon is common in such treatments.

C. Conserving approximations

In this subsection we return to the approximate treatment begun in Sec. IV A, consider response functions, and address the question of whether the constraints are satisfied. The central issue is the use of a so-called conserving approximation, that is an approximation that satisfies the relevant Ward identities, which express the symmetry under U(N) or W_∞ generated by the constraint operators $G(\mathbf{q})$.

The appropriate conserving approximation to use for, say, the density-density response in a normal Fermi liquid, depends on the approximation used for the one-particle properties, that is, the conserving property involves consistency of approximations for different properties. It is well-known that the random-phase approximation corresponds in this sense to the Hartree approximation, and perhaps less well known that the generalized RPA, also called the time-dependent HF approximation, corresponds to the HF approximation (for discussion of conserving approximations, see, e.g., Refs. 59 and 60; for the generalized HF approximation in a FL, see Pines and Noziéres, Chap. 5). These are sometimes stated in terms of Φ derivability, that is approximations that can be derived by making an approximation once and for all for the free energy Φ (or for the thermodynamic potential) in the presence of source fields that couple to the observables of interest (such as the density), and then obtaining response functions in the same approximation by taking functional derivatives with respect to the sources, guaranteeing the same sort of consistency.

The importance of the conserving approximation depends on the nature of the problem. In the example of a normal Fermi liquid, the basic symmetry is conservation of total particle number, which is not broken by Hartree or HF approximations. The conserving approximation is then needed to ensure that the Fermi-liquid relations are satisfied, providing detailed relations among physical quantities. By contrast, in a BCS superconductor, the simplest approximation (which can be viewed as an extension of the HF approximation) violates the conservation of the particle number, and the conserving approximation[59] not only restores gauge invariance (number conservation) but also leads to the prediction of a collective mode, the Anderson-Bogoliubov mode (which is the Goldstone mode connected with the spontaneous symmetry breaking in the case of short-range interactions). Thus the use of a correct approximation has much greater physical consequences in the latter case.

Turning now to the present problem, the HF approximation of Sec. IV A does not break conservation of the total particle number $\hat{N}$. However, the symmetry generators $G(\mathbf{q})=\rho^R(\mathbf{q})$ for $\mathbf{q}\neq 0$ are not conserved by the HF approximation as it stands. The easiest way to see this is that $G(\mathbf{q})$ does not annihilate the HF ground state, which is just the Fermi sea $|\mathrm{FS}\rangle$. Thus this state does not satisfy the constraints $G(\mathbf{q})|\mathrm{FS}\rangle=0$ for $\mathbf{q}\neq 0$. It is also clear that the HF effective Hamiltonian [Eq. (4.6], does not commute with these $G(\mathbf{q})$. The solution to this problem will have to use the conserving approximation appropriate to our HF approximation. Since there is a conserved quantity for all $\mathbf{q}$'s, the results will be even more striking than in cases such as the BCS theory where only a global symmetry was broken. We note that the Fermi sea can be made invariant by projecting to an invariant subspace as in Eq. (3.23). However, such a projection necessitates that further work be numerical. Analytical work, and thus conceptual understanding, can be achieved only by persevering with the gauge theory approach. Rather than give up the Fermi sea trial state and the HF energies, and searching for some other, invariant, starting point, we keep it and take care of the constraints by the following conserving approximation.

The conserving approximation will be illustrated here by the calculation of the $\rho^R-\rho^R$, $\rho^R-\rho^L$, and $\rho^L-\rho^L$ imaginary-time response functions (more precisely, the generalized susceptibilities), defined in Fourier space by

$$\chi_{ij}(\mathbf{q},\omega_n)(2\pi)^2\delta(\mathbf{q}+\mathbf{q}')\beta\delta_{\omega_n+\omega_{n'},0} =\langle\rho^i(\mathbf{q},\omega_n)\rho^j(\mathbf{q}',\omega_{n'})\rangle, \tag{4.19}$$

in which i and j can be R or L, ω_n are the usual Matsubara frequencies, and it is implicit that the connected part of the function is taken, thus dropping a δ-function term containing $\langle\rho^i\rangle$'s. The conserving approximation that corresponds to the HF approximation takes the form of the sum of all ring and ladder diagrams. The Green's function lines in the diagrams are the HF Green's functions

$$\mathcal{G}(\mathbf{k},\omega_\nu)=(i\omega_\nu-\xi_{\mathbf{k}})^{-1}. \tag{4.20}$$

The usual Dyson-equation argument leads to formulas in terms of the one-interaction irreducible susceptibilities, as discussed in Sec. II A, defined as those diagrams that do not become disconnected when one interaction line is cut (note that we disregard the Hartree self-energy diagrams that are implicitly included in our HF Green's functions, which means we are treating the diagrams here as skeleton diagrams; such terms would be absent in any case for a long-range interaction due to the neutralizing background). These formulas, which are completely general, are (all χ's have the same arguments $\mathbf{q}$, ω_n)

$$\chi_{LL}=\frac{\chi_{LL}^{\mathrm{irr}}}{1+\tilde{V}(\mathbf{q})\chi_{LL}^{\mathrm{irr}}}, \tag{4.21}$$

$$\chi_{RL}=\frac{\chi_{RL}^{\mathrm{irr}}}{1+\tilde{V}(\mathbf{q})\chi_{LL}^{\mathrm{irr}}}, \tag{4.22}$$

$$\chi_{RR}=\chi_{RR}^{\mathrm{irr}}-\chi_{RL}^{\mathrm{irr}}\frac{\tilde{V}(\mathbf{q})}{1+\tilde{V}(\mathbf{q})\chi_{LL}^{\mathrm{irr}}}\chi_{LR}^{\mathrm{irr}}. \tag{4.23}$$

Note also that $\chi_{LR}(\mathbf{q},\omega_n)=\chi_{RL}(-\mathbf{q},-\omega_n)$. The conserving approximation is now the statement that the various χ^{irr}'s are to be calculated (for $\omega \neq 0$) as the sum of the ladder diagrams, with the HF Green's functions. Since ρ^L is the physical density, $\chi_{LL}^{\rm irr}$ is the one of most physical interest for long-range $\tilde{V}(\mathbf{q})$, such as Coulomb interactions.

We begin with $\chi_{RR}^{\rm irr}$, so as to show that at $\omega \neq 0$ the fluctuations in the constraints $G(\mathbf{q})$ vanish in our approximation. The Feynman rule for the interaction can be read off in the standard way;[57] it includes the wave-vector-dependent phase factor as well as $\tilde{V}(\mathbf{q})$. Also, there is a phase factor in the ρ^R vertices, as in Eq. (3.15). Note that those in the interaction arise from the phase factors in the physical density ρ^L [Eq. (3.16)]. In the ladder diagrams for $\chi_{RR}^{\rm irr}$ the structure of the momenta is such that *all the phase factors cancel*, as the industrious reader will verify. Note that this is an exact statement, and not only valid at small wave vectors, whether internal or external, so the exponential defining the phase factor was not expanded in a Taylor series. Consequently, *for the ladder diagrams for* $\chi_{RR}^{\rm irr}$ *only*, the ladder series is identical to the same approximation to the irreducible susceptibility in the usual density

$$\rho(\mathbf{q}) = \int \frac{d^2k}{(2\pi)^2} c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}}, \tag{4.24}$$

in a model with Hamiltonian

$$H = \frac{1}{2}\int \frac{d^2k_1\, d^2k_2\, d^2q}{(2\pi)^6} \tilde{V}(\mathbf{q}) c^\dagger_{\mathbf{k}_1-(1/2)\mathbf{q}} c^\dagger_{\mathbf{k}_2+(1/2)\mathbf{q}} \times c_{\mathbf{k}_2-(1/2)\mathbf{q}} c_{\mathbf{k}_1+(1/2)\mathbf{q}} \tag{4.25}$$

with no kinetic-energy term. This could be phrased by saying that there is the ordinary, Galilean-invariant kinetic-energy term with zero magnetic field, but the mass m_0 is infinite. We call this latter model the zero-field, infinite-mass (ZFIM) model. Note that the HF approximations in the two models also coincide, because the phase factors disappeared there also. In the ZFIM model, $[\rho(\mathbf{q}), H] = 0$ for all $\mathbf{q}$, so the model possesses a gauge symmetry, whether or not we wish to impose a constraint $\rho =$ const. In fact, if such a constraint were imposed in this model, there would be no states that satisfied it at all. The reason (in classical language) is that in a continuum model, any configuration of point particles clearly has a nonconstant density. In a similar model on a lattice, solutions to the constraint exist only if the value of the particle number required by the constraint at each site is an integer, since these are the eigenvalues of the number operator for each site. This cannot be satisfied if we take the continuum limit (zero lattice spacing) at a fixed average density. In our system representing the LLL, which is in the continuum, many solutions to the constraint do exist, provided we choose (similarly to the lattice ZFIM model) the constrained value of the total number to be the same as the range of the right indices n, as we have done. Therefore, in the ZFIM model, we will consider the gauge symmetry [or conservation of $\rho(\mathbf{q})$], but not require a constraint to be satisfied.

Explicitly, we can write $\chi_{RR}^{\rm irr}$ (or $\chi^{\rm irr}$ in the ZFIM model) in terms of the ladder sum, which is the solution to an integral equation (we define here various quantities to be used afterwards)

$$\begin{aligned}\chi_{RR}^{\rm irr}(\mathbf{q}, i\omega_\nu) &= -\frac{1}{\beta}\sum_n \int \frac{d^2k}{(2\pi)^2} \Lambda(\mathbf{k},\mathbf{q},i\omega_\nu) \mathcal{G}\left(\mathbf{k}+\frac{1}{2}\mathbf{q}, \omega_n+\omega_\nu\right) \mathcal{G}\left(\mathbf{k}-\frac{1}{2}\mathbf{q}, \omega_n\right) \\ &= -\int \frac{d^2k}{(2\pi)^2} \Lambda(\mathbf{k},\mathbf{q},i\omega_\nu) \frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}}) - f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}} - \xi_{\mathbf{k}-(1/2)\mathbf{q}} - i\omega_\nu}.\end{aligned} \tag{4.26}$$

Here $\Lambda(\mathbf{k},\mathbf{q},i\omega_\nu)$ is a one-particle irreducible vertex function,

$$\begin{aligned}\Lambda(\mathbf{k},\mathbf{q},i\omega_\nu) &= 1 - \frac{1}{\beta}\sum_n \int \frac{d^2k_1}{(2\pi)^2} \mathcal{G}\left(\mathbf{k}_1+\frac{1}{2}\mathbf{q}, \omega_n+\omega_\nu\right) \mathcal{G}\left(\mathbf{k}_1-\frac{1}{2}\mathbf{q}, \omega_n\right) \Gamma(\mathbf{k}_1,\mathbf{k},\mathbf{q},i\omega_\nu) \\ &= 1 - \int \frac{d^2k_1}{(2\pi)^2} \frac{f(\xi_{\mathbf{k}_1+(1/2)\mathbf{q}}) - f(\xi_{\mathbf{k}_1-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}_1+(1/2)\mathbf{q}} - \xi_{\mathbf{k}_1-(1/2)\mathbf{q}} - i\omega_\nu} \Gamma(\mathbf{k}_1,\mathbf{k},\mathbf{q},i\omega_\nu),\end{aligned} \tag{4.27}$$

which we have written in terms of the particle-hole scattering series (the ladders with external Green's function lines removed),

$$\begin{aligned}\Gamma(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_\nu) &= \tilde{V}(\mathbf{k}'-\mathbf{k}) - \frac{1}{\beta}\sum_n \int \frac{d^2k_1}{(2\pi)^2} \Gamma(\mathbf{k},\mathbf{k}_1,\mathbf{q},i\omega_\nu) \mathcal{G}\left(\mathbf{k}_1+\frac{1}{2}\mathbf{q}, \omega_n+\omega_\nu\right) \mathcal{G}\left(\mathbf{k}_1-\frac{1}{2}\mathbf{q}, \omega_n\right) \tilde{V}(\mathbf{k}_1-\mathbf{k}') \\ &= \tilde{V}(\mathbf{k}'-\mathbf{k}) - \int \frac{d^2k_1}{(2\pi)^2} \Gamma(\mathbf{k},\mathbf{k}_1,\mathbf{q},i\omega_\nu) \frac{f(\xi_{\mathbf{k}_1+(1/2)\mathbf{q}}) - f(\xi_{\mathbf{k}_1-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}_1+(1/2)\mathbf{q}} - \xi_{\mathbf{k}_1-(1/2)\mathbf{q}} - i\omega_\nu} \tilde{V}(\mathbf{k}_1-\mathbf{k}').\end{aligned} \tag{4.28}$$

(Note that, in this approximation, the scattering function depends only on the difference ω_ν of the Matsubara frequencies in the external fermion lines, and this is why we are able to perform the frequency sums explicitly.)

