Quasiparticles in fractional quantum Hall effect edge theories

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We propose a quasiparticle formulation of effective edge theories for the fractional quantum Hall effect. For the edge of a Laughlin state with filling fraction $\nu=1/m$, our fundamental quasiparticles are edge electrons of charge $-e$ and edge quasiholes of charge $+e/m$. These quasiparticles satisfy exclusion statistics in the sense of Haldane. We exploit algebraic properties of edge electrons to derive a kinetic equation for charge transport between a $\nu=1/m$ fractional quantum Hall edge and a normal metal. We also analyze alternative ''Boltzmann'' equations that are directly based on the exclusion statistics properties of edge quasiparticles. Generalizations to more general filling fractions (Jain series) are briefly discussed. [S0163-1829(98)03544-9]

I. INTRODUCTION AND SUMMARY

Low-energy excitations over (fractional) quantum Hall effect (QHE) ground states are localized near the edge of a sample. Certain aspects of QHE phenomenology can therefore be captured by an *effective edge theory*. The unusual properties (notably, fractional charge and statistics) of bulk excitations over a fractional QHE (FQHE) ground state carry over to analogous properties of the fundamental edge excitations.

In the existing (theoretical) literature on QHE edge phenomena, the fundamental edge quasiparticles have not played an important role. Most, if not all, results have been obtained by exploiting bosonization schemes. For the analysis of edge-to-edge tunneling experiments, a combination of bosonization with techniques from integrable field theories has led to exact results for universal conductance curves.[1,2] What has been missing until now is a description of edge-to-edge transport phenomena directly in terms of excitations that are intrinsic to an edge in isolation. The authors of Ref. 3 have explored such a picture and have derived a number of strong-coupling selection rules for the scattering of edge excitations.

In this paper we take up the challenge of reformulating effective QHE edge theories directly in terms of a set of fundamental edge excitations. For the principal Laughlin states at filling fraction $\nu=1/m$, we select the *edge electron* (of charge $-e$) and the *edge quasihole* (of charge $+e/m$) as the fundamental excitations. (The reason for this asymmetric choice will become clear.) An important complication is then that these fundamental quasiparticles are *not* fermions but instead obey fractional statistics (in a sense to be explained below). A large part of this paper will be devoted to working out the consequences of these unusual statistics.

Following Wen,[4] we shall assume that the edge theory for a $\nu=1/m$ fractional QHE state (Laughlin state) takes the form of a chiral Luttinger liquid (see Ref. 5 for a further justification of this description). The bosonic description of such a theory is centered around the neutral charge density operators $n_q=1/\sqrt{m}(\partial\varphi)_q$, which satisfy a $U(1)$ affine Kac-Moody algebra

$$[n_q,n_{q'}]=\frac{1}{m}\,q\,\delta_{q+q'}. \tag{1.1}$$

The vertex operators

$$\Psi_{\rm e}(z)=:\exp(-i\sqrt{m}\varphi):(z),$$
$$\Psi_{\rm qh}(z)=:\exp\left(i\frac{1}{\sqrt{m}}\varphi\right):(z) \tag{1.2}$$

have charge $-e$ and $+e/m$, respectively, and have been identified as the edge electron and edge quasihole. They are the direct analogs of the bulk (Laughlin) quasiparticles, an important distinction being that the edge excitations are gapless and have linear dispersion.

The first indication for the nontrivial statistics of the operators (1.2) comes from the operator product expansions (OPE)

$$\Psi_{\rm e}(z)\Psi_{\rm e}(w)=(z-w)^m[\Psi'(w)+\cdots],$$
$$\Psi_{\rm qh}(z)\Psi_{\rm qh}(w)=(z-w)^{1/m}[\Psi''(w)+\cdots], \tag{1.3}$$

where $\Psi'(w)$ and $\Psi''(w)$ are operators of charge $-2e$ and $+2e/m$, respectively. For m an odd integer, the right hand side of the first OPE picks up a minus sign under the exchange $z\leftrightarrow w$, in correspondence with the required antisymmetry of the Laughlin wave functions. The second relation features a fractional power of $(z-w)$, which shows that the quasihole operator has fractional ''exchange statistics.''

For $\nu=1$ the operators $\Psi_{\rm e}$ and $\Psi_{\rm qh}$ are fermionic and the edge theory is simply a theory of free fermions. The exchange statistics of $\Psi_{\rm e}$ and $\Psi_{\rm qh}$ at $\nu=1/m$ clearly signal a deviation from free fermion behavior, but they are hardly helpful for the purpose of setting up a quasiparticle formalism that mimics the free fermion treatment of the $\nu=1$ edge. We shall here argue that a much more convenient point of view is that of the ''exclusion statistics'' properties of $\Psi_{\rm e}$ and $\Psi_{\rm qh}$.

In a recent paper,[6] one of us proposed a method to associate exclusion statistics to quasiparticles for conformal field theory (CFT) spectra. We shall here show that, when applied to the $\nu=1/m$ edge theory excitations, this method gives

ideal fractional exclusion statistics (in the sense of Haldane[7]) with $g=m$ for edge electrons and $g=1/m$ for edge quasiholes. We also find that the edge electrons and edge quasiholes can be viewed as independent excitations, in the sense that there is no mutual exclusion between the two.

Our program in this paper is then (1) to establish the exclusion statistics properties of Ψ_e and Ψ_{qh} and (2) to apply them to both equilibrium and transport properties of these edges. As for transport, we shall focus on the setup of the experiment by Chang *et al.*,[8] where electrons are allowed to tunnel from a normal metal into the edge of a $\nu=1/3$ FQHE edge. We shall use algebraic properties of $\nu=1/3$ edge electrons to write an exact kinetic equation for the perturbative I-V characteristics for this system, reproducing the result obtained by other methods. Interestingly, the relevant algebraic properties derive from the so-called $N=2$ superconformal algebra, which has been well-studied in the context of string theory. We shall also study ''naive'' Boltzmann equations that are based on the exclusion statistics properties of the edge quasiparticles. While the latter equations are not exact, we shall argue that they can be used as the starting point in a systematic approximation to the exact transport results. These results then are of general importance, as they illustrate the possibilities and limitations of the concept of exclusion statistics in the analysis of nonequilibrium physics.

The observations made here are easily generalized to composite edges, related to hierarchical FQHE states, in particular those of the Jain series with $\nu=n/(np+1)$. For the Jain series edge theories, two natural pictures emerge. In the first picture, the edge quasiparticles satisfy Haldane's exclusion statistics with G matrix equal to K^{-1}, where K is the topological order matrix of the bulk FQHE state. In the second picture one decouples one charged mode from $n-1$ neutral modes. A possible quasiparticle basis then consists of a single charged Haldane g-on and a collection of n neutral quasiparticles that are related to parafermions in the sense of Gentile.

This paper is organized in the following way. In Sec. II we discuss exclusion statistics and indicate the applications to FQHE states and to CFT spectra. In Sec. III we discuss in some detail how a quasiparticle basis for $\nu=1/m$ FQHE edge states is obtained and how that leads to an assignment of exclusion statistics parameters. In Sec. IV we explain how equilibrium properties are obtained in a quasiparticle approach. In Sec. V we further study the quasiparticle bases and make the link with Calogero-Sutherland quantum mechanics and Jack polynomials. In Sec. VI we study charge transport between a normal metal and a $\nu=1/m$ FQHE edge in terms of kinetic equations that are based on our quasiparticle formalism. Appendix A describes the extension of our quasiparticle formalism to filling fractions in the Jain series, while Appendix B contains explicit results for an important quasiparticle form factor.

II. EXCLUSION STATISTICS

In his by now famous 1991 paper,[7] Haldane proposed the notion ''fractional exclusion statistics,'' as a tool for the analysis of strongly correlated many-body systems. The central assumption that is made concerns the way a many-body spectrum is built by filling available one-particle states. In words, it is assumed that the act of filling a one-particle state effectively reduces the dimension of the space of remaining one-particle states by an amount g. The choices $g=1$, $g=0$ correspond to fermions and bosons, respectively. The thermodynamics for general ''g-ons,'' and in particular the appropriate generalization of the Fermi-Dirac distribution function, have been obtained in Refs. 9–12. The so-called Wu equations[9]

$$n_g(\epsilon)=\frac{1}{[w(\epsilon)+g]},$$

with

$$[w(\epsilon)]^g[1+w(\epsilon)]^{1-g}=e^{\beta(\epsilon-\mu)} \tag{2.1}$$

provide an implicit expression for the one-particle distribution function $n_g(\epsilon)$ for g-ons at temperature T and chemical potential μ. It has been demonstrated that fractional exclusion statistics are realized in various models for quantum mechanics with inverse square exchange[7,13,14] and in the anyon model in a strong magnetic field.[15,9]

A. Exclusion statistics and the fqHe

A natural application of the idea of exclusion statistics is offered by the various fractional quantum Hall effects. One may take the somewhat naive but certainly justifiable point of view that the essence of the $\nu=1/m$ FQHE is that under the appropriate conditions interacting electrons give rise to free quasiparticles with effective statistics parameter $g=m$. A familiar interpretation of these quasiparticles is that they can be viewed as composites of electrons plus an even number of flux quanta. (The familiar terminology''composite fermions'' is somewhat unfortunate in this context, as we argue that the *exclusion* statistics properties of these composite quasiparticles are *not* fermionic. Clearly, their *exchange* statistics, which are determined ''modulo 2π,'' are fermionic.) The familiar Laughlin wave functions describe the ground state configuration for these quasiparticles. The fundamental excitations (the Laughlin quasiparticles) are expected to carry the ''dual'' (see Sec. III C) statistics $g=1/m$.

