1/N expansion for two-dimensional quantum ferromagnets

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The magnetization of a two-dimensional ferromagnetic Heisenberg model, which represents a quantum Hall system at filling factor $\nu=1$, is calculated employing a large N Schwinger boson approach. Corrections of order $1/N$ to the mean-field ($N=\infty$) results for both the SU(N) and the O(N) generalization of the bosonized model are presented. The calculations are discussed in detail and the results are compared with quantum Monte Carlo simulations as well as with recent experiments. The SU(N) model describes both Monte Carlo and experimental data well at low temperatures, whereas the O(N) model is much better at moderate and high temperatures. [S0163-1829(98)00727-9]

I. INTRODUCTION

Progress in materials synthesis has allowed study of a variety of two-dimensional (2D) systems such as thin films, surfaces, and semiconductor quantum wells. These systems as well as the nearly 2D cuprates have led to much interest in 2D quantum magnetism. It has been found that 2D electron gases in quantum wells in the quantum Hall regime are itinerant ferromagnets.[1–3] The strong external magnetic field quenches the kinetic energy, leading to widely separated Landau levels, but because of band-structure effects it couples only weakly to the electron spins. Thus low-energy spin fluctuations play an important role.

These 2D continuum ferromagnets exhibit topological excitations called skyrmions[1,4] in analogy to the Skyrme model of nuclear physics.[5,6] In the quantum Hall system these excitations carry electrical charge.[1,4] At filling factor $\nu=1$, i.e., if the spin-up states in the lowest Landau level are just filled, skyrmions only appear as thermal excitations of the form of skyrmion-antiskyrmion pairs. At filling factors away from unity, however, skyrmions appear even in the ground state.[7] At all filling factors, low-energy spin fluctuations are also present. The combination of spin fluctuations and skyrmions dramatically alters the magnetization[8,2,9] and the specific heat.[10]

For a quantitative understanding it is useful to first study the case of $\nu=1$ to isolate the effect of low-energy spin fluctuations, which are expected to be well described by a Heisenberg model, at least at low enough temperatures. At higher temperatures higher-order gradient terms neglected in the Heisenberg model could become important. Renormalization-group arguments[11] show that in $D=2-\epsilon$ dimensions the magnetization M of the quantum Hall ferromagnet at $\nu=1$ is a universal function, $M/S=f(JS^2/T,B/T)$, where S is the spin, J is the exchange coupling, and B is the external magnetic field. For $D=2$ this universality is violated by logarithmic corrections.[11] In the Heisenberg model the magnetization only depends on the *three* dimensionless quantities S, J/T, and B/T. Read and Sachdev[11] have evaluated the magnetization using SU(N) and O(N) Schwinger boson formulations of mean-field (MF) theory, i.e., $N\to\infty$, for the Heisenberg model. In a recent communication[12] we have presented analytic results for the leading $1/N$ corrections to the magnetization and results of extensive quantum Monte Carlo simulations. In the present paper we present details of the $1/N$ theory. Details of the Monte Carlo simulations are given elsewhere.[13] An alternative microscopic approach that includes spin-wave corrections to the electronic self-energy has also recently been developed.[14]

Schwinger boson theories[15,16] have proved useful in finding MF theories that respect the symmetry of the Hamiltonian. Formal results to any order in $1/N$ have also been obtained.[17,18] However, numerically evaluating the first-order ($1/N$) corrections is not an easy task. Trumper *et al.*[19] have evaluated various ground-state quantities of a frustrated antiferromagnet in the absence of external fields. Although they are not using the large N formalism, their method is equivalent to a $1/N$ expansion to first order.

There are a number of subtle pitfalls in the $1/N$ calculations, e.g., regarding normal ordering of operators. It seems worthwhile to present the calculations in some detail for the benefit of readers interested in using $1/N$ expansion methods. We also hope to make the physical interpretation of these theories clearer and shed some light on the level of accuracy of $1/N$ expansions.

In the following we give an overview of this work. First, the Heisenberg Hamiltonian is mapped onto an equivalent boson system. There are several ways of doing this. One is the Holstein-Primakoff representation,[20] which has a number of disadvantages, e.g., the square root of operators it introduces, and we do not employ it here. Instead we introduce Schwinger bosons[15] in two different ways. The first, presented in Sec. II A, makes use of the SU(2) symmetry in spin space of the Heisenberg model (which is explicitly broken by an external field). The second utilizes the local equivalence between the groups SU(2) and O(3) to write down an equivalent O(3) boson model (Sec. III A). Subsequently, the two models are generalized to SU(N) and O(N), respectively, which contain N bosons at each site. At this point a remark may be in order on what we do *not* mean by the O(N) model. It is not an N component vector model, e.g., an N component quantum nonlinear σ model. Rather, the spin operators are generators of the Lie group O(N). Only for $N=3$ are the generators antisymmetric 3×3 matrices, which are dual to (axial) vectors. Thus our results are not easily compared to expansions in the number of components of the spin vectors, as developed by Garanin[21] for a classical system.

It is now possible to expand in $1/N$ as a small parameter. MF theory becomes exact for both SU(∞) and O(∞).[17] The $1/N$ expansion is a saddle-point expansion around this MF solution, not a perturbative expansion in the interaction. For this reason it is, in principle, equally valid at all temperatures. Also, it does respect the symmetry of the Heisenberg model. This property makes even the MF results qualitatively correct. In particular, the absence of long-range order if no external field is present is correctly predicted. After rederiving the MF magnetization in Sec. II B for the SU(N) model and in Sec. III B for O(N), we calculate the $1/N$ corrections using a diagrammatic approach[17] (Secs. II C and III C). These corrections take fluctuations around the MF result into account. We will make use of gauge invariance to simplify our task. Here we also have to discuss the effect of normal ordering. In principle, terms to any order in $1/N$ can be obtained in the same way.

The system without exchange interaction can be solved exactly for any value of the spin S and for any N in both the SU(N) and the O(N) model. It can be used to check the $1/N$ expansion. However, the interaction introduces a number of additional complications.

II. SU(N) MODEL

A. General considerations

We start from a Heisenberg model with nearest-neighbor interaction on a square lattice in a constant magnetic field,

$$H=-J\sum_{\langle ij\rangle}\mathbf{S}(i)\cdot\mathbf{S}(j)-B\sum_i S^z(i), \tag{1}$$

where the sum over $\langle ij\rangle$ is over all nearest-neighbor bonds. A factor of $g\mu_B$ has been absorbed into the field B. The total spin at each site is S; $\mathbf{S}(i)\cdot\mathbf{S}(i)=S(S+1)$. We express the spins in terms of Bose operators using a Schwinger boson representation, where two Bose fields a and b are introduced according to[15,22]

$$S^+=a^\dagger b,\quad S^-=b^\dagger a,\quad S^z=(a^\dagger a-b^\dagger b)/2. \tag{2}$$

To restrict the Hilbert space to the physical states, the constraint $a^\dagger a+b^\dagger b=2S$ is introduced, which corresponds to $\mathbf{S}\cdot\mathbf{S}=S(S+1)$ for the original Hamiltonian. The boson Hamiltonian is

$$H=-\frac{J}{2}\sum_{\langle ij\rangle}[a^\dagger(i)a(i)a^\dagger(j)a(j)+a^\dagger(i)b(i)b^\dagger(j)a(j)+b^\dagger(i)a(i)a^\dagger(j)b(j)+b^\dagger(i)b(i)b^\dagger(j)b(j)]-\frac{B}{2}\sum_i[a^\dagger(i)a(i)-b^\dagger(i)b(i)], \tag{3}$$

neglecting a constant. For this Hamiltonian to be equivalent to the Heisenberg model, the spin operators expressed in terms of bosons have to have the correct commutation relations. This is easily shown to be the case.

Utilizing the SU(2) symmetry group of the spins we write the Hamiltonian in a more compact form by first defining a SU(2) spin matrix

$$\mathsf{S}\equiv\begin{pmatrix}a^\dagger a & a^\dagger b\\ b^\dagger a & b^\dagger b\end{pmatrix}, \tag{4}$$

with the constraint Tr$\mathsf{S}=2S$. The Hamiltonian is

$$H=-\frac{J}{2}\sum_{\langle ij\rangle}S^\alpha_\beta(i)S^\beta_\alpha(j)-\frac{B}{2}\sum_i(\sigma^z)^\alpha_\beta S^\beta_\alpha(i), \tag{5}$$

where σ^z is a Pauli matrix and summation over repeated indices is implied. Here, the spin matrices $\mathsf{S}(i)$ should be infinitesimal generators of the SU(2) group, i.e., elements of the corresponding algebra. This is not the case since the generators are traceless. However, if we had defined S as an element of the algebra, the Hamiltonian would only change by a constant and we use the more convenient definition (4).

The group SU(2) is generalized to SU(N) for any even N. The generalization of the Hamiltonian (5) is

$$H=-\frac{J}{N}\sum_{\langle ij\rangle}S^\alpha_\beta(i)S^\beta_\alpha(j)-\frac{B}{2}\sum_i h^\alpha_\beta S^\beta_\alpha(i), \tag{6}$$

where S and h are $N\times N$ Hermitian matrices and S is subject to the constraint Tr$\mathsf{S}=NS$. We choose $h^\alpha_\beta=\delta_{\alpha\beta}(-1)^{\alpha+1}$ so that we regain the SU(2) model for $N=2$. The Schwinger boson representation now requires N boson species b_α,[22] $S^\alpha_\beta=b^\dagger_\alpha b_\beta$, and the constraint is

$$b^\dagger_\alpha b_\alpha=NS. \tag{7}$$

We now go over to the continuum for mathematical convenience. The continuum model may actually give a better description of itinerant magnets but is harder to compare to Monte Carlo simulations on a lattice. Up to a constant we obtain

$$H=\int d^2r\left[\frac{J}{2N}(\partial_j S^\alpha_\beta)(\partial_j S^\beta_\alpha)-\frac{B}{2a^2}h^\alpha_\beta S^\beta_\alpha\right], \tag{8}$$

where $b_\alpha(\mathbf{r})$ is a continuous Bose field with the commutator $[b_\alpha(\mathbf{r}),b^\dagger_\beta(\mathbf{r}')]=a^2\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')$, ∂_j is the two-component gradient, a is the lattice constant, and summation over j is implied. After bosonization we find

$$H=\int d^2r\left[JS(\partial_j b^\dagger_\alpha)(\partial_j b_\alpha)-\frac{J}{N}b^\dagger_\alpha(\partial_j b^\dagger_\beta)b_\beta(\partial_j b_\alpha)-\frac{B}{2a^2}h^\alpha_\beta b^\dagger_\beta b_\alpha\right], \tag{9}$$

which is normal ordered, as necessary for the functional integral. This is basically the Hamiltonian of the complex projective CP^{N-1} model.[18] We have used the fact that the lattice Hamiltonian (6) can be normal ordered trivially since spins at different sites commute so that $S^\alpha_\beta(i)S^\beta_\alpha(j) = :S^\alpha_\beta(i)S^\beta_\alpha(j):$, where :: denotes normal ordering.