Before analyzing these equations in detail, we pause to point out that for $\mathbf{q}$, $\omega = i\omega_\nu$ small and real, they have the form standard in Fermi liquid theory [see Pines and Nozières (PN),[61] and especially Nozières,[62] for the full, formal treatment], with the approximation that the $f_{\mathbf{kk}'}$ function on the Fermi surface is taken to be the lowest-order approximation as already given in Eq. (4.5), for spinless fermions, and this is just the content of the generalized HF approximation (see PN, Ch. 5). The Landau parameters F_ℓ are then given by

$$F_\ell = \mathcal{N}(0) f_\ell, \tag{4.29}$$

$$f_\ell = \int \frac{d\theta_{\mathbf{kk}'}}{2\pi} f_{\mathbf{kk}'} \cos \ell \theta_{\mathbf{kk}'}, \tag{4.30}$$

for $\ell \geqslant 0$, where, as before, $\mathbf{k}\cdot\mathbf{k}' = k_F^2 \cos\theta_{\mathbf{kk}'}$ for $|\mathbf{k}| = |\mathbf{k}'| = k_F$. In particular, we notice that, since the density of states at the Fermi energy $\mathcal{N}(0) = m^*/2\pi$, and since the bare kinetic energy is zero, comparison with Eq. (4.8) yields

$$F_1 = -1. \tag{4.31}$$

This is a particular case of the relation

$$m^*/m_0 = 1 + F_1 \tag{4.32}$$

in ordinary two-dimensional Galilean-invariant Fermi liquids with bare mass m_0. We can view the ZFIM model as such a system but with $m_0 = \infty$, from which $F_1 = -1$ follows. This is the value that would usually be interpreted as the borderline of stability of the system; however, usually this view is taken because the bare mass is finite and the effective mass vanishes, and the latter causes instability. Here the effective mass is finite, so the system is not unstable, and moreover is held right at this point by this symmetry. We take it as implying that the ladder series must be analyzed with even greater attention than usual to the limit $\omega \to 0$, $\mathbf{q} \to 0$, particularly for the $\ell = 1$ angular mode. We also point out a contrast with HLR, where this formula was invoked, but with the bare (or band) mass m in place of m_0, and was connected with Kohn's theorem and the f-sum rule. There the interesting limit was $m \to 0$ (to send the cyclotron mode to infinite frequency), rather than ∞. The present discussion is clearly distinct, though it must be related at some deeper level.

In Fermi-liquid theory, relations like that above are derived through Ward identities connected with symmetries of the problem, and the symmetries are global, so the relations are most useful only at small $\mathbf{q}$ or ω. Next we will derive a Ward-identity relationship between Λ and the self-energy Σ within the HF approximation, in a way more directly connected with the symmetry generated by the ρ^R's, and valid for all $\omega \neq 0$ and $\mathbf{q}$.

First we express the HF approximation as a pair of self-consistent equations:

$$\mathcal{G}(\mathbf{k},\omega_n) = [i\omega_n - (\Sigma(\mathbf{k}) - \overline{\lambda} - \mu)]^{-1}, \tag{4.33}$$

$$\begin{aligned}\Sigma(\mathbf{k}) &= -\frac{1}{\beta}\sum_n \int \frac{d^2k_1}{(2\pi)^2} \tilde{V}(\mathbf{k}-\mathbf{k}_1)\mathcal{G}(\mathbf{k}_1,\omega_n) \\ &= -\int \frac{d^2k_1}{(2\pi)^2} \tilde{V}(\mathbf{k}-\mathbf{k}_1) f(\xi_{\mathbf{k}_1}),\end{aligned} \tag{4.34}$$

where $\xi_{\mathbf{k}} = \Sigma(\mathbf{k}) - \mu - \overline{\lambda}$ as before (the direct term has been dropped as it plays no role in the following, for the one-interaction irreducible functions; it is absent anyway for the long-range interaction case). Then

$$\begin{aligned}\Sigma\left(\mathbf{k}+\frac{1}{2}\mathbf{q}\right) - \Sigma\left(\mathbf{k}-\frac{1}{2}\mathbf{q}\right) - i\omega_\nu &= -i\omega_\nu - \frac{1}{\beta}\sum_n \int \frac{d^2k_1}{(2\pi)^2}\left[\tilde{V}\left(\mathbf{k}+\frac{1}{2}\mathbf{q}-\mathbf{k}_1\right) - \tilde{V}\left(\mathbf{k}-\frac{1}{2}\mathbf{q}-\mathbf{k}_1\right)\right]\mathcal{G}(\mathbf{k}_1,\omega_n) \\ &= -i\omega_\nu - \frac{1}{\beta}\sum_n \int \frac{d^2k_1}{(2\pi)^2}\tilde{V}(\mathbf{k}-\mathbf{k}_1)\mathcal{G}\left(\mathbf{k}_1+\frac{1}{2}\mathbf{q},\omega_n+\omega_\nu\right) \\ &\quad \times\left[\Sigma\left(\mathbf{k}_1+\frac{1}{2}\mathbf{q}\right) - \Sigma\left(\mathbf{k}_1-\frac{1}{2}\mathbf{q}\right) - i\omega_\nu\right]\mathcal{G}\left(\mathbf{k}_1-\frac{1}{2}\mathbf{q},\omega_n\right)\end{aligned} \tag{4.35}$$

after shifting dummy variables. But from Eqs. (4.27) and (4.28), $-i\omega_\nu\Lambda(\mathbf{k},\mathbf{q},i\omega_\nu)$ obeys the same inhomogeneous integral equation, the solution of which should be unique, so we conclude that

$$\begin{aligned}i\omega_\nu\Lambda(\mathbf{k},\mathbf{q},i\omega_\nu) &= i\omega_\nu - \Sigma\left(\mathbf{k}+\frac{1}{2}\mathbf{q}\right) + \Sigma\left(\mathbf{k}-\frac{1}{2}\mathbf{q}\right) \\ &= i\omega_\nu - \xi_{\mathbf{k}+(1/2)\mathbf{q}} + \xi_{\mathbf{k}-(1/2)\mathbf{q}},\end{aligned} \tag{4.36}$$

which is the desired Ward identity (compare Ref. 59). The left-hand side is the vertex function for $\partial\rho^R(\mathbf{q})/\partial\tau$, which should vanish since $\rho^R(\mathbf{q})$ commutes with the Hamiltonian. This implies that, if Λ is viewed as the scattering amplitude for a fermion scattering off a potential coupling to ρ^R, or for creating or destroying a particle-hole pair, then the amplitude vanishes if both particles are on shell, that is if their frequencies $i\omega_n$ satisfy $i\omega_n = \xi_{\mathbf{k}}$. This suggests (following a similar argument in Ref. 63, that was inspired by Ref. 64) that in the on-shell states (energy eigenstates), if they satisfy the constraints $G(\mathbf{q}) = 0$, then the latter property is actually preserved in the time evolution, in spite of its apparent violation in the HF states. This of course is because the calculation we have done is not the naive one of looking at the states as noninteracting particles; rather we used the conserving approximation. It appears that the fermion *excitations* can be viewed as real physical excitations after all, satisfying the constraint conditions on physical states, even though the *operators* $c^\dagger$ are not gauge invariant and so would connect invariant to noninvariant states. These physical fermion excitations, which are dressed by the fluctuations around the HF states, are the physical composite or (as we shall see) neutral fermions discussed in Ref. 12 and in Sec. II B.

Now we return to our original goal of calculating χ_{RR}^{irr} in the ladder approximation. Using the Ward identity and Eq. (4.26), and assuming $\omega \neq 0$, we find

$$\chi_{RR}^{\mathrm{irr}}(\mathbf{q},i\omega_\nu)=\frac{1}{i\omega_\nu}\int\frac{d^2k}{(2\pi)^2}[f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})]=0. \tag{4.37}$$

Another response function containing ρ^R that should vanish is $\chi_{RL}^{\mathrm{irr}}(\mathbf{q},i\omega_\nu)$. In this case, the appearance of ρ^L in place of one ρ^R implies that the phase factors do not all cancel, and on using the Ward identity for the ρ^R vertex we obtain

$$\chi_{RL}^{\mathrm{irr}}(\mathbf{q},i\omega_\nu)=\frac{1}{i\omega_\nu}\int\frac{d^2k}{(2\pi)^2}[f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})]e^{i\mathbf{k}\wedge\mathbf{q}}=0, \tag{4.38}$$

since shifting $\mathbf{k}$ by $\mp\frac{1}{2}\mathbf{q}$ has no effect on the phase factor.

As promised, we have shown that the conserving approximation guarantees that there are no fluctuations in $\rho^R(\mathbf{q})$, at least for nonzero frequency. For zero frequency, the Lagrange multiplier fields $\lambda(\mathbf{q})$ (or the subset of diagonal elements, according to the final gauge-fixed form) enter to give the same result, but we will not show this explicitly. Similar issues were addressed extensively in the literature on slave bosons and heavy fermions in the 1980s (see, for example, Refs. 65,63 and 66–69), and later in connection with theories of high-T_c superconductors and quantum magnets. These problems also involve constraints, but these are usually Abelian and generate only U(1). It is still frequently stated incorrectly in the literature that in the functional-integral saddle-point approach to such problems, ''the constraints are satisfied only on the average.'' In fact, as was well known to several workers (such as the cited authors) in the field in the 1980s, the correct RPA or $1/N$ (i.e., conserving) treatment of fluctuations yields just the same sort of results we have just derived, namely the vanishing of the vertex function for, and of all correlation functions containing, the constraint operators [like our $G(\mathbf{q})$], to all orders in the fluctuations. Thus *the average of, and all fluctuations in, the constraints vanish*, which means that the constraints are satisfied in every order of approximation, when this is set up correctly. (The extension to all orders for the present problem will be discussed later.)

It remains to examine χ_{LL}^{irr}. This will be undertaken in Secs. IV D and IV E.

D. Asymptotics of the ladder series

In this subsection we continue the analysis of the conserving approximation of Sec. IV C. We examine the behavior of the ladder series at small $\mathbf{q}$ and ω_ν, first to elucidate the mechanism behind the vanishing of χ_{RR}^{irr}, and then, in Sec. IV E, the results are applied to the calculation of the physical density-density response function χ_{LL}^{irr}.

The equation for Γ can be rewritten

$$\int\frac{d^2k_1}{(2\pi)^2}\left\{(2\pi)^2\delta(\mathbf{k}'-\mathbf{k}_1)+\tilde{V}(\mathbf{k}'-\mathbf{k}_1)\times\left(\frac{f(\xi_{\mathbf{k}_1+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}_1-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}_1+(1/2)\mathbf{q}}-\xi_{\mathbf{k}_1-(1/2)\mathbf{q}}-i\omega_\nu}\right)\right\}\times\Gamma(\mathbf{k},\mathbf{k}_1,\mathbf{q},i\omega_\nu)=\tilde{V}(\mathbf{k}-\mathbf{k}'), \tag{4.39}$$

which shows that it is a Fredholm integral equation, where the integral kernel appears in the curly brackets on the left-hand side, and contains $\mathbf{q}$ and ω_ν as parameters. It implies that Γ is $\tilde{V}$ times the inverse integral operator. The inverse could be calculated by finding the eigenvalues and eigenfunctions of the integral operator on the left.