The above scenario, which was suggested in Haldane's original paper, has been critically analyzed in the literature, where it has been confirmed in the appropriate low-temperature regime (see, for example, Ref. 16). Our purpose in this paper is to setup and analyze a similar picture for *edge excitations* in the FQHE. Since such excitations can be described using the language of CFT, we first turn to a discussion of exclusion statistics in CFT spectra.

B. Exclusion statistics in CFT

Conformal field theories in two dimensions come with two commuting Virasoro algebras, and these infinite dimensional algebras can be used to organize the finite-size spectra of these theories. In such an approach, a CFT partition function is obtained by combining a number of characters of both Virasoro algebras (or extensions thereof). In applications such as string theory, where the conformal symmetry has a geometric origin and the fundamental fields are bosonic coordinate fields, this ''Virasoro approach'' to CFT is entirely natural. In contrast, the prototypical CFT in the condensed

matter arena is a theory of free fermions, with a finite size spectrum that is simply a collection of many-fermion states constructed according to the rules set by the Pauli principle. When facing other CFT's that are relevant for condensed matter systems one may try to follow a similar road, which is to select a number of fundamental quasiparticle operators and to construct the full (chiral) spectrum as a collection of many-(quasi)-particle states. Explicit examples of this are the so-called spinon bases for $\widehat{su(n)}_k$ Wess-Zumino-Witten (WZW) models.[18,20–22]

Let us now imagine that we have a concrete CFT, with explicit rules for the construction of a many-(quasi)-particle basis of the finite size spectrum. It is then natural to try to interpret that result in terms of ''exclusion statistics'' properties of the fundamental quasiparticles. In a recent paper,[6] one of us has proposed a systematic procedure (based on recursion relations for truncated chiral spectra), which leads to one-particle distribution functions for CFT quasiparticles. In many cases, it was established that the CFT thermodynamics are those of a free gas of quasiparticles governed by new, generalized distribution functions. The examples discussed in Ref. 6 include spinons in the $\widehat{su(n)}_1$ WZW models, CFT parafermions, and edge quasiparticles for the fractional QHE.

The example of the CFT for FQHE edge excitations is particularly interesting, since in those cases the generalized distributions derived from the CFT spectra are identical to those obtained from Haldane statistics (with specific values for g). In Sec. III below we show in some detail how these results are established.

Clearly, the identification of Haldane statistics in FQHE edge theories is most useful since it provides a concrete link between rather abstract considerations on the systematics of quasiparticle bases on the one hand and concrete laboratory physics on the other. In particular, it opens up the possibility of analyzing transport phenomena such as edge-to-edge tunneling in the QHE (which has been well studied both theoretically and experimentally) directly in terms of quasiparticles satisfying fractional exclusion statistics. We shall report the results of such an analysis in Sec. VI below.

III. QUASIPARTICLES FOR THE $\nu=1/m$ FQHE EDGE

We consider the finite size spectrum for the CFT describing a single $\nu=1/m$ FQHE edge. In CFT jargon, this theory is characterized as a $c=1$ chiral free boson theory at radius $R^2=m$. We shall consider the chiral Hilbert space corresponding to the following partition function

$$Z^{1/m}(q)=\sum_{Q=-\infty}^{\infty}\frac{q^{Q^2/2m}}{(q)_\infty}, \tag{3.1}$$

with $(q)_\infty=\Pi_{l=1}^{\infty}(1-q^l)$ and $q=e^{-\beta(2\pi/L)(1/\rho_0)}$. [The one-particle energies are of the form $\epsilon_l=l(2\pi/L)(1/\rho_0)$ with l an integer and ρ_0 the density of states per unit length, $\rho_0=(\hbar v_F)^{-1}$.] In this formula, the $U(1)$ affine Kac-Moody symmetry is clearly visible as all states at fixed $U(1)$ charge Q form an irreducible representation of this symmetry.

We should stress that the Hilbert space corresponding to Eq. (3.1) is not the physical Hilbert space for the edge theory of a quantum Hall sample with the topology of a disc. In the latter Hilbert space physical charge is quantized in units of e and, correspondingly, the $U(1)$ charge Q in Eq. (3.1) is restricted to multiples of m.[4] In the geometry of a Corbino disc, i.e., a cylinder, the operator that transfers charge e/m from one edge to the other is physical. Accordingly, the physical Hilbert space is obtained by taking a tensor product of left and right copies of the Hilbert space (3.1) and restricting the total $U(1)$ charge Q_L+Q_R to multiples of m.[4] In the quasiparticle formalism that we present below the various restrictions on Q_L, Q_R are easily implemented.

Our goal here is to understand the collection of states in Eq. (3.1) in a different manner, and to view them as multiparticle states built from the creation operators for edge quasiparticles $\Psi_{\rm e}$ and $\Psi_{\rm qh}$. To simplify our notations, we shall write $G\equiv\Psi_{\rm e}$, $\phi\equiv\Psi_{\rm qh}$. (The notation G is inspired by the fact that the fundamental anticommutation relations for the modes G_{-t} at $\nu=1/3$ are those of the so-called $N=2$ superconformal algebra. See Sec. VI for more on this.) Due to the above-mentioned restrictions on the $U(1)$ charges Q and Q_L+Q_R, the chiral quasihole operator $\phi(z)$ by itself is not a physical operator in the edge theories for the disc or cylinder [in the proper mathematical terminology we call $\phi(z)$ a chiral vertex operator (CVO)] physical states are obtained by restricting the number of ϕ quanta in the appropriate manner.

A. Quasihole states

We start by considering quasihole states that are built by applying only the modes ϕ_{-s} defined via $\phi(z)=\Sigma_s\phi_{-s}z^{s-1/2m}$. Clearly, the index s gives the dimensionless energy of the mode ϕ_{-s}. When acting on the charge-0 vacuum $|0\rangle$, we find the following multi-ϕ states (compare with Ref. 18 for the case $m=2$, see also Ref. 23)

$$\phi_{-(2N-1)/2m-n_N}\cdots\phi_{-3/2m-n_2}\phi_{-1/2m-n_1}|0\rangle$$
$$\text{with}\quad n_N\geqslant n_{N-1}\geqslant\cdots\geqslant n_1\geqslant 0. \tag{3.2}$$

The choice of minimal modes is such that the lowest state of charge $Q(e/m)$ is at energy $(Q^2/2m)$, in agreement with the scaling dimension of the corresponding CFT primary field. Using so-called generalized commutation relations satisfied by the modes ϕ_{-s} one may show[18] that all multi-ϕ states different from Eq. (3.2) are either zero or linearly dependent on Eq. (3.2).

Before writing more general states we shall first focus on the exclusion statistics properties of the quanta ϕ_{-s}. We follow the procedure of Ref. 6 and start by introducing truncated partition sums for quasihole states (3.2). For $s=1/2m$, $3/2m$, etc, we define polynomials $P_s(x,q)$ to keep track of the number of many-body states that can be made using only the modes ϕ_{-k} with $k\leqslant s$, and that have a highest occupied mode with energy s' such that $s-s'$ is an integer. $P_s(x,q)$ is defined as the trace of the quantity x^Nq^E over all these states, where N is the number of quasiholes, E is the dimensionless total energy, and $x=e^{\beta\mu_{\rm qh}}$. For $m=3$ this gives

$$P_{1/6}=xq^{1/6},\quad P_{1/2}=x^2q^{4/6},\quad P_{5/6}=1+x^3q^{9/6},\quad \text{etc.} \tag{3.3}$$

In general, an occupied quasi-hole state of energy s corresponds to a factor xq^s in these generating polynomials.

The systematics of the edge quasihole states (3.2) directly lead to the following recursion relations between the polynomials $P_s(x,q)$:

$$P_s(x,q)=P_{s-1}(x,q)+xq^sP_{s-1/m}(x,q). \tag{3.4}$$

For $m=1$, which is the case corresponding to a $\nu=1$ integer GHE edge, this relation directly implies $P_{l-1/2}(x,q)=\Pi_{j=1}^{l}(1+xq^{j-1/2})$. In that case the partition sum is simply a product and we recognize free fermions. For general m things are not that simple, but we can rewrite the recursion relation in matrix form

$$\begin{pmatrix} P_{l-(2m-1)/2m} \\ \vdots \\ P_{l-1/2m} \end{pmatrix} = M_l^{\rm qh}(x,q) \begin{pmatrix} P_{l-1-(2m-1)/2m} \\ \vdots \\ P_{l-1-1/2m} \end{pmatrix}, \tag{3.5}$$

with $l=1,2,\ldots$, and $M_l^{\rm qh}(x,q)$ the following $m\times m$ matrix:

$$M_l^{\rm qh}(x,q)=\begin{pmatrix} 1 & 0 & \cdots & 0 & xq^{l-(2m-1)/2m} \\ xq^{l-(2m-3)/2m} & 1 & \cdots & 0 & x^2q^{2l-(4m-4)/2m} \\ \vdots & & \ddots & & \vdots \\ x^{m-1}q^{(m-1)l-m/2+(2m-1)/2m} & x^{m-2}q^{(m-2)l-m/2+(4m-4)/2m} & \cdots & xq^{l-1/2m} & 1+x^mq^{ml-m/2} \end{pmatrix}. \tag{3.6}$$

