Shock waves in one-dimensional Heisenberg ferromagnets

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We use a classical approximation to investigate the existence of shock waves in one dimensional ferromagnets. As a result we find two types of shock waves, bright and dark, which can be interpreted as classical analogs of moving magnetic domains. [S0163-1829(98)04745-6]

I. INTRODUCTION

The Heisenberg model for ferromagnetics and antiferromagnets is certainly one of the most important models of condensed matter physics on which a large amount of work has been done.[1] In spite of this, the study of its properties, both at the quantum and at the classical level, is still a nonexhausted subject of ever continuing interest. In the one-dimensional case the model is exactly solvable for $S=1/2$ and its excitation spectrum consists of quantum solitons which can be viewed in the classical limit as bound states of a large number of magnons. On the other hand, it is known that nonlinear lattices, besides solitons, may support other kinds of excitation which behave, at the initial stages of their evolution, as shock waves in liquids or gases. Such waves have been reported both in integrable[2–4] and in nonintegrable lattices.[5–8]

The aim of the present paper is to show within a classical approximation, the existence of shock waves in one-dimensional Heisenberg ferromagnets. To this end we use coherent states[9] to derive classical equation of motion from the original quantum model. Within this approximation the system is shown to be described by a classical discrete nonlinear Schrödinger (DNLS)-like equation with an Hamiltonian structure and a nonstandard Poisson bracket. This classical system preserves the conservation of the classical analog of the z projection of the total spin (for isotropic case also the total spin). We find that for suitable conditions the excitations of this system naturally display shock waves with sharp rectangular profiles moving on uniform backgrounds. Such waves can exist both above background (bright shock) and below background (dark shock) and on the contrary to other excitations, which may decay into soliton trains or background radiation, they are very stable. Similar solutions were found also in a deformable DNLS system[8,10] and in a chain of two level atoms describing the propagation of Frenkel excitons.[11] We give a numerical and an analytical description of these phenomena both in terms of dispersion relations and in terms of a small amplitude multiscale expansion. The shock waves discussed in this paper may be relevant for ferromagnetic chain with large spin or for quasi-one-dimensional (1D) excitations in high quality ferromagnets such as yttrium iron garnet (YIG).

The paper is organized as follows. In Sec. II we show how to derive classical equation of motion from the original quantum spin model by using the coherent state representation in the path-integral formulation and the stationary phase approximation. We then give analytical arguments for shock wave formation in terms of linear analysis and small amplitude multiscale expansions around the background. In Sec. III we compare our analytical results with a direct numerical integration of the system while in Sec. IV we summarize the main results of the paper.

II. DERIVATION OF THE MODEL AND SHOCK WAVES ANALYSIS

We start from the quantum Heisenberg Hamiltonian written as

$$H=-\sum_{\langle m,n\rangle} J(n,m)[(\hat{S}_n^x\hat{S}_m^x+\hat{S}_n^y\hat{S}_m^y)+\lambda\hat{S}_n^z\hat{S}_m^z], \qquad (1)$$

where $\hat{S}_n=(\hat{S}_n^x,\hat{S}_n^y,\hat{S}_n^z)$ are spin operators of spin magnitude S, $J(n,m)$ is the exchange interaction constant and λ is the anisotropy of the exchange XY-like $(\lambda<1)$ and Ising-like $(\lambda>1)$ interactions. In what follows we consider only the ferromagnetic case $(J>0,\lambda>0)$ with nearest-neighbor interaction $J(n,m)=J\cdot(\delta_{n,m+1}+\delta_{n,m-1})$ and we denote, as usual, with $\hat{S}_n^{\pm}=\hat{S}_n^x\pm i\hat{S}_n^y$ the raising and lowering operators. To derive classical equation of motion it is suitable to use SU(2) coherent states

$$|\mu_n\rangle=\frac{\exp(\mu_n\hat{S}_n^-)}{(1+|\mu_n|^2)^S}|\uparrow\rangle_n \qquad (2)$$