At $i\omega_\nu=0$ (which could be viewed as the limit $i\omega_\nu\to 0$), one zero eigenvector can be found for all $\mathbf{q}$ by use of the Ward identity proved in Sec. IV C; it is $\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}$ [see Eq. (4.35)]. Thus for small $i\omega_\nu$, we expect to have, for all $\mathbf{q}$'s, an eigenvector approximately $\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}$, with an eigenvalue tending to zero with $i\omega_\nu$. If $\mathbf{q}\to 0$ also, we obtain

$$\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}\simeq\mathbf{q}\cdot\mathbf{v_k}, \tag{4.40}$$

where $\mathbf{v_k}=\nabla_\mathbf{k}\xi_\mathbf{k}$. At small $\mathbf{q}$, the nontrivial part of the integral kernel becomes

$$\tilde{V}(\mathbf{k}'-\mathbf{k}_1)\frac{\partial f}{\partial\varepsilon}\bigg|_{\xi_\mathbf{k}}, \tag{4.41}$$

which for zero temperature T is concentrated at $k=k_F$ (indeed, for all $\mathbf{q}$, the difference of Fermi functions is nonzero only in a shell of width of order q around k_F). But this limit of the kernel is independent of $\mathbf{q}$, so in addition to the eigenfunction just found, which is proportional to cos $\theta_\mathbf{k}$ on the Fermi surface, there is another proportional to sin $\theta_\mathbf{k}$. Note that these eigenfunctions, in the spirit of a Fermi-liquid analysis in terms of $\delta n_\mathbf{k}$ or a deformation of the Fermi surface, are just rigid displacements of the Fermi sea, respectively parallel and perpendicular to $\mathbf{q}$. The second eigenfunction is not a zero mode for $\mathbf{q}\neq 0$, so is expected to acquire an eigenvalue that is nonzero as $i\omega_\nu\to 0$, but vanishes as $\mathbf{q}\to 0$.

For general values of the ratio $i\omega_\nu/q$ the integral equation and the eigenvalue problem are not easy to analyze, even for $i\omega_\nu$, $\mathbf{q}$ small, where the eigenvalue equation takes the form

$$A(\mathbf{k},\mathbf{q},i\omega_\nu)+\int\frac{d^2k_1}{(2\pi)^2}\tilde{V}(\mathbf{k}-\mathbf{k}_1)\times\frac{\mathbf{q}\cdot\mathbf{v}_{\mathbf{k}_1}}{\mathbf{q}\cdot\mathbf{v}_{\mathbf{k}_1}-i\omega_\nu}\frac{\partial f}{\partial\varepsilon}\bigg|_{\xi_{\mathbf{k}_1}}A(\mathbf{k}_1,\mathbf{q},i\omega_\nu)=\lambda(\mathbf{q},i\omega_\nu)A(\mathbf{k},\mathbf{q},i\omega_\nu). \tag{4.42}$$

This form of equation is standard in Fermi-liquid theory, with $\tilde{V}(\mathbf{k}-\mathbf{k}_1)$ replaced by $-f_{\mathbf{k}\mathbf{k}_1}$. At $T=0$, $\partial f/\partial\varepsilon=-\delta(\xi_{\mathbf{k}})$ and the equation can in principle be solved for $\mathbf{k}$ on the Fermi surface, and *these values of the eigenfunction determine it elsewhere*. Accordingly we might expand both A and $\tilde{V}$ in terms of Fourier modes $\cos\ell\theta_{\mathbf{k}}$, $\sin\ell\theta_{\mathbf{k}}$, $\ell=0,1,\ldots$ for $|\mathbf{k}|=k_F$. For $i\omega_\nu/|\mathbf{q}|v_F\neq 0$, the Fourier modes are mixed by the integral kernel, so that all components of

$$\begin{aligned}-\tilde{V}(\mathbf{k}-\mathbf{k}')&=f_0+2\sum_{\ell=1}^{\infty}f_\ell\cos\ell\theta_{\mathbf{k}\mathbf{k}'}\\&=f_0+2\sum_{\ell=1}^{\infty}f_\ell(\cos\ell\theta_{\mathbf{k}}\cos\ell\theta_{\mathbf{k}'}\\&\quad+\sin\ell\theta_{\mathbf{k}}\sin\ell\theta_{\mathbf{k}'})\end{aligned}\tag{4.43}$$

are involved. We have seen that the $\ell=1$ mode and f_1 are crucial to the analysis and must be kept. The other Landau parameters F_ℓ take no special values, and merely produce finite renormalizations of the response functions (some identities are implied by the existence of the zero mode for all $\mathbf{q}$, but these bring in derivatives of $\mathbf{v_k}$ and thus parameters that lie outside of Fermi-liquid theory). We propose just to drop these effects so as to obtain the simplest possible approximation that is still conserving. This can be done by replacing f_ℓ for $\ell\neq 1$ by zero, or more accurately by assuming that the only eigenfunctions A that are needed are just $\mathbf{q}\cdot\mathbf{v_k}/q$, $\mathbf{q}\wedge\mathbf{v_k}/q$ (which are the correct continuations off $|\mathbf{k}|=k_F$). We will actually use this even to higher order in q, as we will see is necessary.

With this further approximation, the eigenvalues corresponding to the two eigenfunctions can be evaluated. The final result for Γ is

$$\Gamma(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_\nu)=\frac{\mathbf{q}\cdot\mathbf{v_k}\,\mathbf{q}\cdot\mathbf{v_{k'}}}{\omega_\nu^2\chi_0(\mathbf{q},i\omega_\nu)}-\frac{\mathbf{q}\wedge\mathbf{v_k}\,\mathbf{q}\wedge\mathbf{v_{k'}}}{q^2\chi_0^\perp(\mathbf{q},i\omega_\nu)},\tag{4.44}$$

where

$$\chi_0(\mathbf{q},i\omega_\nu)=-\int\frac{d^2k}{(2\pi)^2}\frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}-i\omega_\nu}\tag{4.45}$$

is the "density-density" response function of a Fermi gas with dispersion $\xi_{\mathbf{k}}$, and

$$\begin{aligned}\chi_0^\perp(\mathbf{q},i\omega_\nu)&=-\frac{1}{2}\mathcal{N}(0)v_F^2\\&\quad-\int\frac{d^2k}{(2\pi)^2}\left(\frac{\mathbf{q}\wedge\mathbf{v_k}}{|\mathbf{q}|}\right)^2\frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}-i\omega_\nu}\end{aligned}\tag{4.46}$$

is the transverse "current-current" response function of the same Fermi gas, including the $\mathbf{q}$-, $i\omega_\nu$-independent contact ("diamagnetic") term. χ_0 arose in a similar way from the longitudinal current-current response, on using the continuity equation. Note that what we are calling the "density" and "current," though natural in appearance, are *not* to be identified with the physical density and current.

The above expressions for χ_0 and $\chi_0^\perp$ are valid for any $\mathbf{q}$ and $i\omega_\nu$. On the real frequency axis, at ω/qv_F and $\mathbf{q}$ small, they become

$$\chi_0(\mathbf{q},\omega+i0^+)=\mathcal{N}(0)+i\mathcal{N}(0)\omega/(qv_F),\tag{4.47}$$

$$\chi_0^\perp(\mathbf{q},\omega+i0^+)=q^2\chi_d^*+i\omega k_F/(2\pi q).\tag{4.48}$$

Here χ_d^* is the diamagnetic susceptibility of the Fermi gas with dispersion $\xi_{\mathbf{k}}$. It is a non-Fermi-liquid property that involves derivatives of $\mathbf{v_k}$ at k_F; if $\xi_{\mathbf{k}}$ were $=(\mathbf{k}^2-k_F^2)/2m^*$, then χ_d^* would be $=-1/(12\pi m^*)$. These imply that the eigenvalues of the longitudinal and transverse eigenmodes of the integral kernel above vanish in the ways predicted in this limit. This involved the cancellation of the diamagnetic term in the current-current response in both cases; this cancellation is well known in normal fluids (i.e., nonsuperfluids).

We can now show that even this further approximation is conserving in the sense discussed in Sec. IV C. Using the above form of Γ we can calculate

$$\chi_{RR}^{\text{irr}}=\chi_0-\chi_0(\chi_0)^{-1}\chi_0=0,\tag{4.49}$$

where the second term is the contribution of Γ, for all $\mathbf{q}$ and $i\omega_\nu\neq 0$. In this calculation, the transverse mode in Γ did not contribute. A similar calculation shows that $\chi_{RL}^{\text{irr}}=0$. An exact treatment of the ladder series in the regime $\omega/qv_F\ll 1$ and $q\ll k_F$ yields the same form with all χ_0's replaced by $\chi_0/(1+F_0)$, and the cancellation still occurs, in agreement with Sec. IV C.

E. Physical response functions

In this subsection we calculate χ_{LL}^{irr}, the physical density-density response function, and its limits, the compressibility and longitudinal conductivity. We also consider the scattering of the fermions by an external potential, and the expression for the current density.

1. Density-density response function

As already remarked, the fact that the ρ^L vertex contains the opposite phase factor from that in ρ^R means that not all the phase factors cancel in χ_{LL}^{irr}; instead, those at the two vertices at the ends of the ladder are doubled. We have

$$\chi_{LL}^{\text{irr}}=\chi_0+\int\frac{d^2k\,d^2k'}{(2\pi)^4}\frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}-i\omega_\nu}\Gamma(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_\nu)\frac{f(\xi_{\mathbf{k}'+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}'-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}'+(1/2)\mathbf{q}}-\xi_{\mathbf{k}'-(1/2)\mathbf{q}}-i\omega_\nu}e^{i\mathbf{k}\wedge\mathbf{q}-i\mathbf{k}'\wedge\mathbf{q}}.\tag{4.50}$$

However, by comparison with $\chi_{RR}^{\mathrm{irr}}=\chi_{RL}^{\mathrm{irr}}=\chi_{LR}^{\mathrm{irr}}=0$, this simplifies to

$$\chi_{LL}^{\mathrm{irr}}=-\int\frac{d^2k}{(2\pi)^2}(e^{i\mathbf{k}\wedge\mathbf{q}}-1)(e^{-i\mathbf{k}\wedge\mathbf{q}}-1)\frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}-i\omega_\nu}+\int\frac{d^2k\,d^2k'}{(2\pi)^4}(e^{i\mathbf{k}\wedge\mathbf{q}}-1)$$
$$\times\frac{f(\xi_{\mathbf{k}+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}+(1/2)\mathbf{q}}-\xi_{\mathbf{k}-(1/2)\mathbf{q}}-i\omega_\nu}\Gamma(\mathbf{k},\mathbf{k}',\mathbf{q},i\omega_\nu)\frac{f(\xi_{\mathbf{k}'+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}'-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}'+(1/2)\mathbf{q}}-\xi_{\mathbf{k}'-(1/2)\mathbf{q}}-i\omega_\nu}(e^{-i\mathbf{k}'\wedge\mathbf{q}}-1). \tag{4.51}$$

For small $\mathbf{q}$, we now expand the phase factor. The first term is then the form found in Refs. 24–26. It is the same as putting $\rho^L-\rho^R$ in place of ρ^L, which goes as $\sim i\mathbf{k}\wedge\mathbf{q}$ at small nonzero $\mathbf{q}$. The second term is the ladder series with the insertion $(\mathbf{k}\wedge\mathbf{q})(\mathbf{k}'\wedge\mathbf{q})$ at the two vertices. This exhibits the effectively dipolar nature of the coupling of an external scalar potential to the physical density: the fermions carry a dipole moment $\wedge\mathbf{k}$, as found in Refs. 12 and 24–26 and discussed in Sec. II. In Γ, only the transverse mode now contributes, and we obtain

$$\chi_{LL}^{\mathrm{irr}}=q^2m^*[\overline{\rho}+m^*\chi_0^\perp(\mathbf{q},i\omega_\nu)]-\frac{q^2[\overline{\rho}+m^*\chi_0^\perp(\mathbf{q},i\omega_\nu)]^2}{\chi_0^\perp(\mathbf{q},i\omega_\nu)}$$
$$=-q^2\overline{\rho}[\overline{\rho}+m^*\chi_0^\perp(\mathbf{q},i\omega_\nu)]/\chi_0^\perp(\mathbf{q},i\omega_\nu). \tag{4.52}$$

Note that in the numerator, the $\overline{\rho}$'s occur because of the absence of a ''diamagnetic'' term to cancel them, and in writing the remainder of the numerator as $\chi_0^\perp$ we have neglected the difference between $\mathbf{k}/m^*$ and $\mathbf{v}_\mathbf{k}$, which affects the coefficient of the term in $\chi_0^\perp$ quadratic in $\mathbf{q}$. This term can be neglected anyway in the following. In the small $\omega/(qv_F)$, $\mathbf{q}$ region we then have

$$\chi_{LL}^{\mathrm{irr}}(\mathbf{q},\omega+i0^+)=\frac{\overline{\rho}^2}{-\chi_d^*-i\omega k_F/(2\pi q^3)}. \tag{4.53}$$

This is similar in form to the result obtained by HLR, or the renormalized version of it according to the scenario discussed in Sec. II A, if we note that $\overline{\rho}=1/(2\pi\tilde{\phi})$ in general (and $\tilde{\phi}=1$ here), except that the 1 in the denominator in Eq. (2.5) has been dropped. That 1 came from the Chern-Simons term, which couples longitudinal and transverse fluctuations; by contrast, in the conserving approximation in the present approach, the ladder propagator Γ does not couple these modes. Note that the first term in the first line of Eq. (4.52) is essentially the result of Refs. 24–26,

$$\chi_{LL}^{\mathrm{irr}}=q^2m^*[\overline{\rho}+m^*\chi_0^\perp(\mathbf{q},i\omega_\nu)], \tag{4.54}$$

which behaves differently at low ω and $\mathbf{q}$, as we will see.