Now we write down the partition function as a coherent state functional integral, where the Bose fields are replaced by complex fields,

$$Z=\int D^2 b_\alpha D\lambda \exp\left(-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^2 r \mathcal{L}[b;\lambda]\right), \quad (10)$$

where the functional integral takes each $b_\alpha(\mathbf{r},\tau)$ over the whole complex plane and each $\lambda(\mathbf{r},\tau)$ parallel to the imaginary axis (a constant real part is irrelevant). Here and in the following we neglect constant factors in Z. τ is the imaginary time, β is the inverse temperature, and $\mathcal{L}$ is the Lagrangian

$$\mathcal{L}=\frac{\hbar}{a^2} b^*_\alpha \partial_0 b_\alpha + JS(\partial_j b^*_\alpha)(\partial_j b_\alpha) - \frac{J}{N} b^*_\alpha(\partial_j b^*_\beta) b_\beta(\partial_j b_\alpha)$$

$$- \frac{B}{2a^2} h^\alpha_\beta b^*_\beta b_\alpha + \lambda b^*_\alpha b_\alpha - NS\lambda. \quad (11)$$

The first term is the usual Berry phase (∂_0 is the time derivative) and the last two terms come from the constraint using the identity $2\pi\delta(\phi)=\int_{-\infty}^{\infty} dx e^{ix\phi}$. λ is a Lagrange multiplier at each point $(\mathbf{r},\tau)$.

To decouple the quartic term we introduce a Hubbard-Stratonovich field $\mathbf{Q}(\mathbf{r},\tau)$: Since

$$\int DQ_j \exp\left(-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^2 r \frac{J}{N}[-iNQ_j - (\partial_j b^*_\alpha) b_\alpha] \times [iNQ_j - b^*_\beta(\partial_j b_\beta)]\right) \quad (12)$$

is independent of b_α, we can multiply the partition function with this expression. Q_j can be chosen real since an imaginary part of Q_j would not couple to the b_α fields because $b^*_\beta \partial_j b_\beta$ is purely imaginary. We get

$$Z=\int D^2 b_\alpha D\lambda DQ_j \exp\left(-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^2 r \mathcal{L}'[b;\lambda,\mathbf{Q}]\right) \quad (13)$$

with

$$\mathcal{L}'=\frac{\hbar}{a^2} b^*_\alpha \partial_0 b_\alpha + JS(\partial_j b^*_\alpha)(\partial_j b_\alpha) + NJQ_jQ_j + iJQ_j b^*_\alpha(\partial_j b_\alpha) - iJQ_j(\partial b^*_\alpha) b_\alpha - \frac{B}{2a^2} h^\alpha_\beta b^*_\beta b_\alpha + \lambda b^*_\alpha b_\alpha - NS\lambda. \quad (14)$$

We see that $\mathbf{Q}$ is a *gauge field*: If we multiply all b_α by a local phase factor, $b_\alpha(\mathbf{r},\tau) \to e^{i\theta(\mathbf{r},\tau)} b_\alpha(\mathbf{r},\tau)$, we reobtain the Lagrangian (14) by letting $Q_j \to Q_j + S\partial_j\theta$. We know from gauge theory that $\mathbf{Q}$ contains more information than is physically relevant; we have the freedom to choose a gauge. We use a transverse gauge,

$$\partial_j Q_j = 0. \quad (15)$$

Of course we obtain the same results if we do not fix the gauge. The gauge freedom then leads to the appearance of zero modes, which turn out not to enter in the magnetization.

B. Mean-field theory

Up to this point the treatment has been exact. In the following we derive mean-field (MF) results, which are exact for $N\to\infty$ and approximate for finite N. This approximation is not the same as standard MF theory for the Heisenberg model. As we will see, SU(N) MF theory captures the low-energy spin-wave physics of the Heisenberg model and correctly predicts the absence of long-range order at finite temperatures.

The MF approximation is the leading order of a stationary phase approximation for the SU(N) partition function. The MF solution is assumed to be homogeneous and static, i.e., $\mathbf{Q}$ and λ are assumed to be constant. This assumption is not justified for all systems,[23] it should hold in ferromagnets, though.[17] The MF values $\overline{\mathbf{Q}}$ and $\overline{\lambda}$ are chosen in such a way that the MF free energy F_0 has a saddle point. If we set λ to its MF value the constraint (7) is no longer satisfied locally but only on average. In order to diagonalize the action we introduce Fourier transforms of the b_α fields,

$$b_\alpha(\mathbf{r},\tau)=\frac{a^2}{2\pi}\int d^2k \sum_{i\omega_n} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_n\tau} b_\alpha(\mathbf{k},i\omega_n), \quad (16)$$

where $i\omega_n = i2\pi n/\hbar\beta$ are bosonic Matsubara frequencies. From now on summation over indices is written out. With Eq. (14) and the definition $h^\alpha_\beta = \delta_{\alpha\beta}(-1)^{\alpha+1}$ the MF partition function is

$$Z_0=\int D^2 b_\alpha(\mathbf{k},i\omega_n) \exp\left(-\mathcal{N}N\beta J\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2 + \mathcal{N}a^2 NS\beta\overline{\lambda} - \int d^2k \sum_{i\omega_n} \mathcal{L}''_0[b]\right), \quad (17)$$

where $\mathcal{N}$ is the total number of sites and

$$\mathcal{L}''_0 = \beta a^2 \sum_\alpha \left(-i\hbar\omega_n + JSk^2a^2 - 2J\overline{\mathbf{Q}}\cdot\mathbf{k}a^2 - \frac{B}{2} h^\alpha_\alpha + a^2\overline{\lambda}\right) b^*_\alpha(\mathbf{k},i\omega_n) b_\alpha(\mathbf{k},i\omega_n). \quad (18)$$

We introduce a number of new symbols,

$$\overline{\Lambda} \equiv a^2\beta\overline{\lambda} - \frac{\beta J}{S}\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2, \quad \tilde{J}\equiv\beta J, \quad \tilde{B}\equiv\frac{\beta B}{2}. \quad (19)$$

Evaluation of the Gaussian integrals yields

$$Z_0 \propto e^{\mathcal{N}NS\overline{\Lambda}} \prod_{\mathbf{k}} \prod_{i\omega_n} \prod_\alpha \left(-i\beta\hbar\omega_n + \tilde{J}Sk^2a^2 - 2\tilde{J}\overline{\mathbf{Q}}\cdot\mathbf{k}a^2 - \tilde{B}h^\alpha_\alpha + \overline{\Lambda} + \frac{\tilde{J}}{S}\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2\right)^{-1}. \quad (20)$$

Writing the product as the exponential of a sum, replacing the $\mathbf{k}$ sum by an integral, $\Sigma_{\mathbf{k}} \to (\mathcal{N}a^2/4\pi^2)\int d^2k$, and shifting $\mathbf{k}$ by $\overline{\mathbf{Q}}/S$, we obtain

$$Z_0 \propto \exp\left(\mathcal{N} N S \bar{\Lambda} - \frac{\mathcal{N} a^2}{4\pi^2} \int d^2k \times \sum_{i\omega_n} \sum_{\alpha} \ln[-i\beta\hbar\omega_n + \tilde{J} S k^2 a^2 - \tilde{B} h^{\alpha}_{\alpha} + \bar{\Lambda}] \right) \tag{21}$$

The MF partition function and thus all MF quantities only depend on $\bar{\lambda}$ and $\bar{\mathbf{Q}}$ through $\bar{\Lambda}$. The saddle-point equation for $\bar{\Lambda}$ is $\partial \ln Z_0/\partial\bar{\Lambda}=0$, resulting in

$$0 = \mathcal{N} N S - \frac{\mathcal{N} a^2}{4\pi^2} \int d^2k \sum_{i\omega_n} \sum_{\alpha} \frac{1}{-i\beta\hbar\omega_n + \tilde{J} S k^2 a^2 - \tilde{B} h^{\alpha}_{\alpha} + \bar{\Lambda}}. \tag{22}$$

The Matsubara sum in this expression is not well defined since the summands do not fall off fast enough. In writing it as a contour integral the contribution from closing the contour does not vanish. The usual procedure is to introduce a convergence factor $e^{\pm i\eta\beta\hbar\omega_n}$ and let $\eta \to 0$ afterwards. The result is ambiguous, depending on the sign in the exponential. Here, Eq. (22) only has solutions for positive sign. Consequently,

$$0 = \mathcal{N} N S - \frac{\mathcal{N} a^2}{4\pi^2} \sum_{\alpha} \int d^2k\, n_B(\tilde{J} S k^2 a^2 - \tilde{B} h^{\alpha}_{\alpha} + \bar{\Lambda})$$

$$= \mathcal{N} N S + \frac{\mathcal{N}}{4\pi \tilde{J} S} \sum_{\alpha} \ln(1 - e^{-\bar{\Lambda} + \tilde{B} h^{\alpha}_{\alpha}}). \tag{23}$$

Here, $n_B(\epsilon) = 1/(e^{\epsilon} - 1)$ is the Bose function. Eventually we find[11]

$$S = -\frac{1}{8\pi \tilde{J} S} [\ln(1 - e^{-\bar{\Lambda} + \tilde{B}}) + \ln(1 - e^{-\bar{\Lambda} - \tilde{B}})]. \tag{24}$$

Equation (24) for $\bar{\Lambda}$ can be evaluated analytically. For given $\bar{\Lambda}$ we have the freedom to choose $\bar{\mathbf{Q}}$, and $\bar{\lambda}$ is then fixed by Eq. (19). This is a consequence of gauge invariance since Eq. (15) specifies the gauge only up to a constant. We choose $\bar{\mathbf{Q}} = 0$. (The square lattice model without continuum approximation runs into problems at this point since the quantity corresponding to $\bar{\mathbf{Q}}$ shows a spurious first-order transition at the MF level.)

The MF magnetization normalized so that $M_0(T=0) = S$ can be obtained from Eq. (21) (Ref. 11)

$$M_0 = \frac{2}{\mathcal{N} N \beta} \frac{d}{dB} \ln Z_0$$

$$= -\frac{1}{8\pi \tilde{J} S} [\ln(1 - e^{-\bar{\Lambda} + \tilde{B}}) - \ln(1 - e^{-\bar{\Lambda} - \tilde{B}})]. \tag{25}$$

Some notes are in order: (i) Equation (24) states that the total number of ''up'' and ''down'' bosons (with $h^{\alpha}_{\alpha} = 1$ and -1, respectively) is conserved, whereas Eq. (25) states that the magnetization is basically the difference of the number of ''up'' and ''down'' bosons. (ii) The dependence of Z_0 on the field B through $\bar{\Lambda}$ is irrelevant at the MF level since $\partial \ln Z_0/\partial\bar{\Lambda} = 0$ by definition. This is not the case at the $1/N$ level. (iii) The normalized magnetization M_0/S exhibits the universality mentioned in Sec. I: It only depends on $\tilde{J} S^2$ and $\tilde{B}$.[11]

Finally we compare the MF magnetization with the original Heisenberg model. From Eqs. (24) and (25) we obtain at low temperatures

$$M_0 - S \cong \frac{1}{4\pi \tilde{J} S} \ln(1 - e^{-\beta B}) \tag{26}$$

up to exponentially small corrections to the field B of order of $\bar{\Lambda} - \tilde{B} \cong e^{-8\pi \tilde{J} S^2}/(1 - e^{-\beta B})$. However, Eq. (26) is just the magnetization of the Heisenberg model neglecting magnon interactions. This means that the SU(N) MF theory captures the correct low-energy spin-wave physics. Consequently, we expect higher-order corrections to be small for low T.