in terms of which we write the state of the system (1) in the form

$$|\Lambda\rangle=\prod_n |\mu_n\rangle. \qquad (3)$$

Here μ_n is a complex variable and $|\uparrow\rangle_n$ denotes the spin up state on site n. The time evolution operator of the Heisenberg Hamiltonian between the initial state $|\Lambda\rangle_i$ at time t_i and a final state $|\Lambda\rangle_f$ at time t_f can then be written in terms of path integral as

$$_i\langle\Lambda|\exp\left(-i\frac{Ht}{\hbar}\right)|\Lambda\rangle_f=\int \partial(\Lambda)\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f}L dt\right), \quad (4)$$

where

$$L=i\hbar\sum_n \frac{S}{1+|\mu_n|^2}\left(\bar{\mu}_n\frac{d\mu_n}{dt}-\mu_n\frac{d\bar{\mu}_n}{dt}\right)-\langle\Lambda|H|\Lambda\rangle. \quad (5)$$

Using the stationary phase approximation[12] one readily obtains from Eq. (4) the following equation of motion:

$$i\hbar\frac{d\mu_n}{dt}=-JS\left(\frac{\mu_{n+1}-\mu_n^2\bar{\mu}_{n+1}}{1+|\mu_{n+1}|^2}+\frac{\mu_{n-1}-\mu_n^2\bar{\mu}_{n-1}}{1+|\mu_{n-1}|^2}\right)+\lambda JS\left(\frac{\mu_n(1-|\mu_{n+1}|^2)}{1+|\mu_{n+1}|^2}+\frac{\mu_n(1-|\mu_{n-1}|^2)}{1+|\mu_{n-1}|^2}\right). \quad (6)$$

It is worth remarking that this equation has a pure classical character (the occurrence of $\hbar$ in it is simply related to the appearance of the gyromagnetic ratio in the classical equation of motion). Moreover, we note that Eq. (6) and its complex conjugated follow from Hamilton's equations

$$\frac{d\mu_n}{dt}=\{\mu_n,H_c\}, \quad \frac{d\bar{\mu}_n}{dt}=\{\bar{\mu}_n,H_c\} \quad (7)$$

with the noncanonical Poisson bracket

$$\{f,g\}=-\frac{i}{2S\hbar}\sum_n (1+|\mu_n|^2)^2\left[\frac{\partial f}{\partial\mu_n}\frac{\partial g}{\partial\bar{\mu}_n}-\frac{\partial f}{\partial\bar{\mu}_n}\frac{\partial g}{\partial\mu_n}\right] \quad (8)$$

and with the classical Hamiltonian H_c given by $\langle\Lambda|H|\Lambda\rangle$, i.e.,

$$H_c=-JS^2\sum_n\left(\frac{2(\bar{\mu}_{n+1}\mu_n+\bar{\mu}_n\mu_{n+1})}{(1+|\mu_n|^2)(1+|\mu_{n+1}|^2)}+\lambda\frac{(1-|\mu_n|^2)(1-|\mu_{n+1}|^2)}{(1+|\mu_n|^2)(1+|\mu_{n+1}|^2)}\right). \quad (9)$$

One readily checks that the conservation of the z component of the quantum total spin is reflected in Eq. (6) in the conservation of the quantity

$$s_z\equiv\langle\Lambda|\sum_n \hat{S}_n^z|\Lambda\rangle=S\sum_n\frac{(1-|\mu_n|^2)}{(1+|\mu_n|^2)} \quad (10)$$

which is just the classical analog of the z component of the total spin (note that in the isotropic case $\lambda=1$ there is also another conserved quantity which is the analog of the total spin squared $s^2\equiv\langle\Lambda|S^2|\Lambda\rangle$). We see, therefore, that the classical system (6) keeps some important symmetry of the original quantum model [note also that the dynamical variables $(\mu_n,\bar{\mu}_n)$ can be related by an inverse stereographic projection to the motion of vectors on a sphere (classical spins)].