We now take various limits of this expression. As $\omega\to0$, we obtain

$$\frac{dn}{d\mu}\equiv\lim_{|\mathbf{q}|\to0}\chi_{LL}^{\mathrm{irr}}(\mathbf{q},0)=-\overline{\rho}^2/\chi_d^*, \tag{4.55}$$

which is finite and positive, so the system is compressible as in HLR, though again the expression differs from that in the scenario of Sec. II A, as given in Eq. (2.13). Though we used the approximate form for Γ, our result is exact within the ladder (conserving) approximation.

To obtain the low-frequency longitudinal conductivity, relevant to the surface acoustic wave experiments, we define a relevant limit:

$$\sigma_{xx}(\mathbf{q})=\lim_{\omega/q\to0}\ \lim_{\substack{q\to0\\ \omega/q\ \mathrm{fixed}}}\frac{-i\omega}{q^2}\chi_{LL}^{\mathrm{irr}}(\mathbf{q},\omega+i0^+) \tag{4.56}$$

for $\mathbf{q}$ parallel to $\hat{x}$ (the conductivity should always be viewed as the response to the total electric field, so it is related to the irreducible response). Here ''lim'' means that we keep the leading nonzero term. This limit corresponds to considering a long-wavelength sound wave, so $|\mathbf{q}|$ is small $\ll k_F$ and $\omega=|\mathbf{q}|v_s$, and then taking the sound velocity v_s to zero (i.e., $v_s\ll v_F$). Then we obtain

$$\sigma_{xx}(q)=\overline{\rho}^2\frac{2\pi q}{k_F}=\frac{q}{2\pi k_F}, \tag{4.57}$$

in exact agreement with HLR for $\tilde{\phi}=1$. There a different procedure was used to define $\sigma_{xx}(q)$, as given by HLR [Eq. (B4.a)]. That and the present definition give the same result both in the RPA of HLR and in the present approximation. This result was expected to be very robust on Fermi-liquid grounds, within the scenario discussed in Sec. II A, since it corresponds to the transverse conductivity of an ordinary Fermi liquid, which is unrenormalized in Fermi-liquid theory. Remarkably, it is the same here, in spite of other differences in the structure of the expressions. This result is not obtained from expression (4.54).[24] It is also remarkable how the factor $\overline{\rho}$, which came from a standard gauge-invariance result for the usual Fermi liquid, here plays one of the roles played in the CS theory by σ_{xy} ($=\overline{\rho}$ in our units). This effect, that the ''current'' response at $\omega/q\to0$ of a Fermi gas to a scalar potential coupled to the dipolar expression for the density gives the Hall conductivity, was pointed out by Störmer.[70]

Finally the spectral density for $\chi_{LL}^{\mathrm{irr}}(\mathbf{q},\omega)$ implied by Eq. (4.53), is, at low frequency,

$$\chi_{LL}^{\mathrm{irr}\,\prime\prime}(\mathbf{q},\omega)=\frac{\omega k_F\overline{\rho}^2/(2\pi q^3)}{\chi_d^{*2}+\omega^2k_F^2/(2\pi q^3)^2} \tag{4.58}$$

[but vanishes for $|\omega|/(qv_F)>1$], and has a peak, an overdamped mode at $\omega\sim|\mathbf{q}|^3$, similar to the result of HLR. As many physicists have noticed, this implies for the various moments, as $q\to0$,

$$\int_0^\infty \chi_{LL}^{\rm irr}{}''\omega^n \sim q^{n+3},\ n\geqslant 1$$
$$\sim q^3 \ln 1/q,\ n=0$$
$$\sim \text{const}\ \ n=-1. \tag{4.59}$$

For $n<-1$, the moments diverge as usual.

The $n=1$ moment can be obtained exactly, because of the Kramers-Kronig relation

$$\chi_{LL}^{\rm irr}(\mathbf{q},\omega+i0^+)=\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\frac{\chi_{LL}^{\rm irr}{}''(\mathbf{q},\omega')}{\omega'-(\omega+i0^+)}$$
$$\sim\frac{-1}{\omega^2}\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\omega'\chi_{LL}^{\rm irr}{}''(\mathbf{q},\omega') \tag{4.60}$$

as $\omega\to\infty$. The high-frequency behavior of $\chi_{LL}^{\rm irr}$ at small q can be obtained by returning to the integral equation for Γ [Eq. (4.39)]. To leading order in qv_F/ω, $\Gamma(\mathbf{k},\mathbf{k}',0,\omega)=\tilde{V}(\mathbf{k}-\mathbf{k}')$, and, from Eqs. (4.43) and (4.51), we obtain

$$\chi_{LL}^{\rm irr}(\mathbf{q},\omega+i0^+)\sim\frac{-q^4k_F^2\overline{\rho}(1+F_2)}{4\omega^2m^*}. \tag{4.61}$$

(The same result except that F_2 is replaced by zero is obtained using our earlier approximation for Γ.) This can be compared with the result in a usual Fermi liquid, which is $-q^2\overline{\rho}(1+F_1)/(\omega^2m^*)=-q^2\overline{\rho}/(\omega^2m)$ on using $1+F_1=m^*/m$. We return in Sec. V below to the question of the general validity of our result, beyond the ladder approximation.

The moments of the spectral density of the full response function χ_{LL} can now also be obtained. For the $n=-1$ moment, one finds $\sim\tilde{V}(\mathbf{q})^{-1}$ for a long-range interaction, as usual in a compressible system. The $n=0$ moment behaves as $q^3\ln 1/q$ again, and gives the LLL "static" (equal time) structure factor $\overline{s(\mathbf{q})}$. It does not go as q^4, as GMP suggested it should in any liquid state. This is because compressible liquids have both low-energy modes and long-range correlations that produce nonanalytic behavior of $\overline{s(\mathbf{q})}$. GMP concluded that fluids in the LLL should be incompressible, but this argument is invalid (this point was also made by Haldane[27]). The $n=1$ moment goes as q^4, as argued by GMP, and using the high-frequency behavior of $\chi_{LL}(\mathbf{q},\omega)$, and because $\tilde{V}(\mathbf{q})$ is less singular than q^{-4},

$$\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\omega'\chi_{LL}''(\mathbf{q},\omega')=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\,\omega'\chi_{LL}^{\rm irr}{}''(\mathbf{q},\omega')$$
$$=\frac{q^4k_F^2\overline{\rho}(1+F_2)}{8m^*}, \tag{4.62}$$

to leading order in q. GMP found a formula for this moment in terms of $V(\mathbf{q})$ and $\overline{s(\mathbf{q})}$, so we obtain a relation among the quantities m^*, F_2, and $\overline{s(\mathbf{q})}$. The result for the $n=1$ moment of $\chi_{LL}^{\rm irr}$ can also be viewed as a sum rule for the leading part at small q of the longitudinal conductivity $\mathrm{Re}\,\sigma_{xx}(\mathbf{q},\omega)=\omega\chi_{LL}^{\rm irr}{}''(\mathbf{q},\omega)/q^2$.

2. Fermion scattering vertex

We now consider the scattering of the fermions by an external potential $V_{\rm ext}(\mathbf{r},t)$. The scattering of a fermion from wave vector $\mathbf{k}+\frac{1}{2}\mathbf{q}$ to $\mathbf{k}-\frac{1}{2}\mathbf{q}$ is given in the same ladder diagram approximation by the vertex function, similar to Λ earlier except for a phase factor,

$$\Lambda^L(\mathbf{k},\mathbf{q},i\omega_\nu)=e^{i\mathbf{k}\wedge\mathbf{q}}-\int\frac{d^2k_1}{(2\pi)^2}e^{i\mathbf{k}_1\wedge\mathbf{q}}\frac{f(\xi_{\mathbf{k}_1+(1/2)\mathbf{q}})-f(\xi_{\mathbf{k}_1-(1/2)\mathbf{q}})}{\xi_{\mathbf{k}_1+(1/2)\mathbf{q}}-\xi_{\mathbf{k}_1-(1/2)\mathbf{q}}-i\omega_\nu}\Gamma(\mathbf{k}_1,\mathbf{k},\mathbf{q},i\omega_\nu) \tag{4.63}$$

after removing the same phase on the external lines as for Λ (only the irreducible part is shown). If the phase factors are replaced by 1, we obtain Λ, so we will first reconsider this briefly.

Earlier we showed that, in the small-$\mathbf{q}$ limit,

$$\Lambda=1-\mathbf{q}\cdot\mathbf{v}_\mathbf{k}/(i\omega_\nu). \tag{4.64}$$

In terms of the asymptotics of Γ, the second term is the correction produced by the longitudinal mode. While the first term is the bare scalar coupling to the external potential, the second term couples to the fermions through their velocity, that is to the "current" (in the same sense as before), and so can be viewed as describing a longitudinal vector potential. Because of the factor $\mathbf{q}/i\omega_\nu$, the vector potential cancels the direct effect of the scalar potential, if we consider the electric field they produce. The system responds by producing a longitudinal response purely in the form of a vector potential, because we chose the gauge such that the scalar potential in the functional integral vanishes at nonzero frequencies. Thus for gauge-invariant response functions, such as χ_{RR} that we considered earlier, these terms produce complete cancellation, as we saw earlier in the example. This should also be true in other calculations, such as for the effect of an external "impurity" potential on the conductivity, if it coupled to ρ^R instead of to ρ^L as it would in fact (such an "impurity" potential would be static, but as usual the same effects would be found there as for all nonzero frequencies, thanks to the zero-frequency Lagrange multiplier or scalar potential field).

Since the vertex functions Λ and Λ^L differ only by the phase factor, we conclude that the phase factors like $e^{i\mathbf{k}\wedge\mathbf{q}}$ can be replaced by $e^{i\mathbf{k}\wedge\mathbf{q}}-1$ when using Λ^L. To first order in $\mathbf{q}$, this gives the dipolar coupling $\mathbf{k}\wedge\mathbf{q}$ with dipole moment $\wedge\mathbf{k}$. The first term in Λ^L is thus the direct coupling of

$V_{\rm ext}$ to the dipole moment of the fermions. This should be contrasted with the direct, minimal coupling to the fermions with charge 1 in the scenario for the low-energy behavior in the approach of HLR , described in Sec. II A. In the second term in Λ^L, where the ladder series Γ contributes, the dipolar coupling brings in the transverse mode in the ladder series, as in the calculation of $\chi^{\rm irr}_{LL}$. This coupling gives, essentially,

$$\mathbf{q}\wedge\mathbf{v}_{\mathbf{k}}[\overline{\rho}+m^*\chi_0^\perp(\mathbf{q},i\omega_\nu)]/\chi_0^\perp(\mathbf{q},i\omega_\nu) \tag{4.65}$$

at small $\mathbf{q}$, $i\omega_\nu$, which is a coupling to the transverse current, and is similar to that found in HLR and also in Ref. 31 in connection with the effects of an impurity potential, that is the $i\omega_\nu=\mathbf{0}$ limit. As there, the external potential couples to the density, which induces a transverse vector potential, which, because it is singular at $\mathbf{q}=0$, scatters the fermions much more effectively than the direct minimal coupling to the potential, let alone the dipolar coupling. The scattering produced can be simplified by comparison with the physical density ρ^L induced by the same external potential, which is $\langle\rho^L\rangle-\overline{\rho}=\chi^{\rm irr}_{LL}V_{\rm ext}(\mathbf{q},i\omega_\nu)e^{-(1/4)|q|^2}$. This shows that if the induced transverse vector potential is denoted $\mathbf{a}+\mathbf{A}$, then we have

$$\nabla\wedge\mathbf{a}=-\langle\rho^L\rangle/\overline{\rho}=-2\pi\tilde{\phi}\langle\rho^L\rangle, \tag{4.66}$$

which is exactly the equation in the CS theory. This shows that *the fermions experience a vector potential that obeys Eq.* (4.66), *where* ρ^L *is the physical charge density*, even though there is no CS term in the effective gauge field coupling and the fermions behave as dipoles. This agrees with the use in Refs. 10 and 12 of the Berry phase argument of Ref. 23 to obtain the vector potential seen by the fermions, which in no way assumed that there are flux tubes attached to the particles, unlike the CS approach. Note that, since we also have

$$\rho^L=\overline{\rho}-\nabla\wedge\mathbf{g}, \tag{4.67}$$

this is consistent with $\mathbf{a}+\mathbf{A}=\mathbf{g}/\overline{\rho}$ for the longitudinal part. There should also be an equation $-\dot{\mathbf{a}}-\nabla a_0=2\pi\tilde{\phi}\wedge\mathbf{j}^L$, where $\mathbf{j}^L$ is the physical current density. The problem of the form of $\mathbf{j}^L$ in the present approach will be considered in Sec. IV E 4.