The grand partition function for the quasihole states (3.2) is then given by

$$Z^{\rm qh}(x,q)=(1 \quad 1 \quad \cdots \quad 1)\left(\prod_{l=1}^{\infty} M_l^{\rm qh}(x,q)\right)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

We propose that the quasihole modes ϕ_{-s} with $s=l-(2m-1)/2m,\ldots,l-1/2m$ be viewed as a single (m-fold degenerate) level in the one-particle spectrum. [This convention is natural since a single quasiparticle over the ground state can only occupy one of these m levels.] The $m\times m$ matrix $M_l^{\rm qh}$ is then a level-to-level transfer matrix and replaces the free fermion ($m=1$) factor $(1+xq^{l-1/2})$. Clearly, the thermodynamics of the states (3.2) will be dominated by the largest eigenvalues $\lambda_l^+(x,q)$ of the matrices $M_l^{\rm qh}(x,q)$. These satisfy the characteristic equations

$$(\lambda_l^+-1)^m-x^mq^{ml-m/2}(\lambda_l^+)^{m-1}=0. \tag{3.8}$$

Instead of trying to solve these equations, we can derive from them a result for the one-particle distribution functions

$$n^{\rm qh}(l)\equiv x\partial_x\,\ln(\lambda_l^+)=x\frac{\partial_x\lambda_l^+}{\lambda_l^+}. \tag{3.9}$$

We find

$$n^{\rm qh}(l)=\frac{\lambda_l^+-1}{1+(1/m)(\lambda_l^+-1)},$$

$$(xq^l)^{-1}=(\lambda_l^+-1)^{-1}(\lambda_l^+)^{1-1/m}. \tag{3.10}$$

Comparing with Eq. (2.1) and identifying $g=1/m$ and $w(\epsilon)=(\lambda_l^+-1)^{-1}$, we see that the distribution function $n^{\rm qh}(l)$ becomes identical to $n_{g=1/m}(\epsilon=l)$. In other words, the exclusion statistics properties of the $\nu=1/m$ quasiholes are those of ''ideal g-ons'' in the sense of Haldane, with $g=1/m$! This identification is consistent with the result of bosonization applied to g-ons,[24] and with the character computations of Ref. 25.

For the case $m=2$, which is not in the category of FQHE edges, the equilibrium distribution is given by

$$n_{1/2}(\epsilon)=\frac{2}{\sqrt{1+4e^{-2\beta(\mu-\epsilon)}}}. \tag{3.11}$$

For $m=3$ the explicit formulas (obtained using the Cardano formula for cubic equations) are quite unpleasant; Fig. 1 shows the distribution $n_{1/3}(\epsilon)$.

B. Edge electron states

The same procedure can be applied to the edge electrons, which are created by modes G_{-t} with $G(z)=\Sigma_t G_{-t}z^{t-m/2}$. Multielectron states take the form

$$G_{-(2M-1)(m/2)-m_M}\cdots G_{-3(m/2)-m_2}G_{-m/2-m_1}|0\rangle$$

with

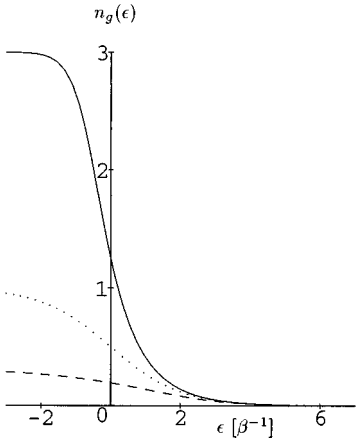

FIG. 1. Distribution functions for fractional exclusion statistics with $g=3$ (dashed line), $g=1$ (dotted line), and $g=\frac{1}{3}$ (solid line), all at the same temperature and at zero chemical potential.

$$m_M \geqslant m_{M-1} \geqslant \cdots \geqslant m_1 \geqslant 0 \tag{3.12}$$

and we have truncated partition sums $Q_t(y,q)$ with t a half-odd integer and $y=e^{\beta\mu_e}$. They satisfy the recursion relations

$$Q_t(y,q)=Q_{t-1}(y,q)+yq^tQ_{t-m}(y,q), \tag{3.13}$$

with the following initial values:

$$Q_{-m/2}=\cdots=Q_{m/2-1}=1. \tag{3.14}$$

The ''transfer matrix'' for the edge electrons $M_k^e(y,q)$ is defined by

$$\begin{pmatrix} Q_K \\ \vdots \\ Q_{K+m-1} \end{pmatrix} = M_k^e(y,q) \begin{pmatrix} Q_{K-m} \\ \vdots \\ Q_{K-1} \end{pmatrix} \tag{3.15}$$

with $K=km-m/2$, $k=1,2,\ldots$, and we have

$$Z^e(y,q)=(1 \quad 0 \quad \cdots \quad 0 \quad 0)\left(\prod_{k=1}^{\infty} M_k^e(y,q)\right)\begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{3.16}$$

In this case, a single action of the transfer matrix comprises a jump of m one-particle levels, and the relevant distribution function will be

$$n^e(k)\equiv y\partial_y \ln[(\mu_k^+)^{1/m}]=\frac{y}{m}\frac{\partial_y\mu_k^+}{\mu_k^+}. \tag{3.17}$$

The characteristic equation for the eigenvalue μ_k^+

$$\prod_{i=0}^{m-1}(\mu_k^+-yq^{mk-i})-(\mu_k^+)^{m-1}=0 \tag{3.18}$$

leads to

$$n^e(k)=\frac{1}{m+(h_k-1)}, \quad (yq^{mk})^{-1}=(h_k-1)^m h_k^{1-m}. \tag{3.19}$$

with $h_k=\mu_k^+/(yq^{mk})$. Identifying $w(\epsilon)=h_k-1$, we again recognize the Wu equations (2.1) for Haldane exclusion statistics, this time with $g=m$, and we may identify $n^e(k)$ with $n_{g=m}(\epsilon=mk)$.

For $m=2$ this gives

$$n_2(\epsilon)=\frac{1}{2}\left(1-\frac{1}{\sqrt{1+4e^{-\beta(\epsilon-\mu)}}}\right). \tag{3.20}$$

See Fig. 1 for the distribution function $n_3(\epsilon)$ at $\mu_e=0$.

C. Duality

Having recognized distribution functions for fractional exclusion statistics with $g=1/m$ and $g=m$, respectively, we expect a particle-hole duality between the two cases (compare with Refs. 10, 12).

Before we come to that, we generalize the results of Secs. III A and III B by considering a chiral $c_{\mathrm{CFT}}=1$ CFT of compactification radius $R^2=r/s$, with $r>s$ and r,s coprime. Choosing ϕ quanta of charge $+(s/r)e$ and G quanta of charge $-e$ as our fundamental excitations, we easily repeat the previous analysis and derive the following recursion relations:

$$X_l(x)=X_{l-r}(x)+xX_{l-s}(x),$$

$$Y_l(y)=Y_{l-s}(y)+yY_{l-r}(y), \tag{3.21}$$

where we put $q=1$ for convenience. (The connection with the quantities P_l and Q_l defined for $r/s=m$ is $X_l \leftrightarrow P_{(2l-1)/2m}$, $Y_l \leftrightarrow Q_{m/2+(l-1)}$.) Proceeding as before we obtain the distribution functions for Haldane statistics with $g=s/r$ (for the ϕ quanta) and $\tilde{g}=r/s$ (for the G quanta).

In Refs. 10, 12 it was recognized that the cases with g and $\tilde{g}=1/g$ are dual in the sense that particles are dual to holes. To recover this duality in our present approach, we note that if $Y_l(y)$ is a solution of the second relation in Eq. (3.21), the expression

$$X_l(x)=Y_l(y=x^{-\tilde{g}})x^{l/s} \tag{3.22}$$

solves the first recursion relation. Assuming $r>s$, we can rewrite both recursion relations in a form involving a $r\times r$ recursion matrix. The largest eigenvalues $\lambda^+(x)$ and $\mu^+(y)$ are then related via

$$\lambda^+(x)=\mu^+(x^{-\tilde{g}})x^{\tilde{g}} \tag{3.23}$$

and the distribution functions

$$n_g(x)\equiv x\partial_x \ln \lambda^+(x), \quad n_{\tilde{g}}(y)\equiv gy\partial_y \ln \mu^+(y), \tag{3.24}$$

satisfy

$$gn_g(x)=1-\tilde{g}n_{\tilde{g}}(y=x^{-\tilde{g}}), \tag{3.25}$$

or, putting $\mu_{\tilde{g}}=-\tilde{g}\mu_g$ and restoring $q\neq 1$

$$gn_g(\epsilon)=1-\tilde{g}n_{\tilde{g}}(-\tilde{g}\epsilon), \tag{3.26}$$

in agreement with the results of Ref. 10. The interpretation of this result is that the $\tilde{g}$ quanta with positive energy act as holes in the ground state distribution of negative energy g quanta. The relative factor $(-\tilde{g})$ between the energy arguments in Eq. (3.26) indicates that the act of taking out r g quanta corresponds to adding s $\tilde{g}$ quanta. This duality further implies that, when setting up a quasiparticle description for fractional QHE edges, we can opt for (i) either quasiholes or edge electrons, with energies over the full range $-\infty<\epsilon<\infty$ or (ii) a combination of both types of quasiparticles, each having positive energies only. The CFT finite size spectrum naturally leads to (ii) (see Sec. III D below), while the analogy with Calogero-Sutherland quantum mechanics naturally leads to option (i) (see Sec. V). When considering transport equations in Sec. VI, we shall be considering both alternatives.