C. 1/N corrections

To take fluctuations in the auxiliary fields λ and $\mathbf{Q}$ into account, we write

$$\lambda(\mathbf{r},\tau) = \bar{\lambda} + i\Delta\lambda(\mathbf{r},\tau), \tag{27}$$

$$Q_j(\mathbf{r},\tau) = 0 + \Delta Q_j(\mathbf{r},\tau). \tag{28}$$

The fluctuations in λ are imaginary since λ has to be integrated along the imaginary axis in Eq. (10). The fluctuations in Q_j are real. They are subject to the gauge constraint in Eq. (15).

We follow the procedure outlined by Auerbach.[17] The exact partition function is

$$Z = \int D\Delta\lambda D\Delta Q_j \exp(-N\mathcal{S}), \tag{29}$$

where the action $\mathcal{S}$ is expanded in a series for small fluctuations r_ℓ with r_ℓ standing for any mode $\Delta\lambda(\mathbf{r},\tau)$ or $\Delta Q_j(\mathbf{r},\tau)$,

$$\mathcal{S} = \sum_{n=0}^{\infty} \frac{1}{n!} S^{(n)}_{\ell_1 \cdots \ell_n} r_{\ell_1} \cdots r_{\ell_n}, \tag{30}$$

where summation over repeated field indices ℓ_i is here and in the following implied. On the other hand, the action can be written as $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\text{dir}} + \mathcal{S}_{\text{loop}}$ with[17]

$$\mathcal{S}_0 = \frac{1}{N} \operatorname{Tr} \ln G_0^{-1}, \tag{31}$$

$$\mathcal{S}_{\text{dir}} = \frac{1}{N\hbar} \int_0^{\hbar\beta} d\tau \int d^2r (N J \mathbf{Q} \cdot \mathbf{Q} - N S \lambda), \tag{32}$$

$$\mathcal{S}_{\text{loop}} = \frac{1}{N} \operatorname{Tr} \ln(1 + G_0 v_\ell r_\ell), \tag{33}$$

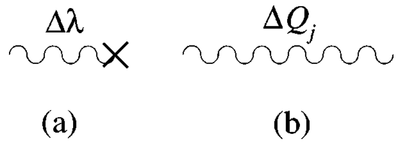

FIG. 1. Diagrams contributing to $\mathcal{S}_{\rm dir}$.

where the trace sums over space, time, and boson flavor, G_0 is the MF bosonic Green function, and v_ℓ is a vertex factor coupling the fluctuation r_ℓ to two bosons.

The first term, $\mathcal{S}_0$, has the standard form for a noninteracting system. It stems from the **k** integral part of the MF free energy; see Eq. (21). The Green function can be read off from the MF partition function,

$$G_0^\alpha(\mathbf{k},i\omega_n)=(-i\hbar\omega_n+JSk^2a^2-Bh_\alpha^\alpha/2+a^2\overline{\lambda})^{-1}. \tag{34}$$

The second term, $\mathcal{S}_{\rm dir}$, comes from the constant part of the MF free energy but also contains fluctuations in the fields r_ℓ which do not involve bosons. The constant part conspires with $\mathcal{S}_0$ to form the MF free energy $-\beta F_0=N\mathcal{S}^{(0)}$. The fluctuating part contains a first-order term in $\Delta\lambda$, corresponding to the coupling of λ to the constant NS in Eq. (14), and a second-order term in ΔQ_j from the $\mathbf{Q}\cdot\mathbf{Q}$ term. The corresponding diagrams are shown in Fig. 1.

The third term, $\mathcal{S}_{\rm loop}$, contains the contribution of fluctuations r_ℓ coupling to bosons. It is the result of a linked-cluster expansion. By expanding the logarithm we obtain the contribution from $\mathcal{S}_{\rm loop}$ to $\mathcal{S}^{(n)}$,

$$\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}\Big|_{\rm loop}=\frac{1}{N}\frac{(-1)^{n+1}}{n}\sum_{P_n}\,{\rm Tr}(G_0 v_{\ell_1}\cdots G_0 v_{\ell_n}). \tag{35}$$

The sum Σ_{P_n} runs over all permutations of the n vertices. The first few terms $\mathcal{S}^{(n)}$ are shown diagrammatically in Fig. 2. Solid lines with arrows denote MF boson Green functions G_0 and the dots correspond to vertex factors v_ℓ. The wriggly lines are external legs r_ℓ. Disconnected diagrams are taken care of by a linked cluster expansion, which puts the whole series into the exponential. No internal r_ℓ lines appear since as far as the action is concerned the $r_\ell(\mathbf{r},\tau)$ are external variables.

For $n\geqslant 3$ Eq. (35) is the only contribution, whereas $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ contain contributions from $\mathcal{S}_{\rm dir}$ and $\mathcal{S}_{\rm loop}$. The total first-order term $\mathcal{S}^{(1)}$ can be shown to vanish as it should since we are expanding around a saddle point.

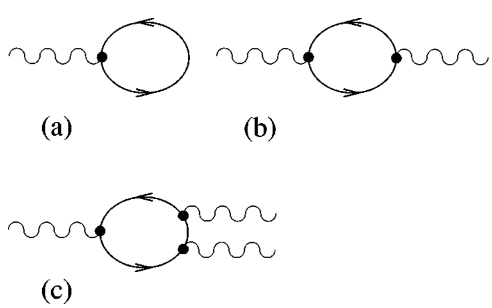

FIG. 2. Diagrams contributing to $\mathcal{S}_{\rm loop}$.

To find the vertex factors v_ℓ we write the exact partition function Z of Eq. (13) in terms of Fourier transforms, where the b_α dependent part of the Lagrangian is

$$\begin{aligned}\mathcal{L}''=\beta a^2\sum_\alpha&\left(-i\hbar\omega_n+JSk^2a^2-\frac{B}{2}h_\alpha^\alpha\right)\\&\times b_\alpha^*(\mathbf{k},i\omega_n)b_\alpha(\mathbf{k},i\omega_n)+\frac{\beta a^4}{2\pi}\\&\times\sum_\alpha\int d^2q\sum_{i\nu_n}[-2J\mathbf{Q}(\mathbf{q},i\nu_n)\cdot\mathbf{k}a^2\\&+a^2\lambda(\mathbf{q},i\nu_n)]b_\alpha^*(\mathbf{k},i\omega_n)b_\alpha(\mathbf{k}-\mathbf{q},i\omega_n-i\nu_n).\end{aligned} \tag{36}$$

The first expression in parentheses is the inverse Green function $(G_0^\alpha)^{-1}$. The same prefactors have to be included in the vertex factors, which are the coefficients of the terms $r_\ell b_\alpha^* b_\alpha$. Consequently,

$$v_{\Delta\lambda}=\frac{a^2}{2\pi}\frac{4\pi^2}{\mathcal{N}a^2}ia^2=\frac{2\pi}{\mathcal{N}}ia^2, \tag{37}$$

$$v_{\Delta Q_j}=\frac{a^2}{2\pi}\frac{4\pi^2}{\mathcal{N}a^2}(-2J)a^2k_j=-\frac{2\pi}{\mathcal{N}}2Ja^2k_j. \tag{38}$$

The factor $4\pi^2/\mathcal{N}a^2$ in both cases stems from the integral over **q**. The factor of i in $v_{\Delta\lambda}$ comes from Eq. (27).

We now consider the expectation value $\langle b_\alpha^\dagger b_\alpha\rangle$ for any α (no summation implied). From this we obtain two important quantities: The average number of bosons per site $\overline{n}=\Sigma_\alpha\langle b_\alpha^\dagger b_\alpha\rangle$, and the magnetization $M=N^{-1}\Sigma_\alpha h_\alpha^\alpha\langle b_\alpha^\dagger b_\alpha\rangle$. Inserting a source term $\Delta\mathcal{L}[j_\alpha]=\Sigma_\alpha j_\alpha b_\alpha^* b_\alpha$ into the Lagrangian (14), where the source current j_α is constant, we find

$$\langle b_\alpha^\dagger b_\alpha\rangle=-\frac{1}{\mathcal{N}\beta a^2}\frac{1}{Z}\frac{\partial Z}{\partial j_\alpha}\bigg|_{j_\alpha=0}. \tag{39}$$

Inserting the series expansion of Eq. (30), evaluating the derivative, and expanding the exponential of the terms containing $\mathcal{S}^{(n)}$, $n\geqslant 3$, we obtain

$$\begin{aligned}\langle b_\alpha^\dagger b_\alpha\rangle=\frac{N}{\mathcal{N}\beta a^2 Z}&\int D\Delta\lambda D\Delta Q_j\\&\times\left(\sum_{n=0}^\infty\frac{1}{n!}\frac{\partial\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}}{\partial j_\alpha}r_{\ell_1}\cdots r_{\ell_n}\right)\\&\times\sum_{m=0}^\infty\frac{(-N)^m}{m!}\left(\sum_{n=3}^\infty\frac{1}{n!}\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}r_{\ell_1}\cdots r_{\ell_n}\right)^m\\&\times\exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1\ell_2}r_{\ell_1}r_{\ell_2}\right).\end{aligned} \tag{40}$$

All terms are Gaussian integrals, which can be evaluated by pairwise contraction over the fields r_ℓ. Diagrammatically, any contraction is represented by connecting two vertices by a *random phase approximation (RPA) fluctuation propagator* $D=(\mathcal{S}^{(2)})^{-1}$, which we represent by a heavy wriggly line.