3. Effect of impurities

Here we consider the effect of impurity scattering on the density-density response and the longitudinal conducitivity. The HF and ladder approximations can be reconsidered with impurities present. Here we neglect the mechanism of Sec. IV E 2, and take only direct scattering by the impurities, analogously to the bare HF appproximation considered so far. The average self-energy should contain an impurity line (the self-consistent Born approximation), and the ladders contain both impurity lines and interactions as the rungs of the ladder. The effective mass and the diamagnetic susceptibility will generally be renormalized by the impurity effects, but we will not distinguish them from their counterparts in the pure system. Calculations are straightforward, and the results can be written down using well-known formulas. The scattering rate $1/\tau$ is given by the usual expression, but contains m^* from the density of states (this could be replaced by the rate from the mechanism of Sec. IV E 2, but this makes little difference). At $q=0$, we have

$$\sigma_{xx}(0,\omega)=\frac{i\omega\overline{\rho}(\overline{\rho}+m^*\chi_0^\perp)}{\chi_0^\perp}, \tag{4.68}$$

and, in the Drude approximation, recalling that the current-current response is isotropic at $q=0$,

$$\chi_0^\perp(0,\omega+i0^+)=\frac{i\omega\overline{\rho}\tau}{m^*(1-i\omega\tau)}. \tag{4.69}$$

Then

$$\sigma_{xx}(0,\omega)=\overline{\rho}m^*/\tau=\sigma_0, \tag{4.70}$$

independent of ω. This can be viewed as the usual form of resistivity of the fermions, $\rho_{xx}=(\overline{\rho}\tau/m^*)^{-1}$, divided by ρ_{xy}^2, so is consistent for small ρ_{xx} with the result of the CS theory, of adding the fermion and CS resistivities [see Eq. (2.9)]. The frequency independence is also consistent with this, if in the CS approach one uses m^* in place of m, and includes FL corrections as in the scenario described in Sec. II A. The effect of the latter corrections is to replace $1-i\omega\tau$ by $1-i\omega\tau m/m^*$ (see PN, p. 191). As $m/m^*\rightarrow0$, with m^*, τ fixed, the result above is obtained.

For a finite wave vector, we will consider only the small-ω and -q region. With impurities present, $\chi_0^\perp$ is analytic in q^2 and ω,

$$\chi_0^\perp(\mathbf{q},\omega+i0^+)=q^2\chi_d^*+i\omega\overline{\rho}\tau/m^*. \tag{4.71}$$

We then obtain the longitudinal conductivity

$$\sigma_{xx}(\mathbf{q},\omega+i0^+)=\frac{i\omega\sigma_0}{i\omega-Dq^2}, \tag{4.72}$$

which exhibits a diffusion pole, with a diffusion constant

$$D=-m^*\chi_d^*/(\overline{\rho}\tau), \tag{4.73}$$

and σ_0 obeys the Einstein relation $\sigma_0=D\,dn/d\mu$.

4. Physical current density

We turn here to a calculation of the expression for the physical current density within linear response. The most obvious way to obtain the current is by projecting the usual expression to the LLL, as was considered by GMP. This yields

$$\mathbf{j}_c=\wedge\nabla\rho^L/(2m), \tag{4.74}$$

which involves the bare mass, and describes the current due to the cyclotron motion of the particles. Since it clearly obeys $\nabla\cdot\mathbf{j}_c=0$, and gives zero when integrated across a section with a boundary condition of zero density, it does not contribute to transport. This current, when coupled linearly to a change in the vector potential, $\mathbf{A}\cdot\mathbf{j}_c$, describes a magnetic moment on each particle, which should be recovered in the U(1) CS approach, as argued by the authors of Ref. 71, and obtained by SM.[24]

We are concerned with transport and with response functions, and this part of the current contains explicit derivatives, so is of less interest at long wavelengths. We therefore turn to the current due to drift motion of the guiding centers of the cyclotron orbits of the particles, due to both the external one-body potential A_0 and the interparticle two-body interaction. We will not consider fully the response to a change in the physical vector potential $\mathbf{A}$. The existence of both parts of the drift current was recognized by GMP and in Ref. 10; for further discussion, see Refs. 72 and 73. In principle, they can be obtained by carrying the calculation of the projected current to higher order in $1/\omega_c$ (the cyclotron current $\mathbf{j}_c$ being the leading term, of order ω_c), by considering virtual excitation of the particles to higher Landau levels. This was carried out in Ref. 74; it yields two types of terms of order ω_c^0 in the matrix elements of the current within the LLL, for an external potential $V_{\rm ext}$. The first of these, called $\tilde{\mathbf{j}}_L^1$, can be written as a series of derivatives of the LLL-projected potential $V_{\rm ext}$ and of the density ρ^L; the series can be further divided into a series of exponential form that agrees with the ''Noether current'' of Martinez and Stone and another series, beginning with a third-order derivative, that is of the form of an integral of an exponential. The second type of term[74] consists of the modification of the cyclotron current by the effective LLL Hamiltonian to order ω_c^{-1}, so is more complicated. The general expression for the current is thus by no means simple. However, to find the net current for transport purposes, we require only the small-q limit, and for this the result is just

$$\mathbf{j}^L = -\rho^L \wedge \nabla A_0 \tag{4.75}$$

for a slowly varying potential $A_0 = V_{\rm ext}$, which exhibits the Hall conductivity $\sigma_{xy} = \overline{\rho}$ in our system.

For the small-q drift current due to the interaction, we have, in Fourier space,

$$\mathbf{j}^L(\mathbf{q}) = \int \frac{d^2q'}{(2\pi)^2} i \wedge \mathbf{q}' \tilde{V}(\mathbf{q}') :\rho^L(\mathbf{q}+\mathbf{q}')\rho^L(-\mathbf{q}'):. \tag{4.76}$$

Diagrammatically, one can see that to calculate the linear response current to a scalar perturbation within the conserving approximation, it will be sufficient to take the operator itself in the HF approximation. Since $\langle \mathbf{j}^L \rangle = 0$ in the unperturbed ground state, the leading term is obtained by replacing a pair of operators $c^\dagger$ and c by their expectation value in the ground state,

$$\langle c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_2} \rangle = (2\pi)^2 \delta(\mathbf{k}_1 - \mathbf{k}_2)\,\theta(k_F - k_1), \tag{4.77}$$

in all possible ways; that is, two ''direct'' and two ''exchange'' terms. Of the direct terms, one vanishes and the other is seen to give the Hall current produced by the field due to the interaction with the average density of particles at wave vector $\mathbf{q}$,

$$\mathbf{j}^L(\mathbf{q})_{\rm direct} = -i \wedge \mathbf{q}\, \overline{\rho}\, \tilde{V}(\mathbf{q}) \rho^L(\mathbf{q}). \tag{4.78}$$

In calculating the irreducible response to the total field, this term is clearly included automatically. Therefore we can turn to the exchange terms which alone give the irreducible response. Since q is small, we use

$$\theta(k_F - |\mathbf{k} + \tfrac{1}{2}\mathbf{q}|) - \theta(k_F - |\mathbf{k} - \tfrac{1}{2}\mathbf{q}|) = -q \cos\,\theta_{\mathbf{k}} \delta(k - k_F) \tag{4.79}$$

for $\mathbf{q}$ in the $\hat{\mathbf{x}}$ direction, and after some algebra we obtain

$$\begin{aligned}\mathbf{j}^L(\mathbf{q})_{\rm irr} &= \int \frac{d^2k\, d^2k'}{(2\pi)^4} i \wedge (\mathbf{k} - \mathbf{k}')\tilde{V}(\mathbf{k} - \mathbf{k}') c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}} \\ &\quad \times [-q \cos\,\theta_{\mathbf{k}'}\, \delta(k' - k_F)] \\ &= -\int \frac{d^2k}{(2\pi)^2} [i \wedge \mathbf{k}\, \mathbf{q}\cdot\mathbf{k}(1 + F_2)/m^* \\ &\quad + i \wedge \mathbf{q}\, k_F^2 (F_0 - F_2)/(2m^*)] c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}},\end{aligned} \tag{4.80}$$

where the Landau parameters F_{ℓ} were defined earlier in Eq. (4.43). We assumed that only values of $\mathbf{k}$ near k_F will be used, which is true for linear response (thus $k_F^2 = \mathbf{k}^2$).

Interpreting $c^\dagger_{\mathbf{k}-(1/2)\mathbf{q}} c_{\mathbf{k}+(1/2)\mathbf{q}}$ as $\delta n_{\mathbf{k}}(\mathbf{q})$ in FL theory, where $\delta n_{\mathbf{k}}(\mathbf{r})$ is the departure of the distribution of occupied k values at $\mathbf{r}$ from the ground state, and is assumed to be nonzero only for $\mathbf{k}$ near k_F, this can be identified as

$$j^L_\mu(\mathbf{q})_{\rm irr} = -i \varepsilon_{\mu\nu} q_\lambda \Pi_{\nu\lambda}(\mathbf{q}), \tag{4.81}$$

where

$$\Pi_{\mu\nu} = \int \frac{d^2k}{(2\pi)^2}\left[\left(k_\mu k_\nu - \frac{1}{2}k^2 \delta_{\mu\nu}\right)(1 + F_2)/m^* + \frac{1}{2m^*} k^2 \delta_{\mu\nu}(1 + F_0)\right] \delta n_{\mathbf{k}}(\mathbf{q}) \tag{4.82}$$

is the stress or momentum flux tensor of the FL; it is equivalent to that in Ref. 75, modified to two dimensions. Since we have identified $\rho^L(\mathbf{r}) = \overline{\rho} - \nabla\cdot\mathbf{P}$ and $\mathbf{P}(\mathbf{r}) = \wedge \mathbf{g}(\mathbf{r})$, we expect a term in the current $\mathbf{j}^L_{\rm irr} = \dot{\mathbf{P}}(\mathbf{r})$.[25] But, by momentum conservation,

$$\frac{\partial g_\mu}{\partial t} + \partial_\nu \Pi_{\mu\nu} = 0, \tag{4.83}$$

and so we find Eq. (4.81). Since we also wish to identify $\mathbf{a} + \mathbf{A} = \mathbf{g}/\overline{\rho}$, we find

$$\mathbf{j}^L_{\rm irr} = \overline{\rho} \wedge \dot{\mathbf{a}}, \tag{4.84}$$

which is essentially the other CS-like equation.