D. The full spectrum

To complete our quasiparticle description for the $\nu=1/m$ edge, we need to specify how quasihole and electron operators can be combined to produce a complete basis for the chiral Hilbert space (3.1). We consider the following set of states:

$$G_{-(2M-1)m/2+Q-m_M}\cdots G_{-m/2+Q-m_1}$$
$$\times\phi_{-(2N-1)1/2m-Q/m-n_N}\cdots\phi_{-1/2m-Q/m-n_1}|Q\rangle$$

with

$$m_M\geqslant m_{M-1}\geqslant\cdots\geqslant m_1\geqslant 0,$$
$$n_N\geqslant n_{N-1}\geqslant\cdots\geqslant n_1\geqslant 0,\quad n_1>0\quad\text{if } Q<0, \tag{3.27}$$

where $|Q\rangle$ denotes the lowest energy state of charge $Q(e/m)$ with Q taking the values $-(m-1),-(m-2),\ldots,-1,0$. Our claim is now that the collection (3.27) forms a basis of the chiral Hilbert space, so that

$$Z^{1/m}(q)=\sum_{Q=-(m-1)}^{0} q^{Q^2/2m}Z_Q^{\rm qh}(x=1,q)Z_Q^{\rm e}(y=1,q), \tag{3.28}$$

where we added a factor $q^{Q^2/2m}$ to take into account the energy of the initial states and we denoted by $Z_Q^{\rm qh}$ and $Z_Q^{\rm e}$ the generalizations of the partition functions (3.7) and (3.16) to the sector with vacuum charge Q. They are naturally written as

$$Z_Q^{\rm qh}=\sum_{N=0}^{\infty}\frac{q^{(1/2m)(N^2+2QN)+(1-\delta_{Q,0})N}}{(q)_N},$$
$$Z_Q^{\rm e}=\sum_{M=0}^{\infty}\frac{q^{(m/2)M^2-QM}}{(q)_M}, \tag{3.29}$$

with $(q)_L=\Pi_{l=1}^{L}(1-q^l)$.

While the collection of states (3.27) looks rather complicated, it may be understood by considering the special case $m=1$, which is a theory of two real free fermions of charge ± 1. In this case there is only the $Q=0$ vacuum and the allowed ϕ and G modes reduce to the familiar free fermion modes $\psi^{\pm}_{-1/2-n_j}$.

The right hand side of Eq. (3.28) has the form of a so-called ''fermionic sum formula'' [17] and the equality of Eqs. (3.1) and (3.28) is a Rogers-Ramanujan identity. Similar identities relating ''fermionic sums'' to characters in conformal field theories have been studied in the literature, see, for example, Refs. 17–19. We would like to stress that the reasoning leading to these identities is very different between our approach and the work of Ref. 17: in our approach the identities express exclusion statistics properties of CFT fields, while in the work of Kedem *et al.* the identities are based on Bethe ansatz solutions of specific integrable lattice models. The first example where these two approaches have been explicitly connected is that of spinons in $SU(2)_1$ CFT and in the associated Haldane-Shastry spin chains.[20,21,18]

The important conclusion from the above is that, up to a finite sum over vacuum charges, the chiral partition sum factorizes as a product of a quasihole piece and an edge electron piece. This means that the two types of quasiparticles are independent or, in other words, that they do not have any mutual exclusion statistics. This then explains our asymmetric choice of quasiparticles. Had we chosen to work with fundamental quasiparticles of charges $\pm e/m$, we would have come across nontrivial mutual statistics. All of this is nicely illustrated with the case $m=2$ where we can opt for the ''QHE basis'' with independent quasiholes and edge electrons, or for a ''spinon basis'' built from charge $\pm e/2$ quanta, which are identical to the spinons of Refs. 20, 21, 18 and which have a nontrivial 2×2 statistics matrix. The two choices have the quasihole states (3.2) (called ''fully polarized spinon states'' in Ref. 18) in common, but differ in the way negative charges are brought in.

The observations made in this section may be generalized to composite edges such as those of the so-called Jain series with filling fraction $\nu=n/(np+1)$. We refer to Appendix A for a brief discussion.

IV. EQUILIBRIUM QUANTITIES

A. Specific heat

The specific heat of a conformal field theory is well-known to be proportional to the central charge $c_{\rm CFT}$

$$\frac{C(T)}{L}=\gamma\rho_0 k_B^2 T,\quad \gamma=\frac{\pi}{6}c_{\rm CFT}, \tag{4.1}$$

where $\rho_0=(\hbar v_F)^{-1}$ is the density of states per unit length. In Ref. 26 it was shown that the specific heat for g-on excitations, with energies in the full range $-\infty<\epsilon<\infty$, is in agreement with the central charge $c_{\rm CFT}=1$ of the corresponding CFT. The same result should of course come out in a picture where we select positive energy electrons and positive energy quasi-holes as our fundamental excitations. In this picture the total energy carried by the edge quasiparticles takes the form

$$E=\rho_0\int_0^{\infty}d\epsilon\,\epsilon n_g(\epsilon)+\rho_0\int_0^{\infty}d\epsilon\,\epsilon n_{\tilde g}(\epsilon) \tag{4.2}$$

and the corresponding result for the specific heat is

$$\frac{C(T)}{L}=(\gamma_{g,+}+\gamma_{\tilde g,+})\rho_0 k_B^2 T, \tag{4.3}$$

where

$$\gamma_{g,+}=\partial_\beta\int_0^{\infty}d\epsilon\,\epsilon n_g(\epsilon),$$
$$\gamma_{\tilde g,+}=\partial_\beta\int_0^{\infty}d\epsilon\,\epsilon n_{\tilde g}(\epsilon). \tag{4.4}$$

It takes an elementary application of the duality relation (3.26) to show that $\gamma_{\tilde g,+}=\gamma_{g,-}$ and hence

$$(\gamma_{g,+}+\gamma_{\tilde g,+})=\gamma_g=\frac{\pi}{6}, \tag{4.5}$$

confirming once again the value $c_{\rm CFT}=1$.

We would like to stress that the individual contributions $\gamma_{g,+}$ do depend on g and that only for $g=1$ (Majorana fermions) $\gamma_{g,+}$ and $\gamma_{g,-}$ are equal. An exact result is[27]

$$\gamma_{g,+}=\frac{\pi}{6}\frac{L(\xi^g)}{L(1)}, \tag{4.6}$$

with ξ a solution of the algebraic equation

$$\xi^g = 1 - \xi \tag{4.7}$$

and $L(z)$ the Rogers dilogarithm. This gives

$$\gamma_{1/2,+} = \frac{\pi}{6}\frac{3}{5}, \quad \gamma_{2,+} = \frac{\pi}{6}\frac{2}{5},$$

$$\gamma_{1/3,+} = \frac{\pi}{6}0.655,\ldots, \quad \gamma_{3,+} = \frac{\pi}{6}0.344,\ldots, \quad \text{etc.} \tag{4.8}$$

B. Hall conductance

While the specific heat coefficient γ is not sensitive to g, the edge capacitance or, equivalently, the Hall conductance, obviously does depend on the filling fraction ν and thereby on g. In the quasiparticle formulation, this result comes out in a particularly elegant and simple manner.

Let us focus on a $\nu = 1/m$ edge and take as our fundamental quasiparticles the edge electron of charge $q = -e$ and statistics $g = m$ and the edge quasihole of charge $q = e/m$ and statistics $\tilde{g} = 1/m$, all quasiparticles having positive energies only.

Let us first consider zero temperature, where the Haldane distribution functions are step functions with maximal value $n_g = 1/g$. If we now put a voltage $V > 0$ the $q < 0$ quasiparticles will see their Fermi energy shift by the amount qV and all available states at energy up to $-qV$ will be filled. The total charge $\Delta Q(V, T=0)$ that is carried by these excitations equals [we use the symbol ΔQ for the total charge, while we keep Q for the reduced charge (in units of e/m)]

$$\Delta Q(V, T=0) = \frac{1}{g} q \rho_0 (-qV), \tag{4.9}$$

where the factor $1/g$ originates from the maximum of the distribution function and thus represents the statistics properties of the quasi-particles. Clearly, positive-q quasiparticles do not contribute to the response at $T=0$, $V>0$.