In the next step we calculate the j_α derivative of $\mathcal{S}^{(n)}$. The derivative basically replaces G_0 by $-(G_0)^2$ so that we may expect it to be related to $\mathcal{S}^{(n+1)}$. The Green function in the presence of the source term is

$$G_0^\alpha(\mathbf{k},i\omega_n)=(-i\hbar\omega_n+JSk^2a^2-Bh_\alpha^\alpha/2+a^2\overline{\lambda}+a^2j_\alpha)^{-1}$$

so that its derivative is $\partial G_0^\alpha/\partial j_\alpha=-G_0^\alpha a^2 G_0^\alpha$. The vertex factor associated with j_α differs from $v_{\Delta\lambda}$ only in a factor of i, $v_{j_\alpha}=2\pi\mathcal{N}^{-1}a^2$. With Eq. (35) we have

$$\frac{\partial}{\partial j_\alpha}\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}\bigg|_{j_\alpha=0}=\frac{\mathcal{N}}{2\pi}\frac{1}{N}\frac{(-1)^{n+2}}{n}\times\sum_{P_n}\mathrm{Tr}(G_0v_{j_\alpha}G_0v_{\ell_1}\cdots G_0v_{\ell_n}+\cdots+G_0v_{\ell_1}\cdots G_0v_{j_\alpha}G_0v_{\ell_n}). \tag{41}$$

The sum contains $nn!$ terms and not $(n+1)!$ because v_{j_α} cannot appear to the right of v_{ℓ_n}. The invariance of the trace under cyclic rotation allows us to write this expression as a sum over all $(n+1)!$ permutations of the vertices v_{j_α}, $v_{\ell_1},\ldots,v_{\ell_n}$, if we introduce a correction factor for overcounting, $nn!/(n+1)!=n/(n+1)$. We obtain[17]

$$\frac{\partial}{\partial j_\alpha}\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}\Big|_{j_\alpha=0}=\frac{\mathcal{N}}{2\pi}\frac{1}{N}\frac{(-1)^{n+2}}{n+1}\times\sum_{P_{n+1}}\mathrm{Tr}(G_0v_{j_\alpha}G_0v_{\ell_1}\cdots G_0v_{\ell_n})=\frac{\mathcal{N}}{2\pi}\mathcal{S}^{(n+1)}_{j_\alpha;\ell_1\cdots\ell_n}. \tag{42}$$

Equation (17.25) in Ref. 17 differs from this result because of different definitions of vertex factors. It follows that

$$\langle b_\alpha^\dagger b_\alpha\rangle=\frac{N}{2\pi\beta a^2Z}\int D\Delta\lambda D\Delta Q_j\times\left(\sum_{n=0}^\infty\frac{1}{n!}\mathcal{S}^{(n+1)}_{j_\alpha;\ell_1\cdots\ell_n}r_{\ell_1}\cdots r_{\ell_n}\right)\times\sum_{m=0}^\infty\frac{(-N)^m}{m!}\left(\sum_{n=3}^\infty\frac{1}{n!}\mathcal{S}^{(n)}_{\ell_1\cdots\ell_n}r_{\ell_1}\cdots r_{\ell_n}\right)^m\times\exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1\ell_2}r_{\ell_1}r_{\ell_2}\right). \tag{43}$$

In principle, we can evaluate the integral for any term in this series. The contraction of two variables gives

$$\frac{1}{Z}\int Dr_\ell r_{\ell_1}r_{\ell_2}\exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1'\ell_2'}r_{\ell_1'}r_{\ell_2'}\right)=\frac{1}{N}(\mathcal{S}^{(2)})^{-1}_{\ell_1\ell_2}, \tag{44}$$

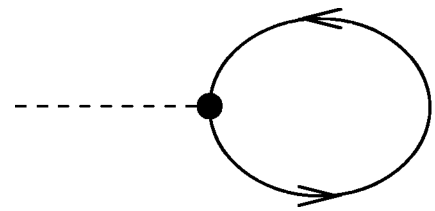

FIG. 3. The diagram for the MF magnetization.

where the r_ℓ are assumed to be real. [In fact they are real only in direct space but complex in Fourier space. We can use the Gaussian integral for complex fields, which has an additional factor of 2, and note that $\Delta\lambda(-\mathbf{q},-i\nu_n)$ and $\Delta\lambda(\mathbf{q},i\nu_n)$ are not independent since $\Delta\lambda(-\mathbf{q},-i\nu_n)=\Delta\lambda^*(\mathbf{q},i\nu_n)$ and similarly for $\Delta\mathbf{Q}$. Thus we have to restrict the sum over $\mathbf{q}$ to one half-space. The factor of 1/2 obtained in this way cancels the factor 2 from the Gaussian integral.] The RPA propagator D is the inverse of the matrix $\mathcal{S}^{(2)}$. We obtain all terms in the expansion (43) by writing down all allowed diagrams consisting of any number of boson loops with one external j_α leg, represented by a dashed line, and an even number of internal vertices, and connecting the latter by RPA propagators in all possible ways. Allowed diagrams are connected and do not contain loops with only one or two internal vertices and no external vertex because first-order terms cancel in an expansion around a saddle point and the loop with two internal legs is already included in the RPA propagator.

To figure out which terms are of which order in $1/N$, note that the magnetization $M=N^{-1}\Sigma_\alpha h_\alpha^\alpha\langle b_\alpha^\dagger b_\alpha\rangle$ contains an explicit factor of $1/N$. Furthermore, each loop contributes a factor of N from summation over flavors (the loop with the external vertex does not contain a sum but has an explicit N), and each RPA propagator contributes a factor of $1/N$; see Eq. (44).

The leading term is depicted in Fig. 3. It is of order N^0 in the magnetization. This term reproduces the MF magnetization (25). Contributions of order $1/N$ in the magnetization have to contain the same number of loops and propagators. The only two allowed diagrams are shown in Fig. 4. Similarly, we could write down the diagrams to any order.

The two relevant contributions are, from Fig. 4(a),

$$\mathcal{D}_1^{(0)}\equiv\frac{N}{2\pi\beta a^2Z}\times\int Dr_\ell\tfrac{1}{2}\mathcal{S}^{(2+1)}_{j_\alpha;\ell_1\ell_2}r_{\ell_1}r_{\ell_2}\exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1'\ell_2'}r_{\ell_1'}r_{\ell_2'}\right)=\frac{1}{4\pi\beta a^2}\mathcal{S}^{(2+1)}_{j_\alpha;\ell_1\ell_2}(\mathcal{S}^{(2)})^{-1}_{\ell_1\ell_2}, \tag{45}$$

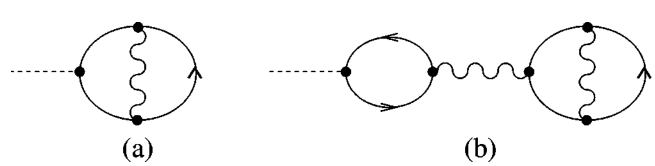

FIG. 4. $1/N$ diagrams contributing to the magnetization. Diagram (a) corresponds to $\mathcal{D}_1^{(0)}$ in Eq. (45) and (b) corresponds to $\mathcal{D}_2^{(0)}$ in Eq. (46).

and from Fig. 4(b),

$$\mathcal{D}_2^{(0)} \equiv \frac{N}{2\pi\beta a^2 Z} \int Dr_\ell \mathcal{S}^{(1+1)}_{j_\alpha;\ell_1} r_{\ell_1} \left(-\frac{N}{6}\right) \times \mathcal{S}^{(3)}_{\ell_2\ell_3\ell_4} r_{\ell_2} r_{\ell_3} r_{\ell_4} \exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1'\ell_2'} r_{\ell_1'} r_{\ell_2'}\right) = -\frac{1}{4\pi\beta a^2} \mathcal{S}^{(1+1)}_{j_\alpha;\ell_1} \mathcal{S}^{(3)}_{\ell_2\ell_3\ell_4} (\mathcal{S}^{(2)})^{-1}_{\ell_1\ell_2} (\mathcal{S}^{(2)})^{-1}_{\ell_3\ell_4}. \quad (46)$$

$\mathcal{D}_2^{(0)}$ contains a sum over all three possible pairings of the four r_ℓ. We have utilized the symmetry of $\mathcal{S}^{(3)}$ in each pair of indices. The evaluation of the relevant $\mathcal{S}^{(n)}$ is relegated to Appendix A. With the results found there we can write down the two terms in $\langle b_\alpha^\dagger b_\alpha \rangle = \mathcal{D}_1^{(0)} + \mathcal{D}_2^{(0)}$,

$$\mathcal{D}_1^{(0)} = \frac{1}{2\mathcal{N}} \sum_{\mathbf{q},i\nu_n} \left(\frac{\sigma_0^{\alpha\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\sigma_\perp^{\alpha\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right), \quad (47)$$

$$\mathcal{D}_2^{(0)} = -\frac{1}{\mathcal{N}N} \frac{n_B(\bar{\Lambda} - \tilde{B} h_\alpha^\alpha)}{n_B(\bar{\Lambda} - \tilde{B}) + n_B(\bar{\Lambda} + \tilde{B})} \times \sum_{\mathbf{q},i\nu_n} \left(\frac{\Sigma_\beta \sigma_0^{\beta\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\Sigma_\beta \sigma_\perp^{\beta\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right), \quad (48)$$

where we have introduced new symbols,

$$\sigma_0^\alpha \equiv \sum_{\mathbf{k}} \frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}, \quad (49)$$

$$\sigma_\perp^\alpha \equiv \sum_{\mathbf{k}} k_2^2 a^2 \frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}, \quad (50)$$

$$\sigma_0^{\alpha\prime} \equiv \sum_{k} \frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}, \quad (51)$$

$$\sigma_\perp^{\alpha\prime} \equiv \sum_{k} k_2^2 a^2 \frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2} \quad (52)$$

with $\epsilon^\alpha_{\mathbf{k}} \equiv \tilde{J}Sk^2a^2 - \tilde{B}h^\alpha_\alpha + \bar{\Lambda}$ and the derivative $n_B^{(1)}$ of the Bose function.

Before we turn to the magnetization we look at the total number of bosons per site $\bar{n} = \Sigma_\alpha \langle b_\alpha^\dagger b_\alpha \rangle$. The MF contribution is $\bar{n}_0 = NS$ since $\bar{\Lambda}$ was chosen that way. To the next order,

$$\bar{n} = NS + \frac{1}{2\mathcal{N}} \sum_{\mathbf{q},i\nu_n} \left(\frac{\Sigma_\alpha \sigma_0^{\alpha\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\Sigma_\alpha \sigma_\perp^{\alpha\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right) - \frac{1}{\mathcal{N}N} \frac{N}{2} \frac{n_B(\bar{\Lambda} - \tilde{B}) + n_B(\bar{\Lambda} + \tilde{B})}{n_B(\bar{\Lambda} - \tilde{B}) + n_B(\bar{\Lambda} + \tilde{B})} \times \sum_{\mathbf{q},i\nu_n} \left(\frac{\Sigma_\beta \sigma_0^{\beta\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\Sigma_\beta \sigma_\perp^{\beta\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right) = NS + 0. \quad (53)$$

We thus see explicitly that the constraint (7) is still satisfied at the $1/N$ level. This is a special case of Auerbach's general proof[17] that the constraint is satisfied to any order. Thus we need not ''shift the saddle point,'' i.e., adjust $\bar{\Lambda}$ so that the constraint is still satisfied.

We now calculate the $1/N$ contribution to the magnetization,

$$M = M_0 - \frac{1}{2\mathcal{N}N} \times \sum_{\mathbf{q},i\nu_n} \left(\frac{\Sigma_\alpha (c_1 - h_\alpha^\alpha) \sigma_0^{\alpha\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\Sigma_\alpha (c_1 - h_\alpha^\alpha) \sigma_\perp^{\alpha\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right), \quad (54)$$

where M_0 is the MF magnetization (25) and

$$c_1 \equiv \frac{n_B(\bar{\Lambda} - \tilde{B}) - n_B(\bar{\Lambda} + \tilde{B})}{n_B(\bar{\Lambda} - \tilde{B}) + n_B(\bar{\Lambda} + \tilde{B})}. \quad (55)$$

Expressing the momentum sum by an integral we find

$$M = M_0 - \frac{1}{N} \frac{a^2}{8\pi^2} \int d^2q \times \sum_{i\nu_n} \left(\frac{\Sigma_\alpha (c_1 - h_\alpha^\alpha) \sigma_0^{\alpha\prime}}{\Sigma_\beta \sigma_0^\beta} + \frac{\Sigma_\alpha (c_1 - h_\alpha^\alpha) \sigma_\perp^{\alpha\prime}}{\mathcal{N}N/2\tilde{J} + \Sigma_\beta \sigma_\perp^\beta} \right) \equiv M_0 - \frac{1}{N} \frac{a^2}{8\pi^2} \int d^2q \sum_{i\nu_n} [\Delta M_0(\mathbf{q}, i\nu_n) + \Delta M_\perp(\mathbf{q}, i\nu_n)]. \quad (56)$$

The first term in the parentheses is due to fluctuations $\Delta\lambda$ while the second comes from $\Delta\mathbf{Q}$.