We should also add to the Hamiltonian the potential terms

$$\int \frac{d^2q}{(2\pi)^2} [a_0(\mathbf{q})\rho^R(-\mathbf{q}) + \tilde{A}_0 \rho^L(-\mathbf{q})], \tag{4.85}$$

where a_0 is the scalar potential introduced earlier, which implements the constraint $\rho^R = \overline{\rho}$, and for which we chose the gauge $\dot{a}_0 = 0$, and $\tilde{A}_0(\mathbf{q}) = e^{-(1/4)q^2} A_0(\mathbf{q}) = \tilde{V}_{\rm ext}(\mathbf{q})$ is the externally applied potential. Then the right-hand side of the momentum conservation equation becomes

$$-[\overline{\rho} \nabla a_0 + \rho^L(\mathbf{r}) \nabla A_0] \tag{4.86}$$

at long wavelengths. Here the coefficient $\bar{\rho}$ arises from ρ^R on using the constraint. There is also a similar Hall contribution to $-\rho^L \wedge \nabla A_0$ to the current density $\mathbf{j}^L$. Expressing the total physical current $\mathbf{j}^L$ in terms of $\dot{\mathbf{g}}=\dot{\mathbf{a}}/\bar{\rho}$, we obtain

$$\mathbf{j}^L=\bar{\rho}\wedge(\dot{\mathbf{a}}+\nabla a_0), \tag{4.87}$$

which is manifestly gauge invariant and of the CS form.

If we consider the current in the right coordinates, $\mathbf{j}^R$, in a similar way, we find that in the absence of a_0 it vanishes identically, because ρ^R commutes with H. This result of vanishing current was already invoked in Sec. II D. It can be interpreted by breaking the current into the pieces $\mathbf{g}/m^*$ and $(\mathbf{a}+\mathbf{A})\bar{\rho}/m^*$ shown there. The first term represents the velocity of the fermions, while the second represents the usual backflow correction in a FL, which in the present case of $F_1=-1$ exactly cancels the first part. The same effect occurs in the ZFIM model: the total current carried by each fermion is $\mathbf{k}/m_0$ by Galilean invariance, and $m_0=\infty$, so it vanishes. (In the presence of a_0, we find $\mathbf{j}^R=\bar{\rho}\wedge\nabla a_0$, a Hall current. This does not affect our argument in Sec. II D, which uses only the irreducible part of the current, from interactions.) A similar calculation can be given for $\mathbf{j}^L$. The velocity term and the leading part of the backflow are the same as for $\mathbf{j}^R$, and so cancel. The subleading terms then give the result as calculated above. This cancellation of the leading terms is (perhaps not surprisingly) similar to what occurred in the formula for the density ρ^L on using the constraint on ρ^R.

The irreducible longitudinal current density-density response function $\chi^{\rm irr}_{j^L_x\rho^L}$ should be ω/q times $\chi^{\rm irr}_{LL}$. This can be verified in terms of the ladder series expressions for both, if one consistently either keeps or drops the Landau parameters F_ℓ for $\ell\neq 1$ in both the ladder series and the $\mathbf{q}\cdot\mathbf{j}^L$ vertex. In particular, in the small-q/ω limit, the $(1+F_2)/m^*$ term in $\mathbf{j}^L$ reproduces that in $\chi^{\rm irr}_{LL}$. However, if we consider the longitudinal current-current response, which should be ω^2/q^2 times $\chi^{\rm irr}_{LL}$, we see that the two-point current correlation function starts at higher order in q/ω than the required term (two-point correlation functions always vanish as $\omega\to\infty$). A similar difficulty is familiar in the usual Fermi liquid, and is resolved by the presence of a term in the current $-\bar{\rho}\mathbf{A}/m$ (the ''diamagnetic current'') that is linear in the applied vector potential perturbation, so that the response function ($\chi_0^\perp$ in the noninteracting case) consists of a constant plus the two-point function of the current without the $\mathbf{A}$ term. A similar effect should occur here. The term required in $\mathbf{j}^L$ is of order q^2. One might attempt to find such possible terms by making the stress tensor expression [Eq. (4.82)] gauge invariant by replacing all $\mathbf{k}$'s (including $k_F^2=\mathbf{k}^2$) by $\mathbf{k}-\mathbf{a}-\mathbf{A}$. This does not affect the other calculations done up to now because, in the absence of a perturbation in the external $\mathbf{A}$, the net $\mathbf{a}+\mathbf{A}$ does not contribute in linear response. But further work is required to check the form of this tensor, since the gauge invariance under SU(N) or W_∞ reduces to conventional U(1) gauge theory only at long wavelengths, while this expression for $\mathbf{j}^L$ is higher order in derivatives. In any case, such minimal coupling terms do not produce the necessary factors of q, and so should be absent. A way to find the part of the *longitudinal* current linear in a change in $\mathbf{A}$, which should be correct at long wavelengths, is to add a term $-\delta\mathbf{A}\cdot\mathbf{j}^{L(0)}$ [where $\mathbf{j}^{L(0)}$ is the exact expression (4.76) of zeroth order in the perturbation $\delta\mathbf{A}$] to the Hamiltonian, then calculate the longitudinal current through first-order terms in $\delta\mathbf{A}$ by commuting ρ^L with H. The resulting first-order term can be seen to give the correct high-frequency limit of the response, because it is given by a double commutator of H with $\rho^L(\pm\mathbf{q})$, which is what appears in the sum rule for the first moment of the spectral density of $\chi^{\rm irr}_{LL}$, and we have seen that it is also related to $(1+F_2)/m^*$. Thus the correct term is obtained, and must be used in the longitudinal current-current response for all ω/q to ensure agreement with the density-density response.

We now consider the full conductivity tensor at $\mathbf{q}=0$. The longitudinal part has already been considered. The full conductivity tensor can be written in the Kubo form

$$\sigma_{\mu\nu}(0,\omega+i0^+)=\bar{\rho}\varepsilon_{\mu\nu}+\frac{1}{i(\omega+i0^+)}\chi^{\rm irr}_{j^L_\mu j^L_\nu}(0,\omega+i0^+), \tag{4.88}$$

where the first term is the Hall conductivity, and $\chi^{\rm irr}_{j^L_\mu j^L_\nu}$ is the current-current two-point function for the irreducible part of the current. This form was proposed by Lee.[25] We may also consider the conductivity tensor when impurities are present. Note that the q^2 term in $\mathbf{j}^L$ does not contribute when $\mathbf{q}=\mathbf{0}$, even when impurities are present. However, we expect an additional contribution to $\mathbf{j}^L$ from the impurity potential, which we have not explicitly calculated. Because averaging (using Gaussian disorder) produces diagrams like those for interactions, except that no frequency is transferred along impurity lines, it should be similar to that derived above. It will represent the loss of conservation of momentum when disorder is present. Only the off-diagonal part of $\chi^{\rm irr}_{j^L_\mu j^L_\nu}$, or the corresponding transverse response to a scalar perturbation, has not so far been calculated. Because the ladder diagrams in the interaction and impurity lines do not violate parity (reflection symmetry), there can be no off-diagonal terms unless the impurity current vertices that we have not calculated contain pieces both parallel and perpendicular to $\mathbf{q}$. If such terms are absent, then $\sigma_{xy}=\bar{\rho}$, unaffected by impurities in this approximation. As emphasized by Lee,[25] this differs from the result of the U(1) CS approach mentioned in Sec. II A. It was argued in Ref. 76 that in the U(1) CS fermion approach, applied to the $\nu=\frac{1}{2}$ case, particle-hole symmetry implies that $\sigma_{xy}=\frac{1}{2}$ exactly, which is only satisfied by the scenario described in Sec. II A if $\sigma_{\psi xy}$ of the CS fermions is $-\frac{1}{2}$. Assuming our results also apply to $\nu=\frac{1}{2}$, there is clearly no problem with particle-hole symmetry in our self-consistent Born approximation (SCBA). We should point out, however, that in this or the similar approximation for the U(1) CS approach, the results do agree at leading order in $\rho_{\psi xx}/\rho_{xy}$, and the condition $\sigma_{\psi xy}=-\frac{1}{2}$ is only needed to guarantee $\sigma_{xy}=\frac{1}{2}$ to *all* orders in this expansion. Thus the contrast between the naive SCBA result $\sigma_{\psi xy}=0$ and the required $\sigma_{\psi xy}=-\frac{1}{2}$ is not such a dramatic singular correction as it might appear at first sight. At higher orders there will of course be other correction terms not included in the

SCBA, which can drive the system into the critical regime representing the transition between quantized Hall plateaus.

V. EXTENSION TO ALL ORDERS IN THE INTERACTION, AND DISCUSSION

In this section, we consider the extension of the results of Sec. IV to all orders in the interaction, and describe the structure of the results we expect, in a scenario which replaces the previous U(1) CS scenario described in Sec. II A. First we consider a more complicated conserving self-consistent approximation, with special attention to long-range interactions. Then we explain the FL theory structure for sufficiently long-range interactions.

In the HF and generalized HF approximations of Sec. IV, the exchange diagrams contained the bare interaction $\tilde{V}(\mathbf{q})$, and this led to a vanishing m^* at k_F for Coulomb or longer-range interactions. An obvious improvement to make is to insert the ladder series into the Coulomb vertex, as in Sec. IV E 2. The longitudinal part of the ladder series Γ renders the coupling to the fermions dipolar at long wavelengths, which removes the divergence in $1/m^*$ for interactions less singular than $1/q^3$. At the same time, we can insert the ladder series inside the interaction line itself, thus screening the interaction. We can also *replace* the interaction line in the exchange diagram by Γ. Finally, we make this approximation self-consistent by making these replacements for *all* interaction lines, including those in Γ, thus iterating to self-consistency. This approximation, applied to response functions as well as the self-energy, is once again conserving in the same sense as in Sec. IV C, and the conclusions there, which follow from $F_1=-1$, still apply.

This approximation is clearly not as tractable as that of HF, but we can still make some general statements. The system should still be compressible for all interactions considered (those less singular than $1/q^2$ as $q\rightarrow 0$). The longitudinal mode in the ladder just produces the dipolar coupling effects already mentioned, which do not cause a breakdown of FL theory, though the effect of the exchange self-energy that contains Γ in place of V has not been calculated. The transverse mode in Γ produces singularities in the self-energy for Coulomb or shorter-range interactions. The self-consistent summation proposed here is the same as regards the transverse mode as that studied in Refs. 11 and 39 (and similar to that in Ref. 37). We have nothing to add here to the previous discussion of this case, except to emphasize that these singular effects should be treated *after* the other FL renormalizations discussed in this paper, and that, in relation to the U(1) CS approach, the effects incorporated in this paper are related to the longitudinal, not transverse, CS gauge field fluctuations (see Sec. II A). For interactions longer range than Coulomb, there is no breakdown of FL theory, since m^* remains finite and the quasiparticle decay rate vanishes faster than the renormalized excitation energy $\xi_{\mathbf{k}}$ as $k\rightarrow k_F$, though not as fast as $(k-k_F)^2$.

We can now discuss the general structure expected in the results to all orders in the interaction; some of this is implicit in the foregoing discussion. We consider only interactions longer range than the Coulomb interaction, so there is no breakdown of FL theory. For the Coulomb interaction, the results are probably still useful, since the only other effect is a logarithmic divergence in m^*, which is very weak.

For such interactions, we again separate in the response functions the ''direct'' or reducible diagrams, which represent the long-range self-consistent field produced by the expectation value of the density. The remaining diagrams are analyzed in terms of the fermion-hole irreducible scattering vertex, which at $\mathbf{q}$, $\omega\rightarrow 0$ is nonsingular and defines the parameters f_ℓ and hence $F_\ell = m^* f_\ell / 2\pi$. A Ward identity, now valid to all orders, implies that $F_1=-1$. In fact an identity for the ρ^R vertex, like that in Sec. IV C, is valid to all orders and for all $\mathbf{q}$ and $\omega_\nu\neq 0$, and expresses the fact that $[\rho^R,H]=0$. In the general diagrams that contribute to these vertex functions, the phase factors in the interaction vertex do not all cancel, so the system is not equivalent to the ZFIM model. The results nonetheless have the same structure as in Sec. IV, and at long wavelengths can be interpreted in terms of an infinitely strongly coupled gauge field, coupled to the FL. *There are no parity violating effects in the long-wavelength dynamics of this system*, because the Landau interaction $f_{\mathbf{kk}'}$ is even under exchange of $\mathbf{k}$ and $\mathbf{k}'$. The only parity-violating effects come in the coupling to external electromagnetic fields, where the Hall effect appears, and the physical density and current obey the CS-like equations. The self-consistent field produced by the long-range interaction (the reducible terms) also produces Hall currents, but there is no parity violation because interactions within the system couple to the density at both ends. The fluctuations in the longitudinal part of the gauge field can be reconsidered by changing to the gauge $\nabla\cdot\mathbf{a}=0$, in which it is the scalar potential a_0 that fluctuates (at all frequencies). This absorbs the F_0 we had previously, and the condition $\rho^R=\overline{\rho}$ is maintained through an effective F_0 that is now infinite (the Landau parametrization is not gauge invariant). The longitudinal part of the ladder series at low ω/q gives an effective interaction between the fermions, which is of order the inverse density of states, that is the Landau A_0 parameter $A_0=F_0/(1+F_0)=1$ (this is similar to effects in the local Fermi liquid in the Kondo problem; see Ref. 65). Because the leading ''monopolar'' part of the ρ^L density fluctuations is suppressed by this, the leading nontrivial part is described by the subleading, dipolar part of the exact density expression ρ^L [note that this subleading coupling is not described by the minimally coupled long-wavelength Hamiltonian in equation (2.30)]. A noteworthy feature of our approach is that this is not obtained separately from the transverse gauge field effects, nor inserted at the beginning, but emerges later. The dipole moment $\wedge\mathbf{k}$ on each fermion is not renormalized, because the momentum is a conserved quantity. This really deserves an explicit proof, but it will be omitted because of the similarity to results in standard FL theory (see, e.g., Nozières[62]); quite generally, conserved quantities are not renormalized.