For the $\nu = 1/m$ FQHE edges, the result for $V > 0$ is

$$\Delta Q(V>0, T=0) = \frac{1}{m}(-e)\rho_0(eV) = -\frac{e^2}{m}\rho_0 V \tag{4.10}$$

while for $V < 0$

$$\Delta Q(V<0, T=0) = m\frac{e}{m}\rho_0\left(-\frac{e}{m}V\right) = -\frac{e^2}{m}\rho_0 V. \tag{4.11}$$

Clearly, the edge capacitance

$$\frac{\Delta Q(V, T=0)}{V} = -\rho_0 \frac{e^2}{m} \tag{4.12}$$

is independent of the sign of V and we establish the correct value of the Hall conductance

$$G = \frac{1}{\rho_0 h}\frac{|\Delta Q|}{V} = \frac{1}{m}\frac{e^2}{h}. \tag{4.13}$$

To show that the results (4.12), (4.13) hold for finite temperatures as well we write the general expression

$$\Delta Q(V,T) = -e\rho_0 \int_0^\infty d\epsilon\, n_m(\epsilon + eV) + \frac{e}{m}\rho_0 \int_0^\infty d\epsilon\, n_{1/m}\left(\epsilon - \frac{e}{m}V\right) \tag{4.14}$$

and evaluate $\partial_\beta \Delta Q(V,T)$. Using once again the duality relations (3.26), we derive

$$\partial_\beta \Delta Q = \frac{e}{m}\,\rho_0 \partial_\beta \int_{-\infty}^{\infty} d\epsilon\, n_{1/m}(\epsilon) \propto \int_0^\infty dx\, \ln(x)\,\partial_x n_{1/m}(x) \tag{4.15}$$

with $x = e^{-\beta\epsilon}$. Using Eq. (3.9) the last line turns into

$$\propto \lim_{x_0 \to \infty} [\ln \lambda^+(x) - n_{1/m}(x)\ln(x)]_0^{x_0} \tag{4.16}$$

and by using the asymptotic behavior for $x \to \infty$

$$\lambda^+(x) \approx x^m, \quad n_{1/m} \approx m \tag{4.17}$$

we conclude that $\partial_\beta \Delta Q$ is indeed zero.

V. JACK POLYNOMIALS AND BEYOND

The quasiparticle basis that we specified in Eq. (3.27) has some arbitrariness to it. For example, we could have chosen to act first with the G_{-t} and then with ϕ_{-s}, which would have lead to a different set of states. Also, one quickly finds that the states (3.27) as they stand are not mutually orthogonal. For the purpose of establishing the thermodynamics of the FQHE edge theory, what matters is the counting of the number of states with given charge and energy, and this information can be extracted from Eq. (3.27). However, for the analysis of more detailed questions, in particular those concerning transport, the precise form of the multi-quasi-particle states is of crucial importance.

In this section, we shall present an ''improved'' set of multiparticle states, which are mutually orthogonal and which are faithful to the statistics properties of the quasiparticles ϕ_{-s} and G_{-t}. The idea will be to specify an operator $H_{\rm CS}$ that acts on the CFT spectrum, and to modify the multiparticle states in such a way that they become eigenstates of $H_{\rm CS}$. The operator $H_{\rm CS}$, which was first given by Iso in Ref. 23, will be nothing else than a CFT version of the Hamiltonian of so-called Calogero-Sutherland (CS) quantum mechanics with inverse square exchange. The analogy with CS quantum mechanics confirms the assignment of $g = 1/m$ ($g = m$) exclusion statistics to ϕ_{-s} and G_{-t}, which are the CFT analogs of the particles and holes of the CS system. It also links the Jack polynomial eigenstates of the CS system to the quasiparticle basis of the FQHE edge theory.

We would like to stress that, in the context of the $\nu = 1/m$ QHE edge, we do not assign physical significance to the operator $H_{\rm CS}$. We merely use this operator as a device to select an optimal set of multiparticle states, where ''optimal'' is meant in the sense of mutual orthogonality and of a

relatively simple form of matrix elements of physical operators between the states.

The need for improving the form of the multiparticle states (3.27) can be phrased in yet another way. Let us, as an example, consider a multiparticle state containing two ϕ quanta. If we were to work in position space, putting the two ϕ fields at positions z_2 and z_1, the *exchange statistics* properties of the field $\phi(z)$ would result in simple phase factors associated to the interchange $z_2 \leftrightarrow z_1$ in a correlator. Working instead in energy space, with ϕ quanta ϕ_{-s_2} and ϕ_{-s_1}, we expect that the *exclusion statistics* properties of ϕ will imply simple behavior under the interchange $s_2 \leftrightarrow s_1$ in a correlator or form factor. In particular, one expects that interchanging $s_2 \leftrightarrow s_1$ in a form factor involving a state $|\ldots,s_2,s_1,\ldots\rangle$ will result in a phase factor $e^{i(\pi/m)}$. We shall show below [see, e.g., Eq. (5.10)], that the form factors of the true ''Jack polynomial'' multiparticle states $|\ldots,s_2,s_1,\ldots\rangle$ indeed satisfy this simple property, which is not valid for the naive multiparticle state $(\ldots\phi_{-s_2}\phi_{-s_1}\ldots)|0\rangle$. In mathematical terms, the issue is to define the correct coproduct in a situation where, due to fractional statistics, the relevant symmetry is not a Lie algebra but rather a quantum group. In the context of the spinon basis for the $m=2$ theory, this quantum group is a so-called yangian, and it has been established that the ''Jack-polynomial'' coproduct agrees with the coproduct that is dictated by the quantum group symmetry.[21,18]

A. The operator $H_{\rm CS}$

To specify the operator $H_{\rm CS}$, we employ the free boson $\varphi(z)$, which already featured in our formula (1.2). Following Ref. 23, we define

$$H_{\rm CS}=\frac{m-1}{m}\sum_{l=0}^{\infty}(l+1)(i\sqrt{m}\partial\varphi)_{-l-1}(i\sqrt{m}\partial\varphi)_{l+1}+\frac{1}{3m}[(i\sqrt{m}\partial\varphi)^3]_0, \tag{5.1}$$

where $\partial\varphi(z)=\Sigma_l(\partial\varphi)_l z^{-l-1}$ and where the second term on the right-hand side denotes the zero mode of the normal ordered product of three factors $(i\sqrt{m}\partial\varphi)(z)$. As a first result, one finds the following action of $H_{\rm CS}$ on states containing a single quasiparticle of charge e/m or $-e$:

$$H_{\rm CS}\phi_{-1/2m-n}|0\rangle=h_\phi(n)\phi_{-1/2m-n}|0\rangle,$$

with

$$h_\phi(n)=\left[\frac{1}{3m}+mn\left(n+\frac{1}{m}\right)\right]$$

and

$$H_{\rm CS}G_{-m/2-n}|0\rangle=h_G(n)G_{-m/2-n}|0\rangle,$$

with

$$h_G(n)=\left[-\frac{m^2}{3}-n(n+m)\right]. \tag{5.2}$$

We would like to stress that the fact that both ϕ_s and G_t diagonalize $H_{\rm CS}$ is quite nontrivial. If one evaluates $H_{\rm CS}$ on any vertex operator $\phi^{(Q)}$ [of charge $Q(e/m)$], one typically runs into the field product $(T\phi^{(Q)})(z)$, where $T(z)=-\frac{1}{2}(\partial\varphi)^2(z)$ is the stress energy of the scalar field φ. Only for $Q=1$ and $Q=-m$ do such terms cancel and do we find that the quasiparticle states are eigenstates of $H_{\rm CS}$.

We can now continue and construct eigenstates of $H_{\rm CS}$ which contain several ϕ or G quanta. What one then finds is that the simple product states such as Eq. (3.2) are not $H_{\rm CS}$ eigenstates, but that they rather act as head states that need to be supplemented by a tail of subleading terms. As an example, one finds two-ϕ eigenstates to be of the form[18]

$$|n_2,n_1\rangle=\phi_{-3/2m-n_2}\phi_{-1/2m-n_1}|0\rangle+\sum_{l=1}^{\infty}a_l[\phi_{-3/2m-n_2-l}\phi_{-1/2m-n_1+l}|0\rangle] \tag{5.3}$$

with coefficients a_l that can be computed. The connection of the coefficients a_l with the Jack polynomials that feature in the eigenfunctions in CS quantum mechanics has been made explicit in Ref. 18. For the $H_{\rm CS}$ eigenstate headed by the multiparticle state (3.27) (with unit coefficient), we shall use the notation

$$|\{m_j\},\{n_i\}\rangle \tag{5.4}$$

so that

$$H_{\rm CS}|\{m_j\},\{n_i\}\rangle=\left[\sum_{j=1}^{M}h_G[(j-1)m+m_j]+\sum_{i=1}^{N}h_\phi\left(\frac{1}{m}(i-1)+n_i\right)\right]|\{m_j\},\{n_i\}\rangle. \tag{5.5}$$

Clearly, the states (5.4), with the m_j and n_i as specified in Eq. (3.27), form a complete and orthogonal basis for the chiral Hilbert space.

B. Norms and form factors

Of importance for later calculations are the norms of the states (5.4) and the matrix elements of physical operators between these states. For the explicit evaluation of such quantities we used the connection with Jack polynomials, relying on results that are available in the mathematical literature[28] (see also Ref. 29).