The same result (56) can be found by writing down the one-loop contribution to the free energy and taking the derivative with respect to magnetic field. In this way we find physical interpretations for the two $1/N$ diagrams: The term coming from the explicit B dependence of the free energy corresponds to Fig. 4(a), whereas the indirect dependence through $\bar{\Lambda}(B)$ corresponds to Fig. 4(b).

Although the expression (56) for the magnetization is formally correct, great care is needed in evaluating the frequency sum. We will first show briefly how naive evaluation leads to a spurious divergence of the integral over momentum $\mathbf{q}$ and then present the solution to this problem. The solution involves carefully taking into account normal ordering of boson operators. First, we derive the contributions to the magnetization for large momentum $\mathbf{q}$ of the (in Fig. 4

vertical) RPA propagator. We express the momentum sum in σ_0^α and $\sigma_\perp^\alpha$ by an integral and shift the variable $\mathbf{k}$ in the two summands by $\mathbf{k}\to\mathbf{k}\mp\mathbf{q}/2$,

$$\sigma_{0,\perp}^\alpha=\frac{\mathcal{N}a^2}{4\pi^2}\times\int d^2k\begin{Bmatrix}1\\ k_2^2a^2\end{Bmatrix}\left(\frac{1}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2+\tilde{J}Sq^2a^2}\right)\times n_B(\tilde{J}Sk^2a^2-\tilde{B}h_\alpha^\alpha+\bar{\Lambda}), \tag{57}$$

where the upper (lower) expression in { } pertains to σ_0^α ($\sigma_\perp^\alpha$), and $\sigma_0^{\alpha\prime}$ and $\sigma_\perp^{\alpha\prime}$ are obtained by replacing n_B by $n_B^{(1)}$. Formally, we expand for small k_1 since large $\mathbf{k}$ are exponentially suppressed. Odd powers of k_1 vanish so that

$$\sigma_{0,\perp}^\alpha=\frac{\mathcal{N}a^2}{4\pi^2}\sum_{m\,\mathrm{even}}\left(\frac{1}{(-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2)^{m+1}}-\frac{1}{(-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2)^{m+1}}\right)\times\int d^2k\begin{Bmatrix}1\\ k_2^2a^2\end{Bmatrix}(2\tilde{J}Sqk_1a^2)^m n_B(\tilde{J}Sk^2a^2-\tilde{B}h_\alpha^\alpha+\bar{\Lambda}). \tag{58}$$

This is an asymptotic series and does not converge. However, for our argument we only need the first two nonvanishing terms, which are well defined.

The leading term in $\Sigma_\alpha\sigma_0^\alpha$ reads

$$\sum_\alpha\sigma_0^\alpha\cong-\frac{\mathcal{N}}{4\pi\tilde{J}S}\left(\frac{1}{-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2}\right)\sum_\alpha\ln(1-e^{-\bar{\Lambda}+\tilde{B}h_\alpha^\alpha})=\mathcal{N}NS\left(\frac{1}{-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2}\right), \tag{59}$$

where the last step follows from the saddle-point equation (24). Thus the $m=0$ term is independent of magnetic field. Similarly we find $\Sigma_\alpha(c_1-h_\alpha^\alpha)\sigma_0^{\alpha\prime}\cong 0$ to the same order, where we have used Eq. (55). This result is not surprising since $\Sigma_\alpha(c_1-h_\alpha^\alpha)\sigma_0^{\alpha\prime}$ is, up to a factor, the magnetic field derivative of $\Sigma_\alpha\sigma_0^\alpha$. The leading nonvanishing ($m=2$) term is

$$\sum_\alpha(c_1-h_\alpha^\alpha)\sigma_0^{\alpha\prime}\cong\frac{\mathcal{N}a^2}{4\pi^2}\left(\frac{1}{(-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2)^3}-\frac{1}{(-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2)^3}\right)\times\int d^2k(2\tilde{J}Sqk_1a^2)^2\sum_\alpha(c_1-h_\alpha^\alpha)\times n_B^{(1)}(\tilde{J}Sk^2a^2-\tilde{B}h_\alpha^\alpha+\bar{\Lambda}) \tag{60}$$

and consequently, for large $\mathbf{q}$,

$$\Delta M_0(\mathbf{q},i\nu_n)=\frac{\Sigma_\alpha(c_1-h_\alpha^\alpha)\sigma_0^{\alpha\prime}}{\Sigma_\beta\sigma_0^\beta}\cong\frac{\tilde{J}a^4}{2\pi^2N}\times\left(\frac{1}{-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2}\right)\times\sum_\alpha(c_1-h_\alpha^\alpha)\int d^2k k_1^2 n_B^{(1)}\times(\tilde{J}Sk^2a^2-\tilde{B}h_\alpha^\alpha+\bar{\Lambda}). \tag{61}$$

Adding the two fractions under the sum and performing the Matsubara sum we find

$$\sum_{i\nu_n}\left(\frac{1}{-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2}\right)=\sum_{i\nu_n}\frac{-2\tilde{J}Sq^2a^2}{(\beta\hbar\nu_n)^2+(\tilde{J}Sq^2a^2)^2}=-\coth\frac{\tilde{J}Sq^2a^2}{2}. \tag{62}$$

For large momentum $\mathbf{q}$ we thus obtain

$$\sum_{i\nu_n}\Delta M_0(\mathbf{q},i\nu_n)\cong-\frac{1}{4\pi N\tilde{J}S^2}\sum_\alpha(c_1-h_\alpha^\alpha)\times\ln(1-e^{-\bar{\Lambda}+\tilde{B}h_\alpha^\alpha})=c_1-\frac{M_0}{S}. \tag{63}$$

The large momentum limit of the $\Delta\mathbf{Q}$ contribution is found more easily. We have to consider $\Delta M_\perp(\mathbf{q},i\nu_n)$ for large momenta; see Eq. (56). The leading order ($m=0$) term of $\Sigma_\beta\sigma_\perp^\beta$ is negligible compared to the constant added to it. Furthermore, the $m=0$ term in $\Sigma_\alpha(c_1-h_\alpha^\alpha)\sigma_\perp^{\alpha\prime}$ does not vanish. Thus the leading term is

$$\frac{\tilde{J}a^4}{2\pi^2N}\left(\frac{1}{-i\beta\hbar\nu_n-\tilde{J}Sq^2a^2}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2}\right)\times\sum_\alpha(c_1-h_\alpha^\alpha)\int d^2k k_2^2 n_B^{(1)}(\tilde{J}Sk^2a^2-\tilde{B}h_\alpha^\alpha+\bar{\Lambda}). \tag{64}$$

This expression is equal to Eq. (61). Consequently, for large momenta the integrand of the external momentum integral in Eq. (56) is

$$2(c_1 - M_0/S), \tag{65}$$

which is independent of momentum. This term would lead to a strong UV divergence of the **q** integral in Eq. (56).

We now discuss the cure. As mentioned above, we have to normal order the operators before we can write the partition function as a functional integral. Careful treatment of ordering is often essential to resolve ambiguities in the expectation value of operator products at equal times. Here we are interested in $\langle b_\alpha^\dagger(\mathbf{r},\tau) b_\alpha(\mathbf{r},\tau)\rangle$. One way of dealing with these ambiguities is to split the time of the operators in the Lagrangian in such a way that the creation operator is at an infinitesimally later time than the annihilation operator. The time-ordered product in the definition of the Green function then takes care of normal ordering at equal times.

If the action contains a term like

$$\Delta\mathcal{S} = \frac{1}{N\hbar}\int_0^{\hbar\beta} d\tau \int d^2r \sum_\alpha c_\alpha(\mathbf{r},\tau) b_\alpha^*(\mathbf{r},\tau+\eta) b_\alpha(\mathbf{r},\tau), \tag{66}$$

where $\eta > 0$ is small, then, after Fourier transformation,

$$\Delta\mathcal{S} = \frac{\beta a^6}{2\pi N}\int d^2k d^2q \sum_{i\omega_n, i\nu_n}\sum_\alpha c_\alpha(\mathbf{q}, i\nu_n) \times e^{i\omega_n\eta} b_\alpha^*(\mathbf{k}, i\omega_n) b_\alpha(\mathbf{k}-\mathbf{q}, i\omega_n - i\nu_n). \tag{67}$$

The coefficient c_α obtains a phase factor $e^{i\omega_n\eta}$, where $i\omega_n$ is the frequency of the boson created at this point. We split the time in this way in the exact Lagrangian (14) as well as in the source term $j_\alpha b_\alpha^* b_\alpha$. As a consequence, phase factors containing the frequency of the outgoing boson appear at all vertices. The only places where they are relevant turn out to be in $\mathcal{S}^{(3)}$ and $\mathcal{S}^{(2+1)}$. In the first term in Eq. (A7) we obtain a total factor of $e^{i\omega_n\eta} e^{i(\omega_n+\nu_n)\eta} e^{i\omega_n\eta} = e^{3i\omega_n\eta} e^{i\nu_n\eta}$ from the three vertices. The factor containing $i\omega_n$ is irrelevant since the sum over $i\omega_n$ is unambiguous anyway. We are left with an overall factor of $e^{i\nu_n\eta}$. The second term from the symmetrization in Eq. (A7) obtains a factor $e^{-i\nu_n\eta}$. If the two terms are added, the terms which have the denominators squared obtain a prefactor of $2i \sin \nu_n\eta$, which vanishes in the limit $\eta\to 0$ (the denominators are already of second order in frequency so that ambiguities or divergences do not arise) and the remaining terms are

$$\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta\lambda(-\mathbf{q},-i\nu_n),\Delta\lambda(\mathbf{q},i\nu_n)} = \frac{(2\pi)^3}{\mathcal{N}^3 N} i\beta a^2 \times \sum_{\mathbf{k}}\sum_\alpha \frac{e^{-i\nu_n\eta} n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - e^{i\nu_n\eta} n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2} \times(-\beta^2 a^4), \tag{68}$$

$$\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta Q_2(-\mathbf{q},-i\nu_n),\Delta Q_2(\mathbf{q},i\nu_n)} = \frac{(2\pi)^3}{\mathcal{N}^3 N} i\beta a^2 \sum_{\mathbf{k}}\sum_\alpha \times \frac{e^{-i\nu_n\eta} n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - e^{i\nu_n\eta} n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}\, 4\tilde{J}^2 a^4 k_2^2. \tag{69}$$

$\mathcal{S}^{(2+1)}$ follows as above.