The compressibility is given by

$$\frac{dn}{d\mu} = -\frac{\overline{\rho}^2}{\chi_d^*}, \tag{5.1}$$

where χ_d^* is the fully renormalized (irreducible) diamagnetic susceptibility, and is the only non-Fermi surface quantity to make an appearance in the response in the regime of small $\mathbf{q}$ and ω. The other quantities mentioned in Sec. IV are given by the same forms as there, when written in terms of $\overline{\rho}$, k_F, m^*, F_{ℓ}, and χ_d^*. In particular, we mention the longitudinal conductivity in the regime $q^3 < \omega < q v_F$, relevant to surface acoustic waves. The result, which is identical to that of HLR, is exact in the same way, and for the same reason, as the low-frequency transverse conductivity of the usual FL. Also, the high-frequency behavior, or $n=1$ moment of the spectral density, of the irreducible density-density response, is given by the same sum-rule-like form as in Sec. IV E 1, as long as we consider only excitation of a single quasiparticle-quasihole pair (in the FL sense). If multiple quasiparticle-hole pairs do not contribute at this order in q, then this ''sum rule'' is exact. In the usual FL, multiple quasiparticle-hole pairs contribute to spectral densities at $O(q^4)$, by considerations of phase space, and the f-sum rule is for the q^2 part (and higher-order terms actually vanish in this particular case). Thus it is not certain in our case that our sum rule is exact. The same phase-space considerations apply, and if we assume that the squared matrix element of the density ρ^L is of order q^2 (i.e., dipolar) for matrix elements to multiple quasiparticle-hole excitations, as we have seen it is for single quasiparticle-quasihole excitations, then these other contributions can be neglected. This seems likely to be correct, but as we do not have a proof, we will leave it as a conjecture that *Eq. (4.62) is an exact relation*, which we call the ''F_2 sum rule,'' and that it holds for both the irreducible and reducible responses, as in the generalized HF approximation. If correct, we also obtain a relation of $(1+F_2)/m^*$ to the LLL structure factor $\overline{s}(\mathbf{q})$ and $\tilde{V}$, as noted already in Sec. IV E 1.

When impurities are included, an improved approximation is obtained by treating them diagrammatically similarly to the interaction lines as described at the beginning of this section. In this Drude- (or SCBA-)-like approximation, the conductivity takes the same form as in Sec. IV E 3. Based on the existing results,[24–26] we also expect that similar results hold for $\tilde{\phi}>1$, with $\overline{\rho}=(2\pi\tilde{\phi})^{-1}$.

We expect that the direct interaction of the particle with its correlation hole (or attached vortices), described in Refs. 10, 12, and 24–26 is contained in this description, but may not be easily obtainable diagrammatically. If it is obtained in some approximation, the effects stemming from $F_1=-1$ will still be present when the approximation is conserving.

One other way that the FL picture could break down is by a pairing instability as in the theory of superconductivity. The interaction in the quasiparticle-quasiparticle channel with quasiparticles of wave vectors $\mathbf{k}$, $-\mathbf{k}$ can be considered using the ladder approximation. The dipolar nature of the coupling gives rise to an attractive interaction, as noticed by the authors of Refs. 26 and 27. Since the system is compressible, this interaction is screened. In addition, the ladder series Γ, representing transverse and longitudinal gauge field fluctuations, can be exchanged between the fermions, and the transverse part can be combined with the interaction V. The transverse gauge field is believed to be pair-breaking when included in an Eliashberg-equation treatment.[77] The longitudinal part gives an extra repulsive short-range interaction, which also suppresses pairing, especially in the s-wave channel. Therefore the question of whether pairing is actually expected to occur requires careful consideration. There is unpublished evidence that it does occur for bosons at $\nu=1$ for some interactions.[26,27] If pairing does occur, the system will become incompressible at low energies and long wavelengths, essentially because of the Meissner effect in the superfluid Fermi system: the diamagnetic susceptibility now behaves as $\chi_d^* \sim -1/q^2$, which, inserted in our result for $dn/d\mu$, shows the system is incompressible. This shows that it is not just the symmetries of the Hamiltonian that make the ground state compressible in the FL-like state, but it is the fact that the state is assumed to be a normal (nonsuperfluid) liquid.

Assuming the system is a FL, the scenario we have described here and in Sec. II D is essentially a FL coupled to an infinitely strongly coupled gauge field (that represents $F_1=-1$), with no CS term. The central point was the Ward identity that gave $F_1=-1$. We connected this with the gauge invariance under $\mathrm{U}(N)_R$, or equivalently with conservation of $G(\mathbf{q})$. Other authors have very recently commented on ''translational invariance in momentum space,''[24,27,51,25] and its relation to some sort of gauge symmetry. We will try to make this more precise. The Hamiltonian [Eq. (4.1)] is invariant under shifts of the wave vectors of all the fermions by $\mathbf{Q}$: $\mathbf{k}_i \to \mathbf{k}_i+\mathbf{Q}$. The generator of a translation of the wave vectors of all fermions is

$$\frac{1}{2} i \int \frac{d^2k}{(2\pi)^2} [c_{\mathbf{k}}^\dagger \nabla_{\mathbf{k}} c_{\mathbf{k}} - (\nabla_{\mathbf{k}} c_{\mathbf{k}}^\dagger) c_{\mathbf{k}}]. \tag{5.2}$$

In first quantization and in position space, it is simply $\Sigma_i \mathbf{r}_i$. This is related to Galilean invariance in ordinary systems with finite bare mass m_0. If we rescale the generator of Galilean transformation[78] to obtain shifts in $\mathbf{k}_i$ instead of in $\mathbf{v}_i=\mathbf{k}_i/m_0$, we obtain

$$\sum_i (\mathbf{r}_i - t\mathbf{p}_i/m_0), \tag{5.3}$$

and the second term can be dropped when $m_0 \to \infty$. However, in this limit we obtain the ZFIM model, and the Galilean symmetry is enlarged to the local gauge symmetry generated by $\rho(\mathbf{q})$, already discussed. In our system, by contrast, the gauge symmetry is generated by $\rho^R(\mathbf{q})$,

$$\begin{aligned} \rho^R(\mathbf{q}) &= \int \frac{d^2k}{(2\pi)^2} e^{-(1/2)i\mathbf{k}\wedge\mathbf{q}} c_{\mathbf{k}-(1/2)\mathbf{q}}^\dagger c_{\mathbf{k}+(1/2)\mathbf{q}} \\ &= \hat{N} + \int\int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \mathbf{q}\cdot[i\wedge\mathbf{k} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \\ &\quad + c_{\mathbf{k}}^\dagger \nabla_{\mathbf{k}} c_{\mathbf{k}} - (\nabla_{\mathbf{k}} c_{\mathbf{k}}^\dagger) c_{\mathbf{k}}], \end{aligned} \tag{5.4}$$

keeping *all* terms to linear order in $\mathbf{q}$. Using the similar expansion of $\rho^L(\mathbf{q})$, the generator of shifts in $\mathbf{k}$ can be written as the first-order term in $\rho^R+\rho^L$. The other, unused, pieces are the particle number $\hat{N}$ and the momentum $\int \mathbf{k} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$, which are also conserved quantities (note that the terms in ρ^L and ρ^R linear in $\mathbf{q}$, are generators of magnetic translations in the left and right coordinates, respectively, written in momentum space). Thus the ''shifting'' symmetry is part of the gauge symmetry, *in combination with other global symmetries*, and not just part of the gauge symmetry as stated by

SM. Even so, for some purposes, viewing it just as part of the gauge symmetry can be useful, as we saw in Sec. II D, and will again in the next paragraph.

In unpublished work,[27] Haldane proposed to write the effective Hamiltonian of the quasiparticles, for the case of a finite system on a torus (say a square torus of side L), as

$$H_{\rm eff}=\frac{1}{4m^*N}\sum_{ij}(\mathbf{k}_i-\mathbf{k}_j)^2, \tag{5.5}$$

which possesses the shifting symmetry. In this system, shifting all the momenta by the smallest possible amount $2\pi/L$ changes the total momentum by $2\pi N/L$, and gives a state equivalent to the original one.[79] The latter fact is assumed in numerical calculations, and such calculations seem to confirm this form of the Hamiltonian. We may identify this Hamiltonian as similar to our

$$\sum_i (\mathbf{k}_i-\mathbf{a}-\mathbf{A})^2/(2m^*), \tag{5.6}$$

in the case of a spatially constant $\mathbf{a}+\mathbf{A}$, since (by an equation of motion) $\mathbf{a}+\mathbf{A}=\mathbf{g}/\overline{\rho}=\Sigma_i\mathbf{k}_i/N$. The shift transformation is a gauge transformation (up to caveats just discussed) that does not change the physical states; this fact goes beyond the simple symmetry property possessed by Eq. (5.5). Our Hamiltonian is preferable because, when $\mathbf{a}$ is allowed to vary spatially, it represents a local interaction, unlike Eq. (5.5). Integrating out $\mathbf{a}$, and using the constraint on the density ρ^R, we obtain a Hamiltonian like that in SM, except that we have the effective mass m^*, whereas in their work it appears at a stage where they instead have the bare mass m. This Hamiltonian is also the starting point for the arguments of Ref. 51.

VI. CONCLUSION

In this paper we have developed a truly lowest-Landau-level theory for the Fermi-liquid-like state of charged bosons at $\nu=1$. We used a formalism of Pasquier and Haldane[53], in which the composite fermion fields depend on two complex coordinates, one of which is the coordinate of the boson, and the other is in effect the coordinate of a vortex in the wave function of the other bosons, attached to the boson. The wave functions in both these coordinates are restricted to the lowest Landau level, and there are operator constraints which fix the density in the vortex coordinates. The constraints imply that the system is a gauge theory. The effective theory for low-energy, long-wavelength phenomena is a Fermi liquid in which the fermions couple to a gauge field, for which there are no bare terms in the action.[52] The ladder series treatment in Sec. IV, with the approximate form Eq. (4.44), is equivalent to the RPA applied to this gauge field. Since there is no Chern-Simons term in the gauge field action, the longitudinal and transverse modes decouple. The longitudinal part, within the RPA, gives rise to an effective scalar interaction at small momentum exchange of order the inverse density of states. This enforces the fixed-density constraint. The transverse part couples to the physical density, the first nontrivial term in which is dipolar in form and parity violating. Each fermion carries a dipole moment equal to its wave vector. The result is a finite compressibility, and a low-frequency longitudinal conductivity that agrees with that in HLR. The gauge field obeys the same Chern-Simons equations relating it to the physical density and current as in the U(1) Chern-Simons fermion approach of HLR. Because there is no CS term in the action, the results nonetheless differ in form from those in the scenario for the fully renormalized theory based on HLR. Although the gauge theory reduces to an ordinary U(1) theory at long wavelengths, this has to be supplemented by the expression for the density, which is a nonminimal coupling from the U(1) point of view. The form of the expression for the physical current intimates that this is not the whole story, and we expect that the full W_∞ gauge group will be involved in general. In view of existing results of other authors,[24,25] the results obtained here for $\tilde{\phi}=1$ (bosons at $\nu=1$) are also expected to apply for other cases of the FL-like state, when written in terms of $\overline{\rho}=(2\pi\tilde{\phi})^{-1}$ and other parameters. There are many possible extensions and applications of the present methods, to which we hope to return elsewhere.