In order to investigate shock solutions of Eq. (6) it is suitable to consider excitations propagating against a nonzero background of the form $\mu_n^{(0)}=\rho\exp(-i\omega t+ikn)$ where $\rho<1$ and

$$\omega=2JS[\lambda-\cos(k)]\frac{1-\rho^2}{1+\rho^2}. \quad (11)$$

(here and in the following we fix $\hbar=1$). The stability of the background can be studied with the help of the substitution $\mu_n=(1+\psi_n)\rho\exp(-i\omega t+ikn)$, where $|\psi_n|\ll|\mu_n|$, in Eq. (6). By linearization we obtain the dispersion relation $\Omega(K)$ associated with the linear equations for $\psi_n,\bar{\psi}_n$ [$\propto\exp(i\Omega t-iKn)$],

$$\Omega=2\frac{1-\rho^2}{1+\rho^2}JS\sin(k)\sin(K)\pm\frac{2\sqrt{2}}{1+\rho^2}JS\sin\left(\frac{K}{2}\right)\times\{\cos^2(k)(1+\rho^2)^2-\cos^2(k)\cos(K)(1-\rho^2)^2-4\lambda\rho^2\cos(k)\cos(K)\}^{1/2}. \quad (12)$$

Thus the background is stable (i.e., Ω is real at all K), if $\cos k>\lambda$ and $0>\cos k>-[2\rho^2/(1+\rho^4)]\lambda$. Naturally, in what follows the analysis will be restricted to this region of the parameters. In order to get the equation governing the initial stages of the evolution of a shock wave we use the small amplitude expansion $\mu_n=(\rho+a_n)\exp[i(-\omega t+kn-\phi_n)]$, where the two real quantities a_n and ϕ_n are considered depending on slow variables $X=\gamma n$, $T=\gamma t$, and $\tau=\gamma^3 t$, (with $\gamma\ll 1$) and are represented in the form $a_n=\gamma^2 a_n^{(0)}+\gamma^4 a_n^{(1)}+\cdots$, $\phi_n=\gamma\phi_n^{(0)}+\gamma^3\phi_n^{(1)}+\cdots$. Collecting all the terms of the same order in γ we arrive at a series of equations. In the zero order we recover the dispersion relation (11). In the second and third orders we get the following equations:

$$\frac{\partial\phi^{(0)}}{\partial T}=\frac{8\rho JS}{(1+\rho^2)^2}(\cos k-\lambda)a^{(0)}-2JS\sin(k)\frac{1-\rho^2}{1+\rho^2}\frac{\partial\phi^{(0)}}{\partial X}, \quad (13)$$

$$\frac{\partial a^{(0)}}{\partial T}=\rho JS\cos(k)\frac{\partial^2\phi^{(0)}}{\partial X^2}-2\sin(k)JS\frac{1-\rho^2}{1+\rho^2}\frac{\partial a^{(0)}}{\partial X}. \quad (14)$$

It is suitable to introduce new variables $(\xi_\pm,T)$ instead of (X,T), where $\xi_\pm=X-c_\pm T$ and the velocities $c_\pm$ are given by

$$c_\pm=\frac{2JS}{(1+\rho^2)}[(1-\rho^2)\sin k\pm\rho\sqrt{2\cos k(\cos k-\lambda)}]. \quad (15)$$

Comparing this result with Eq. (12) one sees that $c_\pm=d\Omega_\pm/dK$ at $K=0$, i.e., $c_\pm$ are group velocities of two branches of the spectrum in the center of the BZ. Then it follows from Eqs. (13) and (14) that solutions $a^{(0)}=a^{(0)}(\xi_\pm)=a_\pm$ and $\phi^{(0)}=\phi^{(0)}(\xi_\pm)$ are related by

$$a_{\pm} = \mp (1+\rho^2) \frac{\sqrt{\cos k}}{2\sqrt{2(\cos k - \lambda)}} \frac{\partial \phi^{(0)}}{\partial \xi_{\pm}}. \tag{16}$$