As an example, we focus on multi-quasi-hole states $|\{n_i\}\rangle$. To make contact with the Jack's, we view the ordered set $\{n_i\}$ as a Young tableau λ. The norm squared of the state $|\{n_i\}\rangle$ then becomes

$$\langle\{n_i\}|\{n_i\}\rangle=j_{\lambda'}, \tag{5.6}$$

where λ' is the Young tableau dual to λ and the j_λ are taken from Ref. 28. Explicit examples are

$$\langle n_1|n_1\rangle = j_{(1^{n_1})} = \frac{(n_1+1/m-1)(n_1+1/m-2)\cdots 1/m}{n_1(n_1-1)\cdots 1},$$

$$\langle n_2,n_1|n_2,n_1\rangle = j_{(2^{n_1},1^{n_2-n_1})} = \frac{(n_2+2/m-1)(n_2+2/m-2)\cdots(n_2+2/m-n_1)}{(n_2+1/m)(n_2+1/m-1)\cdots(n_2+1/m-n_1+1)} \frac{(n_2-n_1+1/m-1)\cdots 1/m}{(n_2-n_1)\cdots 1} \frac{(n_1+1/m-1)\cdots 1/m}{n_1\cdots 1}, \quad (5.7)$$

etc. In the limit where all $n_i \gg 1$, one finds

$$\langle\{n_i\}|\{n_i\}\rangle \approx \prod_{i=1}^{N} \frac{{n_i}^{1/m-1}}{\Gamma(1/m)}. \quad (5.8)$$

Of interest for the analysis of processes where electrons or holes tunnel into a FQHE edge is the form factor

$${}_N\langle\{n_m,\ldots,n_2,n_1\}|G^{\dagger}_{-m/2-n}|0\rangle = f(n_m,\ldots,n_1)\,\delta_{n,n_m+\cdots+n_1}. \quad (5.9)$$

where the subscript N indicates that the state has been properly normalized. This form factor describes the amplitude by which an incoming hole (described by the operator $G^{\dagger}$ and of charge $+e$) creates a state that has m quasiholes excited over the ground state. Explicit computation in the limit where all $n_i \gg 1$ yields (for simplicity we give the result for $m=3$, see Appendix B for the general case)

$$f(n_3,n_2,n_1) \approx \frac{\Gamma(1/3)^{1/2}}{\Gamma(2/3)} \left(\frac{(n_3-n_2)(n_3-n_1)(n_2-n_1)}{n_3 n_2 n_1}\right)^{1/3}. \quad (5.10)$$

Remarkably, this result takes the form of a ''Jastrow factor'' in the energy variables n_i. The order-$(\frac{1}{3})$ zeros when two n_i come near reflect the $g=\frac{1}{3}$ exclusion statistics properties of the fundamental quasiholes. Note that the expression (5.10) is invariant under global scalings of all energies n_i. The form (5.10) of the form factor can be viewed as a limit in (chiral) CFT of a result on correlation functions for the ''classical'' model of quantum mechanics with inverse square exchange. This result was conjectured by Haldane[30] and later proven in Refs. 13, 29.

VI. TRANSPORT PROPERTIES

Having checked that the thermodynamics of FQHE edges is correctly reproduced in the quasiparticle language we are now ready to move on and consider transport properties. Following the setup of a number of recent experiments, we shall consider a situation where electrons (or holes) from a Fermi-liquid reservoir are allowed to tunnel into a $\nu=1/m$ FQHE edge. The dc I-V characteristic for this setup, which were first computed by Kane and Fisher[31] (see also Ref. 32), show a crossover from a linear (thermal) regime into a power-law behavior at high voltages and thus presents a clear fingerprint of the Luttinger-liquid features of the FQHE edge. The experimental results from Ref. 8 are in agreement with these predictions. (See Ref. 2 for a further theoretical analysis of these data.)

The calculations by Kane and Fisher were based on bosonization and on the Keldysh formalism for nonequilibrium transport. Our goal here is to see if we can reproduce their results in an approach directly based on the edge quasiparticle formalism. Before going into this, we would like to stress that the ''thermodynamic Bethe ansatz (TBA) quasiparticles'' behind the approach of Ref. 2 are quite different from what we have here, the most important distinction being that the TBA quasiparticles are a combination of degrees of freedom of both sides of the tunneling barrier; they do not exist for a $\nu=1/m$ edge in isolation.

If the $\nu=1/m$ FQHE edge were to behave as a Fermi liquid, we could calculate charge transport across a barrier using a simple (Boltzmann) kinetic equation of the form

$$I(V,T) \propto e\int_{-\infty}^{\infty} d\epsilon W\{f_1(\epsilon-eV)F_2(\epsilon)-F_1(\epsilon-eV)f_2(\epsilon)\}, \quad (6.1)$$

with $f(\epsilon)$ and $F(\epsilon)$ the Fermi-Dirac distributions for electrons and holes, respectively, and W the probability for an electron or hole of energy ϵ to cross the barrier and enter the edge. As is well known, this Boltzmann equation leads to an ohmic (linear in V) and temperature-independent current. Now that we have seen that the non-Fermi-liquid features of the $1/m$ edge can be captured via the statistics of the edge quasiparticles we can try to write a ''Boltzmann equation'' for transport to and from FQHE edges by putting in appropriate generalizations $h(\epsilon)$ and $H(\epsilon)$ of the quantities $f_2(\epsilon)$ and $F_2(\epsilon)$, respectively. Before giving precise results (in Sec. VI A below) we shall consider a ''naive'' expression based on the intuition from the quasiparticle approach. In first approximation, the factor $h(\epsilon)$, which describes the probability for an electron to *leave* a $\nu=1/m$ edge, comprises two effects.

(1) A correlation effect, which can be traced to the nontrivial scaling dimension of the edge electron operator [see, for example, Eq. (1.3)]. At zero temperature, this is the so-called tunneling density of states

$$A^{+}(\epsilon) \propto \epsilon^{m-1}. \quad (6.2)$$

(2) A temperature dependence related to the exclusion statistics properties of the edge electrons. As we have seen, the natural factor associated to the *presence* of an edge electron is the distribution function

$$n_{g=m}(\epsilon). \quad (6.3)$$

Combining these factors, we come to the naive expressions

$$h^{(0)}(\epsilon) = \epsilon^{m-1} n_{g=m}(\epsilon), \quad (6.4)$$

and by similar reasoning we obtain

$$H^{(0)}(\epsilon)=\epsilon^{m-1}e^{\beta\epsilon}n_{g=m}(\epsilon), \tag{6.5}$$

where the thermal factor $e^{\beta\epsilon}n_m(\epsilon)$ has been dictated by the requirement of detailed balance. (This same thermal factor was proposed in Ref. 33, which proposes a generalization of the Boltzmann equation to the case of fractional statistics. In our view, the proposal of Ref. 33 is incomplete, as it ignores the correlation effects which are unavoidable for quasiparticles obeying fractional exclusion statistics.)

One quickly finds that the Boltzmann equation with factors $h^{(0)}$ and $H^{(0)}$ is not exact at finite temperature. In Sec. VI B we shall further comment on this equation and argue that it can be viewed as a first stage in a systematic approach. Before we come to that, we shall in the next section present a particularly simple derivation of the exact perturbative I-V characteristics for tunneling from a Fermi liquid to a $\nu=\frac{1}{3}$ FQHE edge. This derivation uses the idea of a kinetic equation, together with the algebraic properties of the edge electrons.

A. Kinetic equation for interedge transport

A careful derivation, based directly on the form of the tunneling Hamiltonian [we write Ψ for the edge electron operator (denoted by Ψ_e in Sec. II and by G in Sec. III), and we indicate the filling fraction by an explicit subscript]

$$H_{\rm int}\propto t\int d\epsilon[\Psi^{\dagger}_{\nu=1}(\epsilon)\Psi_{\nu=1/3}(\epsilon)+\text{H.c.}], \tag{6.6}$$

leads to the following kinetic equation (see, e.g., Ref. 32):

$$I(V,T)\propto et^2\int_{-\infty}^{\infty}d\epsilon[f(\epsilon-eV)H(\epsilon)-F(\epsilon-eV)h(\epsilon)], \tag{6.7}$$

where h,H are one particle Green's functions

$$H(\epsilon)=\langle\Psi^{\dagger}_{\nu=1/3}(\epsilon)\Psi_{\nu=1/3}(\epsilon)\rangle_{V,T},$$

$$h(\epsilon)=\langle\Psi_{\nu=1/3}(\epsilon)\Psi^{\dagger}_{\nu=1/3}(\epsilon)\rangle_{V,T} \tag{6.8}$$

for edge electrons in the $\nu=\frac{1}{3}$ FQHE edge, taken at $V=0$. Note that the expression (6.7) is perturbative as it gives the lowest nontrivial order in the parameter t.