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APPENDIX A: NONCOMMUTATIVE GEOMETRY FOR PEDESTRIANS

In this appendix we explain the formalism we use for states in and operators acting in the Hilbert space of a single particle in the lowest Landau level, in the simplest case of the infinite plane with uniform magnetic field, and a magnetic length equal to 1 (see also Ref. 49). This is equivalent to the ''noncommutative plane'' in noncommutative geometry. In particular we explain the ''noncommutative Fourier transform'' which we use extensively.

The normalized basis states in coordinate representation in the symmetric gauge are

$$u_m(z)=\frac{z^m e^{-(1/4)|z|^2}}{\sqrt{2\pi 2^m m!}}. \tag{A1}$$

A general state in the Hilbert space thus has wave function $\psi(z)=f(z)e^{-(1/4)|z|^2}$, where f is a complex analytic function that does not grow too fast at infinity, so that $\int|\psi|^2$ is finite. All operators can be written as integral kernels, so that an operator $\hat{a}$ is represented by the kernel $a(z,\overline{z'})$, which acts on states $\psi(z)$ as

$$\hat{a}\psi(z)=\int d^2z'\,a(z,\overline{z'})\,\psi(z'), \tag{A2}$$

and matrix products become the ''star product'' $\hat{a}*\hat{b}$, the integral kernel of which is

$$\hat{a}*\hat{b}(z,\overline{z'})=\int d^2z_1\, a(z,\overline{z_1})b(z_1,\overline{z'}). \tag{A3}$$

The operators themselves can, of course, be expanded as

$$a(z,\overline{z'})=\sum_{m,n=0}^{\infty} a_{mn}u_m(z)\overline{u_n(z'),} \tag{A4}$$

so that a_{mn} are elements of infinite matrices.

Arbitrary operators in the larger Hilbert space of states in all Landau levels, that is all square-integrable complex functions in the plane (really, sections of the appropriate bundle), can be projected to the LLL. In particular, the identity $\delta(\mathbf{r}-\mathbf{r}')$ has matrix elements δ_{mn} in the orthonormal basis, and the corresponding operator as an integral kernel is

$$\delta(z,\overline{z'})\equiv\sum_m u_m(z)\overline{u_m(z')}$$
$$=\frac{1}{2\pi}\exp\left(-\frac{1}{4}|z|^2-\frac{1}{4}|z'|^2+\frac{1}{2}z\overline{z'}\right). \tag{A5}$$

As befits the identity, this obeys $\hat{\delta}\psi=\psi$, $\hat{\delta}*\hat{a}=\hat{a}*\hat{\delta}=\hat{a}$. This operator also implements projection to the LLL.

Another operator is defined by multiplication by the plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$. Its projection to the LLL is

$$\int d^2z_1\,\delta(z,\overline{z_1})e^{i\mathbf{k}\cdot\mathbf{r}_1}\delta(z_1,\overline{z'})$$
$$=\delta(z,\overline{z'})e^{(1/2)i(\overline{k}z+k\overline{z'})-(1/2)|k|^2}, \tag{A6}$$

where, in this appendix, $k=k_x+ik_y$ (elsewhere in the paper $k=|\mathbf{k}|$ for all vectors $\mathbf{k}$). It is convenient to define

$$\tau_{\mathbf{k}}(z,\overline{z'})=\delta(z,\overline{z'})e^{(1/2)i(\overline{k}z+k\overline{z'})-(1/4)|k|^2}. \tag{A7}$$

Thus $\hat{\tau}_{\mathbf{k}}=e^{i\mathbf{k}\cdot\hat{\mathbf{R}}}$, the adjoint of which is $\hat{\tau}_{-\mathbf{k}}$, so $\overline{\tau_{\mathbf{k}}(z',\overline{z})}=\tau_{-\mathbf{k}}(z,\overline{z'})$. The operator $\tau_{\mathbf{k}}$ has the effect of magnetic translation (i.e., translation which commutes with the Landau-level index) by $-ik$ or $\wedge\mathbf{k}$ in the plane.[80] It obeys the well-known magnetic-translation relation

$$\hat{\tau}_{\mathbf{k}}*\hat{\tau}_{\mathbf{k}'}=\hat{\tau}_{\mathbf{k}+\mathbf{k}'}e^{(1/4)(\overline{k}k'-k\overline{k'})}. \tag{A8}$$

Here $\frac{1}{4}(\overline{k}k'-k\overline{k'})=\frac{1}{2}i\,\mathrm{Im}\,\overline{k}k'=\frac{1}{2}i\mathbf{k}\wedge\mathbf{k}'$, which is i times the (signed) area of the triangle formed by $\mathbf{k}$, $\mathbf{k}'$, $-(\mathbf{k}+\mathbf{k}')$.

$\tau_{\mathbf{k}}$ are the natural functions for use in defining a ''noncommutative Fourier transform.'' The motivation is that functions (like the operator kernels) of z and z' are like wave functions for a single particle in zero magnetic field, for which the plane waves make sense. For such a function $a(z,\overline{z'})$, we write

$$a(z,\overline{z'})=\int\frac{d^2k}{2\pi}a_{\mathbf{k}}\tau_{\mathbf{k}}(z,\overline{z'}), \tag{A9}$$

and, for the inverse transformation,

$$a_{\mathbf{k}}=\int \hat{a}*\hat{\tau}_{-\mathbf{k}}, \tag{A10}$$

where the integral is defined by $\int\hat{b}=\mathrm{Tr}\,\hat{b}=\int d^2z\, b(z,\overline{z})$. The inversion theorem for this transform is easily proved by Gaussian integration. We note the orthonormality and completeness relations,

$$\int \hat{\tau}_{\mathbf{k}}*\hat{\tau}_{\mathbf{k}'}=2\pi\delta(\mathbf{k}+\mathbf{k}'), \tag{A11}$$

$$\int\frac{d^2k}{2\pi}\tau_{\mathbf{k}}(z,\overline{z'})\tau_{-\mathbf{k}}(w,\overline{w'})=\delta(z,\overline{w'})\,\delta(w,\overline{z'}). \tag{A12}$$

The ''noncommutativity'' of the transform shows up when one has convolutions where the relation (A8) must be used.

In the main text the above formalism is applied to second quantized operators c, $c^\dagger$, ρ^L, and ρ^R, where it concerns their dependence on the z and w variables, and has nothing to do with the Fock space in which they act as operators. In the case studied in this paper, the Fourier transform can be applied to c and $c^\dagger$ because the net magnetic-field strength vanishes for $\nu=1/\tilde{\phi}=1$. (For $\nu\neq1$, one would require the full set of Landau-level states in the net, effective magnetic field,[6] projected to the z and w variables, in place of the plane waves which project to $\tau_{\mathbf{k}}$. The Fourier transform would still apply to ρ^L and ρ^R, of course.) For $\nu=1$ we define

$$c(z,\overline{w})=\int\frac{d^2k}{(2\pi)^{3/2}}c_{\mathbf{k}}\tau_{\mathbf{k}}(z,\overline{w}), \tag{A13}$$

$$c_{\mathbf{k}}=(2\pi)^{1/2}\int \hat{c}*\hat{\tau}_{-\mathbf{k}}; \tag{A14}$$

the normalization has been chosen so as to obtain the conventional anticommutators in Eq. (3.14). For ρ^L and ρ^R we use the normalization given above for an arbitrary $\hat{a}$, and the properties of the $\tau_{\mathbf{k}}$'s lead to Eqs. (3.16) and (3.15). We also note that for the diagonal values $z=z'$,

$$\rho^L(z,\overline{z})=\int\frac{d^2q}{(2\pi)^2}\rho^L(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}-(1/4)|q|^2},$$

$$\rho^L(\mathbf{q})=e^{(1/4)|q|^2}\int d^2r\,\rho^L(z,\overline{z})e^{-i\mathbf{q}\cdot\mathbf{r}},$$

and similarly for ρ^R. This exhibits the connection with GMP.

Finally we note that other formulas of noncommutative geometry can be obtained in the integral kernel formalism. For example, the commutator in the star product,

$$\hat{a}*\hat{b}-\hat{b}*\hat{a}=[\hat{a}\overset{*}{,}\hat{b}], \tag{A15}$$

defines the ''Weyl-Moyal bracket'' that generalizes the Poisson bracket of functions on the classical phase space to the quantum case. It is usually written as an infinite series of derivatives. Our integral kernel formulation avoids such series and allows generalization to other (e.g., compact) Riemann surfaces, or to nonuniform field strengths. In all cases, one can begin with an orthonormal set of LLL states, i.e., holomorphic sections of the appropriate bundle. A crucial operator is the ''reproducing kernel'' analogous to $\delta(z,\overline{z'})$. This can be easily obtained for the sphere and the torus, for a uniform field strength.

APPENDIX B: HUBBARD-STRATONOVICH TRANSFORMATION AND THE 1/M EXPANSION

Here we show how to reproduce the results of the HF and ladder approximations as the saddle-point and Gaussian fluctuations in a Hubbard-Stratonovich field. First, one may replace the interaction term in the imaginary-time action by

$$\int \prod_{i=1}^{4} d^2 z_i \, [c^\dagger(z_1,\overline{z_2}) c(z_3,\overline{z_4}) V(\mathbf{r}_2-\mathbf{r}_3)\sigma(z_4,\overline{z_3},z_2,\overline{z_1}) + \tfrac{1}{2}|\sigma(z_4,\overline{z_3},z_2,\overline{z_1})|^2 V(\mathbf{r}_2-\mathbf{r}_3)], \tag{B1}$$

(the τ dependence and τ integration is implicit) where σ is a fourth-rank tensor field, written in the coordinate notation using LLL orthonormal functions as for c, $c^\dagger$; it is Hermitian,

$$\overline{\sigma(z_4,\overline{z_3},z_2,\overline{z_1})} = \sigma(z_1,\overline{z_2},z_3,\overline{z_4}), \tag{B2}$$

and is integrated over functionally. Performing the latter functional integral reproduces the interaction term. The field σ decouples the interaction in the exchange channel. The saddle-point approximation for the σ integral (along with the Lagrange multipliers) reproduces the exchange, but not the Hartree, part of the Hartee-Fock interaction. Gaussian fluctuations in σ around the saddle point reproduce the ladder series. Thus the ladder series becomes the RPA in the σ field. It should be possible to identify part of the σ fluctuations as the gauge field, in a manner similar to that in some lattice models.[54]

In other problems, such a saddle point and Gaussian fluctuations are the leading terms in a $1/M$ expansion, where M is the number of components of a field corresponding to our c and $c^\dagger$. We may introduce such components here, and then set $M=1$ at the end, by replacing c_{mn} by $c_{mn\alpha}$, where $\alpha = 1, \ldots, M$. The interaction is taken independent of α, so the system has SU(M) symmetry. Then Eq. (B1) now has the form

$$\int \sum_{\alpha=1}^{M} c_\alpha^\dagger c_\alpha V\sigma + \frac{1}{2} M \int |\sigma|^2 V \tag{B3}$$

schematically. This appears suitable for $1/M$ expansion, but there is a problem with the constraints. The latter must still be taken to be

$$\sum_{\alpha=1}^{M} \sum_m c_{nm\alpha}^\dagger c_{mn'\alpha} = \delta_{nn'} \tag{B4}$$

in order to reproduce an M-component system of bosons, whatever the filling factor. To obtain a zero net field for the fermions, we must be at total filling factor $\nu=1$, so we must have $\overline{\rho}=1/2\pi$, that is of order M^0, not M. Therefore not all the terms in the action are of order M, and we can expect problems with the $1/M$ expansion. These are not necessarily completely fatal, however; an expansion can sometimes be obtained even in such cases (see Ref. 65). It is not possible to rescale or redefine the model to avoid this problem. It could be avoided if we could attach $1/M$ of a vortex to each particle (which would now be anyons, so that $c^\dagger$ still creates fermions), as in the U(1) CS approach.[43] However, this is not possible in the present PH formalism.

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