For brevity we drop out the explicit form of the equations appearing in the forth and fifth orders of γ and present simply the condition of their compatibility. This condition has the form of a Korteweg–de Vries (KdV) equation

$$\frac{\partial a_{\pm}}{\partial \tau} + \alpha(k) a_{\pm} \frac{\partial a_{\pm}}{\partial \xi_{\pm}} + \beta(k) \frac{\partial^3 a_{\pm}}{\partial \xi_{\pm}^3} = 0 \tag{17}$$

with

$$\alpha(k) = \frac{4JS}{(1+\rho^2)^2} [-8\rho \sin k + 2\rho\lambda \tan k \pm 3(1-\rho^2)\sqrt{2\cos k(\cos k - \lambda)}] \tag{18}$$

and

$$\beta(k) = \frac{JS}{8\rho(1+\rho^2)\sqrt{\cos k - \lambda}} \left\{ \frac{8}{3}\rho \sin k \sqrt{\cos k - \lambda}(1-\rho^2) \pm \sqrt{2\cos k}\left[\frac{2}{3}\rho^2(\cos k - \lambda) - 4\rho^2\lambda - \cos k(1-\rho^2)^2\right]\right\}. \tag{19}$$

From Eqs. (17) and (19) it follows that if

$$4\sqrt{2}\rho(1-\rho^2)\sin k\sqrt{\frac{\cos k - \lambda}{\cos k}} = \pm 3[(1-\rho^2)^2 \cos k + 4\lambda\rho^2] \mp 2\rho^2(\cos k - \lambda) \tag{20}$$

is satisfied, the coefficient $\beta(k)$ becomes zero and the KdV equation reduces to the well-known equation

$$\frac{\partial a_{\pm}}{\partial \tau} + \alpha(k) a_{\pm} \frac{\partial a_{\pm}}{\partial \xi_{\pm}} = 0 \tag{21}$$

which supports shock solutions.[13] This implies that for parameter values satisfying Eq. (20) shock wave should develop in the classical spin chain.

Note that from expression (21) it follows that at $k=0$ there exists only one background $0<\rho<1$ at which $\beta(k)$ equals to zero while this is not true for $k\neq 0$. Moreover, Eq. (21) is not satisfied for all λ values but there exists a maximal value $\lambda_{\max}=1/7$ above which Eq. (21) does not have physical meaning.

III. NUMERICAL EXPERIMENTS

To check the above predictions, we have numerically integrated Eq. (6) on a long chain (to neglect boundary conditions), taking as initial condition a bell shaped bright or dark pulse of the type

$$\mu_n = \rho e^{ikn}\left(1 \pm \frac{A}{\cosh[(n-n_0)]^2}\right) \tag{22}$$

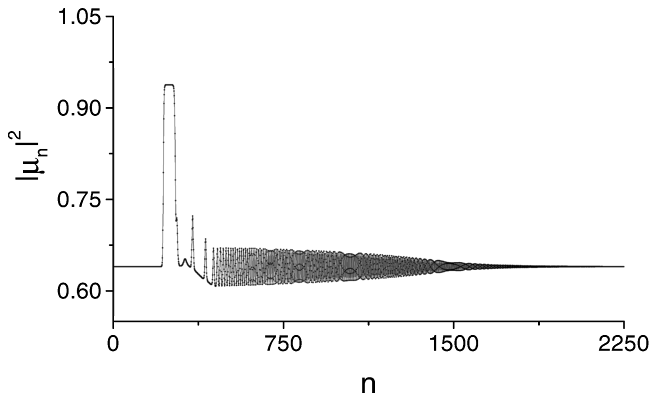

FIG. 1. Evolution of a bright shock against nonzero background with $k=0$, $\rho=0.8$, and for parameter values $JS=0.8$ and λ determined from Eq. (20).

(note that with this initial condition rectangular shock profiles should develop). In Fig. 1 we have reported the profile which develop out from an initial bright pulse of amplitude $|A|=3.6$, after an evolution time of $T=420$. The background is moving in-phase ($k=0$) with $\rho=0.8$ and for parameter values given by $JS=0.8$, and λ derived from Eq. (20). From this figure we see the appearance of a leading rectangular shock profile followed by solitons and background radiation. The shock wave connects the uniform background field with a local plateau with two sharp transitions at the edges. If we define the local magnetization as $M_n=S(1-|\mu_n|^2)/(1+|\mu_n|^2)$ we have that the local magnetization in the rectangular shock waves of Fig. 1 does not change in time and is different from the surroundings. This suggest the interpretation of such solutions as propagating magnetic domains.