The quantities $H(\epsilon)$ and $h(\epsilon)$ can be determined by using two simple observations. The first is that of *detailed balance*, which can be phrased as the requirement that at zero voltage there should be no current flowing. This fixes the ratio of $H(\epsilon)$ and $h(\epsilon)$ according to

$$H(\epsilon)=e^{\beta(\epsilon-eV)}h(\epsilon). \tag{6.9}$$

The second observation uses the *algebraic properties* of the edge electron operator, which include the anticommutation relation

$$\{\Psi^{\dagger}_{\nu=\frac{1}{3}}(\epsilon),\Psi_{\nu=\frac{1}{3}}(\epsilon')\}=\frac{2\pi}{L}\frac{1}{\rho_0}\epsilon^2\delta(\epsilon-\epsilon')+6\frac{E}{\rho_0}+3(\epsilon+\epsilon')\frac{\Delta Q}{e\rho_0}. \tag{6.10}$$

In this formula, E is the operator for the total energy per unit length (proportional to the Virasoro zero mode L_0), and ΔQ is the operator for the total charge per unit length [proportional to the zero mode J_0 of the $U(1)$ Kac-Moody algebra]. Clearly, this anti-commutator fixes the sum $H(\epsilon)+h(\epsilon)$. The expectation values of energy and charge follow directly from our analysis in Sec. IV. We find

$$\langle E\rangle=\rho_0\left(\frac{\pi^2}{6\beta^2}+\frac{(eV)^2}{6}\right),\ \langle\Delta Q\rangle=-e\rho_0\frac{(eV)}{3} \tag{6.11}$$

and obtain the exact expressions

$$H(\epsilon)=\frac{(\epsilon-eV)^2+\pi^2/\beta^2}{e^{-\beta(\epsilon-eV)}+1},\quad h(\epsilon)=\frac{(\epsilon-eV)^2+\pi^2/\beta^2}{1+e^{\beta(\epsilon-eV)}}. \tag{6.12}$$

They lead to I-V characteristics

$$I(V,T)\propto et^2\beta^{-3}\left[\frac{\beta eV}{2\pi}+\left(\frac{\beta eV}{2\pi}\right)^3\right], \tag{6.13}$$

in agreement with the result obtained in different approaches.[31,2]

Clearly, the Green's functions (6.8) can be evaluated in other ways, for example, by using a conformal transformation in the x,t domain.[32] We would like to stress that our derivation is more direct and uses nothing more than the fundamental anticommutation relation of the edge electrons. For $\nu=\frac{1}{3}$, these are particularly simple as they derive from the so-called $N=2$ superconformal algebra, which has been well-studied in other contexts. For other filling fractions the fundamental anti-commutators look more complicated but are available in principle.

B. Interpretation in terms of exclusion statistics

If we compare the exact kinetic equation for $\nu=\frac{1}{3}$ with a naive generalized Boltzmann equation, we see that the mistake in the latter is in the approximation of the Green's function $h(\epsilon)$ by a the product $h^{(0)}(\epsilon)$ of a tunneling density of states times a Haldane distribution for fractional statistics. The reason why this approximation turns out to be rather poor is that the operator $N(\epsilon)=\Psi^{\dagger}_{\nu=1/3}(\epsilon)\Psi_{\nu=1/3}(\epsilon)$ inside a FQHE edge is *not* to be viewed as a simple counting operator weighted by the appropriate power law of ϵ. This fact can be traced to the nontrivial operator terms (proportional to the energy and the charge operators) in the right-hand side of Eq. (6.10). To further illustrate this point we evaluated the expectation value of the operator $N(\epsilon)$ in a (normalized) one-electron state $|\epsilon'\rangle$

$$\langle\epsilon'|N(\epsilon)|\epsilon'\rangle\propto\epsilon^2\delta(\epsilon-\epsilon')+6\frac{(\epsilon'-\epsilon)(\epsilon'^2+\epsilon^2)}{\epsilon'^2}\theta(\epsilon'-\epsilon). \tag{6.14}$$

This result shows an interaction effect in the action of $N(\epsilon)$ on a one-electron state: rather than just counting quanta of energy ϵ, the operator $N(\epsilon)$ is sensitive to the presence of quanta at energy $\epsilon'>\epsilon$ as well. In the Green's function $h(\epsilon)$ (for $\epsilon>0$), the first term on the right-hand side of Eq. (6.14) corresponds to $h^{(0)}(\epsilon)$, while the second term leads to the following correction term:

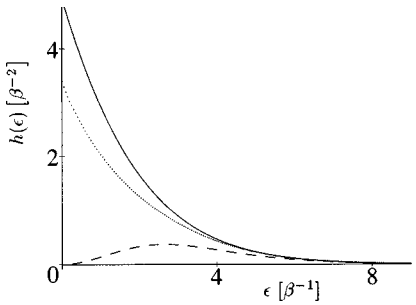

FIG. 2. One-particle Green's function $h(\epsilon)$ for filling fraction $\nu=\frac{1}{3}$ and at zero voltage. The drawn curve is the exact result (6.12); the dashed curve is the approximation $h^{(0)}(\epsilon)$ and the dotted curve corresponds to $[h^{(0)}+h^{(1)}](\epsilon)$.

$$h^{(1)}(\epsilon)=6\int_{\epsilon}^{\infty}d\epsilon'\ \frac{(\epsilon'-\epsilon)(\epsilon'^2+\epsilon^2)}{\epsilon'^2}\,n_3(\epsilon'). \quad (6.15)$$

In Fig. 2 we have plotted the exact result for $h(\epsilon)$ against the approximations $h^{(0)}(\epsilon)$ and $[h^{(0)}+h^{(1)}](\epsilon)$. Clearly, the correction term $h^{(1)}(\epsilon)$ greatly improves the accuracy of the description.

The situation here can be described as follows. As far as thermodynamics goes, the distribution functions $n_3(\epsilon)$ and $n_{1/3}(\epsilon)$ give exact results for quantities such as specific heat and conductance, and we may view the edge system as an ideal gas of fractional statistics quasiparticles. However, the operators $\Psi^{\dagger}_{\nu=1/3}(\epsilon)$, $\Psi_{\nu=1/3}(\epsilon)$ are not one-particle operators in the usual sense, as they do not simply add or extract a single quasiparticle from a many-particle state. In edge tunneling experiments, the edge system communicates with a Fermi liquid *via* the operators $\Psi^{\dagger}_{\nu=1/3}(\epsilon)$ and $\Psi_{\nu=1/3}(\epsilon)$ and we cannot avoid interaction effects. We do believe, however, that a systematic expansion based on the quasiparticle picture is possible.

To avoid reference to a ''filled sea of negative energy edge electrons'' we prefer to discuss transport in the picture where the fundamental quasiparticles are positive energy edge electrons and edge quasiholes, respectively. If we we stick for a moment to the abovementioned ''naive'' zeroth order approximation, we would arrive at the following lowest contribution to the tunneling current at voltage V and temperature T:

$$\begin{aligned}I^{(0)}(V,T)\propto&-e\int_0^{\infty}d\epsilon n_1(\epsilon-V)N_3(\epsilon)\epsilon^2\\&+e\int_0^{\infty}d\epsilon N_1(\epsilon-V)n_3(\epsilon)\epsilon^2\\&+e\int_0^{\infty}d\epsilon_3 d\epsilon_2 d\epsilon_1 n_1\Big(\sum_i\ \epsilon_i+V\Big)\\&\times N_{1/3}(\epsilon_3)N_{1/3}(\epsilon_2)N_{1/3}(\epsilon_1)f^2(\epsilon_3,\epsilon_2,\epsilon_1)\\&-e\int_0^{\infty}d\epsilon_3 d\epsilon_2 d\epsilon_1 N_1\Big(\sum_i\ \epsilon_i+V\Big)\\&\times n_{1/3}(\epsilon_3)n_{1/3}(\epsilon_2)n_{1/3}(\epsilon_1)f^2(\epsilon_3,\epsilon_2,\epsilon_1),\end{aligned} \quad (6.16)$$

where $f(\epsilon_3,\epsilon_2,\epsilon_1)$ is the form factor given in Eq. (5.10) and the integrations are over ordered sets of energies $\epsilon_3\geqslant\epsilon_2\geqslant\epsilon_1\geqslant 0$. We have used the notation

$$N_g(\epsilon)=e^{\beta\epsilon}n_g(\epsilon). \quad (6.17)$$

This result becomes exact in the limit $T\to 0$, where all distributions become step functions and the interaction effects disappear. Note that this formula has a clear asymmetry between electrons and holes: electrons that come into the edge settle as edge electrons, while incoming holes ''decay'' into a total of three edge quasiholes, with relative amplitudes given by the form factor $f(\epsilon_3,\epsilon_2,\epsilon_1)$. For $T=0$, $V>0$ the expression (6.16) reduces to a single term

$$I^{(0)}(V>0,T=0)\propto -e\int_{0\leqslant\epsilon\leqslant V}d\epsilon\epsilon^2\propto V^3, \quad (6.18)$$

while $T=0$, $V<0$ it reduces to

$$I^{(0)}(V<0,T=0)\propto e\int_{0\leqslant\Sigma_i\epsilon_i\leqslant V}d\epsilon_3 d\epsilon_2 d\epsilon_1 f^2(\epsilon_3,\epsilon_2,\epsilon_1)\propto V^3. \quad (6.19)$$

In the latter case, the power law $I\propto V^3$ is a simple consequence of the fact that we perform three independent integrations $\int d\epsilon_3 d\epsilon_2 d\epsilon_1$ over quasihole energies, with a form factor $f(\epsilon_3,\epsilon_2,\epsilon_1)$ that is scale invariant.

Clearly, the expression (6.16) needs corrections. We believe that a systematic expansion, along the lines of the expansion $h(\epsilon)=h^{(0)}(\epsilon)+h^{(1)}(\epsilon)+\cdots$ that we have demonstrated above, is possible. We plan to demonstrate this in more detail in a future publication.