The Matsubara sum of the leading large **q** term is now, instead of Eq. (62),

$$\begin{aligned}
&\sum_{i\nu_n}\left(\frac{e^{-i\nu_n\eta}}{-i\beta\hbar\nu_n - \tilde{J}Sq^2a^2} - \frac{e^{+i\nu_n\eta}}{-i\beta\hbar\nu_n + \tilde{J}Sq^2a^2}\right)_{\eta\to 0^+} \\
&= \sum_{i\nu_n}\left(\frac{-2\tilde{J}Sq^2a^2}{(\beta\hbar\nu_n)^2 + (\tilde{J}Sq^2a^2)^2} \times \cos\nu_n\eta + \frac{2\beta\hbar\nu_n}{(\beta\hbar\nu_n)^2 + (\tilde{J}Sq^2a^2)^2}\sin\nu_n\eta\right)_{\eta\to 0^+} \\
&= -\coth\frac{\tilde{J}Sq^2a^2}{2} + \lim_{\eta\to 0^+}\frac{\sinh \tilde{J}Sq^2a^2(1/2-\eta)}{\sinh \tilde{J}Sq^2a^2/2} \\
&= 1 - \coth\frac{\tilde{J}Sq^2a^2}{2}.
\end{aligned} \tag{70}$$

The sine series exactly cancels the leading term of the naive series for large **q**, making the remaining expression exponentially small. In other words, the phase factors introduced to ensure correct operator ordering just remove the constant (65) from the integrand for large **q**. The exponential factors are irrelevant in all other terms, which are of higher order in $1/i\nu_n$ and are thus unambiguous.

Deriving a convergence factor $e^{\pm i\nu_n\eta}$ is a common method to resolve the ambiguity of a Matsubara sum. What is unusual here is that two different factors appear for the two terms. It is easy to fall into the trap of thinking that one should simply add the two fractions under the frequency sum in Eq. (62), arguing that the sum then looks unambiguous. This loses an essential contribution because of the two different phase factors.

We utilize the above result by calculating the **q** integrand numerically without taking normal ordering into account and then subtracting the constant (65) explicitly. The frequency sum is expressed in terms of a contour integral. First we analyze the analytic structure of σ_0^α etc. in the complex ν plane. If we replace the sum over **k** by an integral in Eq. (49) for σ_0^α we see that the integrand of the k_1 integral has a pole and, consequently, σ_0^α has a branch cut. Furthermore, it can be shown that $\Sigma_\alpha\sigma_0^\alpha$ does not have zeros apart from the trivial case $\mathbf{q}=0$. The quantity $\Sigma_\alpha(c_1 - h_\alpha^\alpha)\sigma_0^{\alpha\prime}$ obviously also has a branch cut along the real axis and does not have poles. Consequently, the $\Delta\lambda$ contribution to the magnetization, ΔM_0, has a branch cut and no poles. The contour of integration, $\mathcal{C}$, is shown in Fig. 5. We have

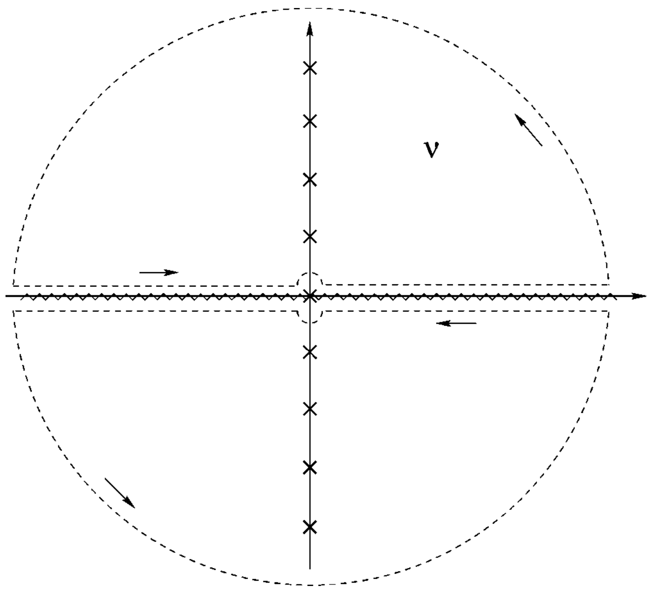

FIG. 5. Contour of integration for the Matsubara sum over $i\nu_n$.

$$\frac{1}{\hbar\beta}\sum_{i\nu_n\neq 0}\Delta M_0=\frac{1}{2\pi i}\oint_{\mathcal{C}}d\nu n_B(\beta\hbar\nu)\Delta M_0$$
$$=\frac{1}{2\pi i}\left[-\int_{\mathcal{C}_\epsilon}d\nu n_B(\beta\hbar\nu)\Delta M_0(0)+\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)d\nu n_B(\beta\hbar\nu)[-iA_0(\nu)]\right],\tag{71}$$

where $A_0(\nu)\equiv -2\,\mathrm{Im}\,\Delta M_0(\nu+i\delta)$ is the spectral function of ΔM_0 and $\mathcal{C}_\epsilon$ is a positively directed circular path of radius $\epsilon\rightarrow 0$ around the origin. Since A_0 vanishes continuously at the origin we get, after moving the last term to the left-hand side,

$$\frac{1}{\hbar\beta}\sum_{i\nu_n}\Delta M_0=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu n_B(\beta\hbar\nu)A_0(\nu).\tag{72}$$

A similar equation holds for the $\Delta\mathbf{Q}$ contribution, $\Delta M_\perp$. We do not discuss the numerical methods in any detail since they are standard. We only note that it is useful to expand the Bose functions n_B in σ_0^α etc. in a geometric series because this allows one to perform the integrals over $\mathbf{k}$ analytically, thereby replacing 2D integrals by numerical summation of well-behaved series.

After evaluating the frequency sum, we have to subtract the constant (65). Numerically the correction term is indeed found to cancel the constant for large $\mathbf{q}$. The new leading term drops off as $1/q^2$ so that the $\mathbf{q}$ integral diverges only logarithmically. We regularize the integral by restricting it to the first Brillouin zone, i.e., by a lattice cutoff. We use a circular Brillouin zone. The integration over the angle of $\mathbf{q}$ is then trivial.

We find that fluctuations in λ and $\mathbf{Q}$ always decrease the magnetization, as is intuitively expected. In fact the magnetization to order $1/N$ can become slightly negative. Of course, the exact magnetization cannot be negative. Apparently the $1/N$ expansion does not work well for SU(2). We can force the result to be positive by putting the fluctuations into an exponential: In writing down the functional integral we should impose the constraint that the total magnetization be positive. This constraint can be implemented by writing the full magnetization as $M=M_0e^g$ and expanding g in powers of $1/N$. If $M=M_0+M_{1/N}+O(1/N^2)$ then $g=1+M_{1/N}/M_0+O(1/N^2)$. Both expansions are *equally valid* to order $1/N$. Of course, this method seems dubious if the $1/N$ term is not small.

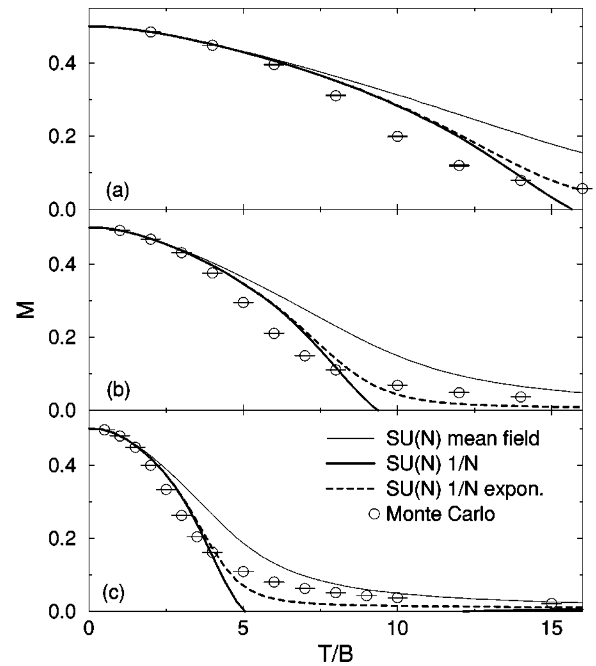

FIG. 6. SU(N) magnetization for magnetic fields (a) $B/J=0.05$, (b) $B/J=0.1$, and (c) $B/J=0.25$ as a function of temperature in units of B. The thin solid line is the MF magnetization, the thick solid line includes $1/N$ corrections linearly, the thick dashed line includes them in the exponential to force positivity, and the circles with error bars are quantum Monte Carlo results for a 32×32 lattice (Refs. 12, 13, and Appendix C).

In Fig. 6 we plot the magnetization for $S=1/2$ and the fields $B/J=0.05,0.1,0.25$ as a function of T/B. At the lowest temperatures the SU(N) model describes the Monte Carlo results quite well, as expected since the SU(N) model captures the correct low-energy physics. The $1/N$ corrections are thus very small here. At high temperatures, however, the $1/N$ term is too large for all considered fields. Although the results with the exponentiation trick are better and show the correct qualitative behavior, they are not quantitatively better than the MF results. We discuss the results further at the end of the next section.

III. O(N) MODEL

A. General considerations

In the last section the Heisenberg model was rewritten in terms of Bose fields and the resulting SU(2) model with two boson flavors was generalized to SU(N). The homomorphism between the groups SU(2) and O(3) opens another way to obtain a large N theory. In this section the Heisenberg model is mapped onto an O(3) model, which is then generalized to O(N). Many concepts are identical to the SU(N) case. However, the O(N) model adds a number of complications.

The group theory involved here can be found elsewhere.[24] In brief, the Lie groups SU(2) and O(3) have the same algebra, up to different representations. This means that the infinitesimal generators of SU(2), namely the Pauli matrices, have the same commutation relations as the three infinitesimal generators of O(3), $(X_k)_{ij}=-2i\epsilon_{ijk}$. Put informally, SU(2) and O(3) have the same local structure, although the global structure is different. Note that we could also talk about the SO(3) model instead of O(3) since the two have the same algebra.

The upshot of this is that we can map the Heisenberg model onto an O(3) model. We introduce three Bose fields b_α and let $S^k=-i\epsilon_{k\alpha\beta}b_\alpha^\dagger b_\beta$, $k=x,y,z$, where we again assume summation over repeated indices. It is easily shown that the commutators of the spin components S^k are correct. To restrict the Hilbert space to the physical states two constraints are needed, $b_\alpha^\dagger b_\alpha=S$ and $b_\alpha^\dagger b_\alpha^\dagger=0$. The second constraint needs explanation. Let us consider the eigenstates of S^z for a single spin. Since S^z is not diagonal in boson flavors we introduce new bosons $c_1\equiv(b_1+ib_2)/\sqrt{2}$, $c_2\equiv(b_1-ib_2)/\sqrt{2}$, and $c_3\equiv b_3$. Then $S^z=-c_1^\dagger c_1+c_2^\dagger c_2$ and the constraints read $c_\alpha^\dagger c_\alpha=S$ and $c_1^\dagger c_2^\dagger+c_2^\dagger c_1^\dagger+c_3^\dagger c_3^\dagger=0$. The eigenstates of S^z are simultaneous eigenstates to $c_1^\dagger c_1$, $c_2^\dagger c_2$, and $c_3^\dagger c_3$. As an example, the following table shows the eigenvalues of the number operators and of S^z for $S=2$. The first constraint means that there are two bosons.