A similar result can be obtained starting from an initial dark profile as shown in Fig. 2. In this case the shock plateau develops below the background and therefore it can be referred to as a dark shock. Notice that in the above context dark and bright pulses correspond respectively to domains with higher and lower magnetization compared with the magnetization of the background.

Following the time evolution of the shock profiles in Figs. 1 and 2 we find that the rectangular waves separate from the other components (solitons and radiation) and stay stable over long time. We have numerical evidence that Eq. (20) is

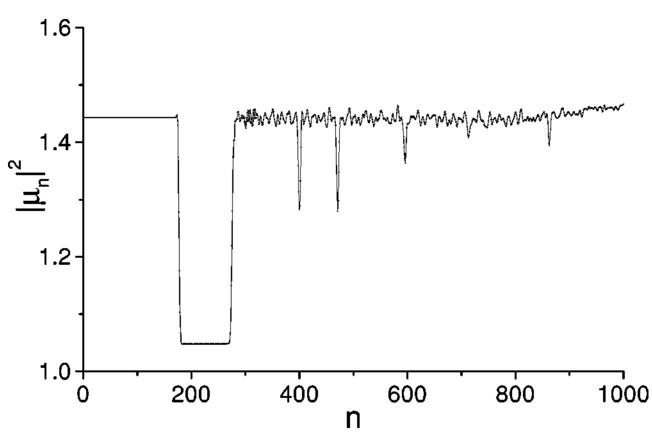

FIG. 2. Same as in Fig. 1 but for a dark initial condition and for parameter values $\rho=0.8$, $JS=1$ and λ determined from Eq. (20).

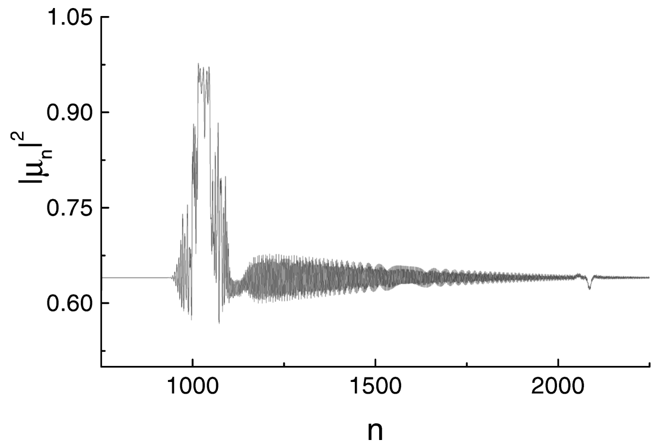

FIG. 3. Evolution profile for the same parameter values as in Fig. 1 but with $\lambda = 0.185$ not satisfying Eq. (20).

a necessary condition for creation of shock waves. In Fig. 3 we have reported the evolution profile for the same parameter values of Fig. 1 except for $\lambda = 0.185$ not satisfying relation (20). We see that the rectangular shock is destroyed and oscillations develop on the wave front. The same is observed for other choices of parameters for which Eq. (20) is not satisfied. We also checked that shocks remain stable upon collision with other excitations (the same was found in Ref. 8) suggesting the presence of a strong soliton component in them. This leads to the interpretation of the above shocks as bound states of many solitons. To check this interpretation, however, further investigation is required.

IV. CONCLUSIONS

In conclusion, we have shown within a classical approximation the existence of shock waves in 1D Heisenberg ferromagnets. These results represent an example of the manifestation of classical fluid dynamics in magnetic systems. Previously, classical behaviors in magnetic systems were reported on multimagnon instabilities and chaos in pure and doped YIG and some antiferromagnets.[14] In order to observe the shock waves obtained here in magnetic systems, generation of macroscopic number of magnons is required. One of the promising candidates to realize this may be highly pumped YIG.

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