VII. CONCLUSIONS

The edge electrons that have been central in this paper are the edge analogs of the composite fermions (CF) used to describe bulk physics. We have made clear that, while the exchange statistics of these particles are fermionic, their exclusion statistics properties are not and are instead captured by nontrivial distribution functions $n_m(\epsilon)$ that take the place of the familiar Fermi-Dirac distribution. We have also investigated to what extent a quasiparticle picture, with edge electrons and edge quasiholes as the fundamental quanta, can be used as a starting point for a quantitative analysis of transport. We have used algebraic properties of the $\nu=\frac{1}{3}$ edge electrons to derive exact results, and we have claimed that in general exclusion statistics properties may be used to set up a systematic expansion. In our view, these results hold some important lessons for other situations where fractional statistics quasiparticles have been proposed (spinons in $d=1$ quantum spin chains, anyons in $d=2$, etc.).

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APPENDIX A: COMPOSITE EDGES—JAIN SERIES

In this appendix we briefly describe a quasiparticle formulation of the composite edge theories corresponding to the filling fractions $\nu=n/(np+1)$ of the Jain series. These edge theories can be written as a collection of n free bosons, coupled via the topological K matrix of the effective bulk Chern-Simons theory.[4] In Ref. 34 it was shown that the effective low-energy CFT for particles satisfying Haldane statistics with $n\times n$ statistical matrix G is a $c=n$ CFT with topological matrix $K=G^{-1}$. Inverting the argument we expect that the fundamental excitations of the CFT for QHE matrix K can be interpreted in terms of pseudoparticles satisfying fractional exclusion statistics with matrix $G=K^{-1}$.

An alternative and more natural approach to the Jain series edges would be to first perform a change of basis which separates a single charged mode from a set of $n-1$ neutral modes.[35,36] The latter are governed by an $\widehat{su(n)}_1$ affine Kac-Moody symmetry, and can be treated separately. An option is to view them as a set of n free parafermions in the sense of Gentile, see Ref. 6. The CFT for the remaining charged mode is of the type that we described in this paper, with $g=\nu$. The entire edge theory is then described by a single (charged) g-on and a set of $\widehat{su(n)}_1$ degrees of freedom.

As an example of how the chiral Hilbert space works out, here is the example of $\nu=2/5$, with K matrix

$$K=\begin{pmatrix}3 & 2\\ 2 & 3\end{pmatrix}. \tag{A1}$$

This theory has two independent $U(1)$ affine Kac-Moody symmetries, giving a factor $[\Pi_{l=1}^{\infty}(1-q^l)]^{-2}$ in the partition function. The various charge sectors are labeled by pairs of integers (l_1,l_2), the energy being given by $E(l_1,l_2)=\frac{1}{10}(3l_1^2-4l_1l_2+3l_2^2)$ [this is the bilinear form defined by the inverse of the K matrix (A1)]. Thus

$$Z_{\nu=2/5}(q)=\sum_{(l_1,l_2)}\frac{q^{E(l_1,l_2)}}{[\Pi_{l=1}^{\infty}(1-q^l)]^2}. \tag{A2}$$

Under the rearrangement into $\widehat{su(2)}_1$ times $U(1)$, the combination $\frac{1}{2}(l_2-l_1)$ plays the role of the $su(2)$ spin, while l_1+l_2 is the charge under the new $U(1)$. The character identity will be[36]

$$Z_{\nu=2/5}(q)=\chi_{j=0}^{su(2)_1}(q)Z_{\text{even}}^{2/5}(q)+\chi_{j=1/2}^{su(2)_1}(q)Z_{\text{odd}}^{2/5}(q), \tag{A3}$$

where the subscript even (odd) on $Z^{s/r}$ means that we restrict to the states with total $U(1)$ charge Q even or odd. Simple expressions for the $su(2)_1$ characters are

$$\chi_{j=0}^{su(2)_1}(q)=\sum_{m+n\ \text{even}}\frac{q^{1/4(m+n)^2}}{(q)_m(q)_n},$$

$$\chi_{j=1/2}^{su(2)_1}(q)=\sum_{m+n\ \text{odd}}\frac{q^{1/4(m+n)^2}}{(q)_m(q)_n}. \tag{A4}$$

For the general case with $\nu=n/(np+1)$, the charged sector is described by a free boson CFT at compactification radius $R^2=\nu^{-1}$, which we write as $R^2=r/s$. The chiral partition sum is

$$Z^{s/r}(q)=\sum_{Q=-\infty}^{\infty}\frac{q^{Q^2/(2rs)}}{\Pi_{l=1}^{\infty}(1-q^l)}, \tag{A5}$$

and restrictions, such as the even/odd in Eq. (A3) are taken into account by restricting the charge quantum number Q.

Our fundamental charged edge quasi-particles will now be the primary fields of $U(1)$ charges $+s$ and $-r$; we shall write the creation and annihilation modes of these fields as ϕ_{-t} and G_{-t}, respectively. Note that for $s\neq 1$ the operators G_{-t} are not the physical edge electrons as the latter can only be written by including nontrivial factors from the neutral sector.

In close analogy with our analysis in Sec. III D, we can now establish that the states

$$G_{-(2M-1)r/2s+Q/s-m_M}\cdots G_{-r/2s+Q/s-m_1}\times\phi_{-(2N-1)s/2r-Q/r-n_N}\cdots\phi_{-s/2r-Q/r-n_1}|Q\rangle$$

with

$$m_M\geqslant m_{M-1}\geqslant\cdots\geqslant m_1\geqslant 0,\quad n_N\geqslant n_{N-1}\geqslant\cdots\geqslant n_1,$$

and

$$n_1\geqslant 0\quad\text{if } Q\geqslant 0,$$
$$n_1>0\quad\text{if } Q<0, \tag{A6}$$

with $Q=-(r-s),\ldots,+(s-1)$, spanning the chiral Hilbert space (A5) of the charged boson. The total energy of the lowest-energy state in the charge sector Q having particle numbers M and N for the quanta of type G and ϕ, respectively, equals

$$E(Q;M,N)=\frac{Q^2}{2rs}+\frac{r}{2s}M^2-\frac{Q}{s}M+\frac{s}{2r}N^2+\left[\frac{Q}{r}+\delta_{Q<0}\right]N \tag{A7}$$

and leads to the following expression for the chiral partition sum:

$$Z^{s/r}(q)=\sum_{Q=-(r-s)}^{(s-1)}\sum_{M,N\geqslant 0}\frac{q^{E(Q;M,N)}}{(q)_M(q)_N}. \tag{A8}$$

The equality of the expressions (A5) and (A8) is an identity of the Rogers-Ramanujan type (see Refs. 17–19 for some similar identities).

In the case $p<0$, the Jain series QHE edge exhibits counterflowing edge modes and it has been claimed that a disorder-driven fixed point dominates the physics.[37,35] It will be most interesting to analyze this scenario in a quasiparticle formulation.

APPENDIX B: FORM FACTOR FOR GENERAL *M*

We briefly explain the exact evaluation of the form factor $f(m_m,\ldots,m_1)$ as defined in Eq. (5.9). Let us consider the special case $m=2$ first. In that case the ''hole operator''

$G(z)$ has conformal dimension 1 and may be identified with one of the currents of the affine Kac-Moody algebra $\widehat{su(2)}_1$. By exploiting the OPE

$$\phi(z)\phi(w)=(z-w)^{+1/2}[G(w)+\mathcal{O}(z-w)] \tag{B1}$$

one obtains

$$G(w)=\oint_{C_w}\frac{dz}{2\pi i}(z-w)^{-3/2}\phi(z)\phi(w). \tag{B2}$$

We also have[21]

$$\phi(z)\phi(w)|0\rangle=(z-w)^{1/2}\sum_{n_2,n_1}P^{(-1/2)}_{n_2,n_1}(z,w)|n_2,n_1\rangle, \tag{B3}$$

where $P^{(-1/2)}_{\{n_i\}}(z,w)$ are the appropriate Jack polynomials. Combining the above, we obtain

$$G(w)|0\rangle=\sum_{n_2,n_1}P^{(-1/2)}_{n_2,n_1}(w,w)|n_2,n_1\rangle \tag{B4}$$

and it follows that

$${}_N\langle n_2,n_1|G^{\dagger}_{-1-n}|0\rangle=\delta_{n,n_2+n_1}[j_{(2^{n_1},1^{n_2-n_1})}]^{1/2}P^{(-1/2)}_{n_2,n_1}(1,1), \tag{B5}$$

with $j_{\lambda'}$ as in Eq. (5.6). For general m one obtains a similar result in terms of Jack polynomials with label $(-1/m)$. Using the explicit result[28,29]

$$P^{(-1/m)}_{\{n_i\}}(1,1)=\prod_{i=0}^{m-1}\frac{\Gamma(1/m)}{\Gamma(1-i/m)}\times\prod_{i<j}\frac{\Gamma[n_j-n_i+(j-i+1)/m]}{\Gamma[n_j-n_i+(j-i)/m]}, \tag{B6}$$

together with the result (5.8) for the $j_{\lambda'}$, we derive the following asymptotic form for $n_i\gg 1$:

$$f(n_m,\ldots,n_1)=\frac{[\Gamma(1/m)]^{m/2}}{\Pi_{i=0}^{m-1}\Gamma(1-i/m)}\frac{\Pi_{i<j}(n_j-n_i)^{1/m}}{\Pi_{i=1}^{m}n_i^{(m-1)/2m}}. \tag{B7}$$

The simple Jastrow form of this form factor is a clear indication that in the limit $n_i\gg 1$ a much simpler derivation, along the lines of ''bosonization in momentum space'' should be possible.

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