$c_1^\dagger c_1$	$c_2^\dagger c_2$	$c_3^\dagger c_3$	S^z
2	0	0	−2
1	0	1	−1
1	1	0	0
0	1	1	1
0	2	0	2
0	0	2	0

The state with $S^z=0$ is obviously counted twice. The second constraint just removes the last state. It can be rewritten as $c_3^\dagger c_3^\dagger|\psi\rangle=-2c_1^\dagger c_2^\dagger|\psi\rangle$, where $|\psi\rangle$ is any state. This means that the state one gets by creating two c_3 bosons is the same as the one produced by creating one c_1 and one c_2. The second constraint thus reduces the Hilbert space by identifying states with one another. For general S, the second constraint removes spurious spin multiplets of lower total spin.

The first constraint does not make sense for half integer spin. We assume S integer. We will not have to do this for even N in O(N) theory.

The O(3) spin matrix should be an element of the algebra, which consists of antisymmetric 3×3 matrices. In three dimensions any antisymmetric matrix is dual to an axial vector. This is in fact the reason why angular momenta can be written as axial vectors in three dimensions. Here we go the opposite way and define the spin matrix by $S_\beta^\alpha\equiv i\epsilon_{\alpha\beta k}S^k=b_\alpha^\dagger b_\beta-b_\beta^\dagger b_\alpha$. Using the antisymmetry of S_β^α, the Hamiltonian (1) becomes

$$H=-\frac{J}{2}\sum_{\langle ij\rangle} S_\beta^\alpha(i)S_\alpha^\beta(j)-\frac{B}{2}\sum_i h_\beta^\alpha S_\alpha^\beta(i) \tag{73}$$

with $h=[(0,i,0),(-i,0,0),(0,0,0)]$.

To generalize the model to O(N), we introduce N Bose fields b_α subject to the constraints

$$b_\alpha^\dagger b_\alpha=\frac{NS}{3}, \tag{74}$$

$$b_\alpha^\dagger b_\alpha^\dagger=0. \tag{75}$$

The O(N) spin matrices are $S_\beta^\alpha=b_\alpha^\dagger b_\beta-b_\beta^\dagger b_\alpha$. The second constraint again restricts the Hilbert space by identifying, say, $b_N^\dagger b_N^\dagger|\psi\rangle$ with another state. The O(N) Hamiltonian is

$$H=-\frac{3J}{2N}\sum_{\langle ij\rangle} S_\beta^\alpha(i)S_\alpha^\beta(j)-\frac{B}{2}\sum_i h_\beta^\alpha S_\alpha^\beta(i), \tag{76}$$

where h contains $N/3$ copies of the O(3) matrix along the diagonal. NS is an integer multiple of 3.

The next steps are similar to the SU(N) model. Going over to the continuum and inserting bosons we get

$$H=\int d^2r\bigg[JS(\partial_j b_\alpha^\dagger)(\partial_j b_\alpha)-\frac{3J}{N}b_\alpha^\dagger(\partial_j b_\beta^\dagger)b_\beta(\partial_j b_\alpha) -\frac{B}{a^2}h_\beta^\alpha b_\beta^\dagger b_\alpha\bigg]. \tag{77}$$

In writing the partition function as a functional integral, the first constraint (74) is implemented using a Lagrange multiplier field λ. Two Lagrange multipliers μ_1 and μ_2 are introduced to enforce the two components of the second constraint (75). They couple to the b_α fields in the form $\mu_1\mathrm{Re}\, b_\alpha b_\alpha+\mu_2\mathrm{Im}\, b_\alpha b_\alpha=\mu^* b_\alpha b_\alpha/2+\mu b_\alpha^* b_\alpha^*/2$, where we have introduced $\mu=\mu_1+i\mu_2$, which is somewhat misleading, though, since both μ_1 and μ_2 have to be integrated along the *imaginary* axis. The partition function reads

$$Z=\int D^2b_\alpha D\lambda D^2\mu\ \exp\bigg(-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\int d^2r\mathcal{L}[b;\lambda,\mu]\bigg), \tag{78}$$

where

$$\mathcal{L}=\frac{\hbar}{a^2}b_\alpha^*\partial_0 b_\alpha+JS(\partial_j b_\alpha^*)(\partial_j b_\alpha)-\frac{3J}{N}b_\alpha^*(\partial_j b_\beta^*)b_\beta(\partial_j b_\alpha) -\frac{B}{a^2}h_\beta^\alpha b_\beta^* b_\alpha+\lambda b_\alpha^* b_\alpha-\frac{NS}{3}\lambda+\frac{1}{2}\mu^* b_\alpha b_\alpha +\frac{1}{2}\mu b_\alpha^* b_\alpha^*. \tag{79}$$

The quartic term is decoupled using a Hubbard-Stratonovich transformation,

$$\mathcal{L}'=\frac{\hbar}{a^2}b_\alpha^*\partial_0 b_\alpha+JS(\partial_j b_\alpha^*)(\partial_j b_\alpha)+3NJQ_jQ_j$$
$$+3iJQ_jb_\alpha^*(\partial_j b_\alpha)-3iJQ_j(\partial_j b_\alpha^*)b_\alpha-\frac{B}{a^2}h_\beta^\alpha b_\beta^* b_\alpha$$
$$+\lambda b_\alpha^* b_\alpha-\frac{NS}{3}\lambda+\frac{1}{2}\mu^* b_\alpha b_\alpha+\frac{1}{2}\mu b_\alpha^* b_\alpha^*, \qquad (80)$$

where Q_j is real and a gauge field. As compared to SU(N), additional complications arise since under gauge changes μ transforms like a charge 2 particle, as discussed below. We choose the transverse gauge, $\partial_j Q_j=0$.

B. Mean-field theory

Again, MF theory is exact for $N\to\infty$. We make a static assumption for λ, $\mathbf{Q}$, and μ. We then express the fields b_α in terms of Fourier transforms. The partition function reads

$$Z_0=\int D^2 b_\alpha(\mathbf{k},i\omega_n)\exp\Bigg(-3\mathcal{N}N\beta J\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2$$
$$+\mathcal{N}a^2\frac{NS}{3}\beta\overline{\lambda}-\int d^2k\sum_{i\omega_n}\mathcal{L}_0''[b]\Bigg) \qquad (81)$$

with

$$\mathcal{L}_0''[b]=\beta a^2\sum_\alpha\,(-i\hbar\omega_n+JSk^2a^2$$
$$-6J\overline{\mathbf{Q}}\cdot\mathbf{k}a^2+a^2\overline{\lambda})b_\alpha^*(\mathbf{k},i\omega_n)b_\alpha(\mathbf{k},i\omega_n)$$
$$-\beta a^2\sum_{\alpha\beta}Bh_\beta^\alpha b_\beta^*(\mathbf{k},i\omega_n)b_\alpha(\mathbf{k},i\omega_n)$$
$$+\beta a^2\sum_\alpha\frac{a^2}{2}\overline{\mu}^* b_\alpha(-\mathbf{k},-i\omega_n)b_\alpha(\mathbf{k},i\omega_n)$$
$$+\beta a^2\sum_\alpha\frac{a^2}{2}\overline{\mu}b_\alpha^*(-\mathbf{k},-i\omega_n)b_\alpha^*(\mathbf{k},i\omega_n), \qquad (82)$$

where sums are again written out. To diagonalize the Zeeman term we substitute new fields,

$$c_{3n+1}=\frac{1}{\sqrt{2}}(b_{3n+1}+ib_{3n+2}),$$
$$c_{3n+2}=\frac{1}{\sqrt{2}}(b_{3n+1}-ib_{3n+2}),\quad c_{3n+3}=b_{3n+3}. \qquad (83)$$

After shifting the integration variable $\mathbf{k}$ to $\mathbf{k}+3\overline{\mathbf{Q}}/S$ we get

$$\mathcal{L}_0''[c]=\beta a^2\sum_\alpha\Bigg(-i\hbar\omega_n+JSk^2a^2-B\hat{h}_\alpha^\alpha$$
$$+a^2\overline{\lambda}-\frac{9J}{S}\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2\Bigg)c_\alpha^*(\mathbf{k},i\omega_n)c_\alpha(\mathbf{k},i\omega_n)$$
$$+\beta a^2\sum_\alpha\frac{a^2}{2}\overline{\mu}^* c_{\overline{\alpha}}(-\mathbf{k},-i\omega_n)c_\alpha(\mathbf{k},i\omega_n)$$
$$+\beta a^2\sum_\alpha\frac{a^2}{2}\overline{\mu}c_{\overline{\alpha}}^*(-\mathbf{k},-i\omega_n)c_\alpha^*(\mathbf{k},i\omega_n), \qquad (84)$$

where $\hat{h}$ is diagonal with the diagonal elements $-1,1,0,-1,1,0,\ldots$ and

$$\overline{\alpha}=\begin{cases}3n+2 & \text{for } \alpha=3n+1\\ 3n+1 & \text{for } \alpha=3n+2\\ 3n+3 & \text{for } \alpha=3n+3.\end{cases} \qquad (85)$$

The partition function depends on $\overline{\lambda}$ and $\overline{\mathbf{Q}}$ only through $\overline{\Lambda}\equiv a^2\beta\overline{\lambda}-(9\beta J/S)\overline{\mathbf{Q}}\cdot\overline{\mathbf{Q}}a^2$. To get rid of the terms mixing $(\mathbf{k},i\omega_n)$ with $(-\mathbf{k},-i\omega_n)$ we note that $c_\alpha(\mathbf{k},i\omega_n)$ is even in ω_n and introduce new fields,

$$d_\alpha(\mathbf{k},i\omega_n)=\frac{1}{\sqrt{2}}[c_\alpha(\mathbf{k},i\omega_n)-ic_\alpha(-\mathbf{k},i\omega_n)]. \qquad (86)$$

Then we have

$$\mathcal{L}_0''[d]=\beta a^2\sum_\alpha\Bigg(-i\hbar\omega_n+JSk^2a^2$$
$$-B\hat{h}_\alpha^\alpha+\frac{\overline{\Lambda}}{\beta}\Bigg)d_\alpha^*(\mathbf{k},i\omega_n)d_\alpha(\mathbf{k},i\omega_n)$$
$$+\beta a^2\sum_\alpha\frac{ia^2}{2}\overline{\mu}^* d_{\overline{\alpha}}(\mathbf{k},i\omega_n)d_\alpha(\mathbf{k},i\omega_n)$$
$$-\beta a^2\sum_\alpha\frac{ia^2}{2}\overline{\mu}d_{\overline{\alpha}}^*(\mathbf{k},i\omega_n)d_\alpha^*(\mathbf{k},i\omega_n). \qquad (87)$$

The fields d_α are now integrated out. We define $\tilde{J}\equiv\beta J$ and $\tilde{B}\equiv\beta B$ (note different definition of $\tilde{B}$). By integrating over the d_α, putting the product into the exponential, and evaluating the flavor sum we obtain

$$Z_0\propto\exp\Bigg[\mathcal{N}\frac{NS}{3}\overline{\Lambda}-\frac{\mathcal{N}a^2}{4\pi^2}\frac{N}{3}\int d^2k$$
$$\times\sum_{i\omega_n}\Big(\tfrac{1}{2}\ln[(-i\beta\hbar\omega_n+\tilde{J}Sk^2a^2+\overline{\Lambda})^2-a^4\beta^2\overline{\mu}^*\overline{\mu}]$$
$$+\ln[(-i\beta\hbar\omega_n+\tilde{J}Sk^2a^2+\overline{\Lambda})^2-a^4\beta^2\overline{\mu}^*\overline{\mu}-\tilde{B}^2]\Big)\Bigg]. \qquad (88)$$

The MF values $\overline{\Lambda}$ and $\overline{\mu}$ are determined by the saddle-point equations

$$\frac{\partial \ln Z_0}{\partial \bar{\Lambda}}=0, \quad \frac{\partial \ln Z_0}{\partial \bar{\mu}}=0, \quad \frac{\partial \ln Z_0}{\partial \bar{\mu}^*}=0. \tag{89}$$

The last two are equivalent. They yield

$$0=a^4\beta^2\bar{\mu}\int d^2k\sum_{i\omega_n}\left(\frac{1}{2}\,\frac{1}{(-i\beta\hbar\omega_n+\tilde{J}Sk^2a^2+\bar{\Lambda})^2-a^4\beta^2\bar{\mu}^*\bar{\mu}}+\frac{1}{(-i\beta\hbar\omega_n+\tilde{J}Sk^2a^2+\bar{\Lambda})^2-a^4\beta^2\bar{\mu}^*\bar{\mu}-\tilde{B}^2}\right). \tag{90}$$

One solution is $\bar{\mu}=0$. For $\bar{\mu}\neq 0$ we get

$$0=\frac{\ln(1-e^{-\bar{\Lambda}-a^2\beta|\bar{\mu}|})-\ln(1-e^{-\bar{\Lambda}+a^2\beta|\bar{\mu}|})}{2a^2\beta|\bar{\mu}|}+\frac{\ln(1-e^{-\bar{\Lambda}-\sqrt{\tilde{B}^2+a^4\beta^2\bar{\mu}^*\bar{\mu}}})-\ln(1-e^{-\bar{\Lambda}+\sqrt{\tilde{B}^2+a^4\beta^2\bar{\mu}^*\bar{\mu}}})}{\sqrt{\tilde{B}^2+a^4\beta^2\bar{\mu}^*\bar{\mu}}}, \tag{91}$$

which does not have solutions for $\bar{\mu}\neq 0$. Such solutions would correspond to broken gauge symmetry; *if* we had $\bar{\mu}\neq 0$, a term like $|(\partial_j-6Q_j/S)\mu|^2$ would appear in the gauge-invariant Lagrangian, which would make the gauge field $\mathbf{Q}$ massive[25] (this is the Anderson-Higgs mechanism). In our case, however, it is massless at the saddle point.

The partition function is now

$$Z_0\propto\exp\left[\mathcal{N}\frac{NS}{3}\bar{\Lambda}-\frac{\mathcal{N}a^2}{4\pi^2}\int d^2k \times\sum_{i\omega_n}\sum_{\alpha}\ln(-i\beta\hbar\omega_n+\tilde{J}Sk^2a^2-\tilde{B}\hat{h}^{\alpha}_{\alpha}+\bar{\Lambda})\right]. \tag{92}$$

The MF equation for $\bar{\Lambda}$ becomes[11]

$$S=-\frac{1}{4\pi\tilde{J}S}[\ln(1-e^{-\bar{\Lambda}+\tilde{B}})+\ln(1-e^{-\bar{\Lambda}})+\ln(1-e^{-\bar{\Lambda}-\tilde{B}})] \tag{93}$$

and the MF magnetization is[11]

$$M_0=\frac{3}{\mathcal{N}N\beta}\frac{d}{dB}\ln Z_0=-\frac{1}{4\pi\tilde{J}S}[\ln(1-e^{-\bar{\Lambda}+\tilde{B}}) -\ln(1-e^{-\bar{\Lambda}-\tilde{B}})], \tag{94}$$

which exhibits the same universality as the SU(N) result. At low temperatures,

$$M_0-S\cong\frac{1}{4\pi\tilde{J}S}\ln(1-e^{-\beta B})+\frac{1}{2\pi\tilde{J}S}\ln(1-e^{-2\beta B}) \tag{95}$$

up to exponentially small corrections to the magnetic field. Thus, although the leading term is the same as the noninteracting magnon approximation, Eq. (26), the second term is different. Thus we expect the correct behavior at the lowest temperatures but deviations already for $T\sim 2B$.

C. 1/*N* corrections

The method used to calculate the $1/N$ corrections is similar to the SU(N) case. However, the second constraint (75) introduces additional problems. The magnetization is now

$$M=\frac{3}{N}\sum_{\alpha\beta}h^{\alpha}_{\beta}\langle b^{\dagger}_{\alpha}b_{\beta}\rangle=\frac{3}{N}\sum_{\alpha}\hat{h}^{\alpha}_{\alpha}\langle c^{\dagger}_{\alpha}c_{\alpha}\rangle, \tag{96}$$

using the definition of c_α in Eq. (83). Fourier transforming the c_α we find $\langle c^{\dagger}_{\alpha}c_{\alpha}\rangle=\langle d^{\dagger}_{\alpha}d_{\alpha}\rangle$ (no summation implied) and

$$M=\frac{3}{N}\sum_{\alpha}\hat{h}^{\alpha}_{\alpha}\langle d^{\dagger}_{\alpha}d_{\alpha}\rangle. \tag{97}$$

In the following we use the representation in terms of fields d_α, Eq. (86). The fluctuations are written as

$$\lambda(\mathbf{r},\tau)=\bar{\lambda}+i\Delta\lambda(\mathbf{r},\tau), \tag{98}$$

$$\mu_1(\mathbf{r},\tau)=0+i\Delta\mu_1(\mathbf{r},\tau), \tag{99}$$

$$\mu_2(\mathbf{r},\tau)=0+i\Delta\mu_2(\mathbf{r},\tau), \tag{100}$$

$$Q_j(\mathbf{r},\tau)=0+\Delta Q_j(\mathbf{r},\tau), \tag{101}$$

where $\Delta\lambda$, ΔQ_j, $\Delta\mu_1$, and $\Delta\mu_2$ are all real. For convenience we use a complex $\Delta\mu=\Delta\mu_1+i\Delta\mu_2$ so that $\mu(\mathbf{r},\tau)=i\Delta\mu(\mathbf{r},\tau)$ and $\mu^*(\mathbf{r},\tau)=-i\Delta\mu^*(\mathbf{r},\tau)$. The SU($N$) methods of Ref. 17 can be adapted to the O(N) model; we write

$$Z=\int D\Delta\lambda D^2\Delta\mu D\Delta Q_j\exp(-N\mathcal{S}) \tag{102}$$

and expand the action $\mathcal{S}$ as in Eq. (30) for SU(N), where r_ℓ can also stand for $\Delta\mu$ or $\Delta\mu^*$. We can also write $\mathcal{S}=\mathcal{S}_0+\mathcal{S}_{\rm dir}+\mathcal{S}_{\rm loop}$ with

$$\mathcal{S}_0=\frac{1}{N}\,\mathrm{Tr}\ln G_0^{-1}, \tag{103}$$

$$\mathcal{S}_{\rm dir}=\frac{1}{N\hbar}\int_0^{\hbar\beta}d\tau\int d^2r\left(3NJ\mathbf{Q}\cdot\mathbf{Q}-\frac{NS}{3}\lambda\right), \tag{104}$$

$$\mathcal{S}_{\rm loop}=\frac{1}{N}\,\mathrm{Tr}\ln\left(1+G_0\sum_{\ell}v_\ell r_\ell\right). \tag{105}$$

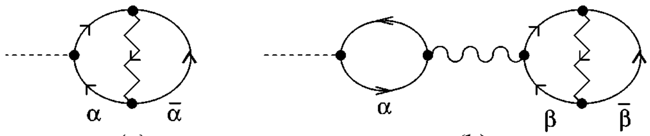

FIG. 7. Diagrams in the $1/N$ magnetization containing $\Delta\mu$ fluctuations.

The (normal) MF Green function can be read off from Eq. (87),

$$G_0^\alpha(\mathbf{k},i\omega_n)=(-i\hbar\omega_n+JSk^2a^2-B\hat{h}_\alpha^\alpha+a^2\overline{\lambda})^{-1}. \tag{106}$$

For $\overline{\mu}=0$ no anomalous Green function exists since there is no $d_\alpha^* d_{\bar{\alpha}}^*$ term.

The constant part of $\mathcal{S}_{\rm dir}$ together with $\mathcal{S}_0$ again gives the MF action $\mathcal{S}^{(0)}$. The first-order terms cancel. The vertex factors can be found in analogy to SU(N),

$$v_{\Delta\lambda}=\frac{2\pi}{\mathcal{N}}ia^2,\quad v_{\Delta\mu}=\frac{2\pi}{\mathcal{N}}\frac{i}{2}a^2,\quad v_{\Delta\mu^*}=\frac{2\pi}{\mathcal{N}}\frac{i}{2}a^2,$$

$$v_{\Delta Q_j}=\frac{2\pi}{\mathcal{N}}(-6J)a^2k_j. \tag{107}$$

The diagrammatics are similar to the SU(N) case. For $\Delta\lambda$ and $\Delta\mathbf{Q}$ fluctuations the only differences are (i) The ΔQ_j vertices contain an additional factor of 3 each, giving 9 in $\mathcal{S}^{(2)}$, (ii) the direct ΔQ_j propagator from $\mathcal{S}_{\rm dir}$ contains an additional factor of 3, (iii) h_α^α is replaced by $\hat{h}_\alpha^\alpha$, (iv) now $\tilde{B}\equiv\beta B$, and (v) $\overline{\Lambda}$ is given by Eq. (93).

In particular, we find $\mathcal{S}^{(2)}_{\Delta Q_2(0,0),\Delta Q_2(0,0)}=0$ as for the SU(N) model. Thus gauge fluctuations are massless for O(N) as well as for SU(N).

The contribution from $\Delta\mu$ requires some thought. From Eqs. (80) and (87) we see that $\Delta\mu$ couples to two ''creation operators'' $d_\alpha^* d_{\bar{\alpha}}^*$, whereas $\Delta\mu^*$ couples to $d_\alpha d_{\bar{\alpha}}$. Consequently, the boson loop in $\mathcal{S}^{(2)}$ can only contain one $\Delta\mu$ vertex and one $\Delta\mu^*$ vertex or neither of them. Thus $\mathcal{S}^{(2)}$ and the RPA propagator do not mix $\Delta\mu$ with other fluctuations. Consequently, the only contributions to $\langle d_\alpha^\dagger d_\alpha\rangle$ at the $1/N$ level correspond to the diagrams in Fig. 7, where the zig-zag line denotes the $\Delta\mu$ RPA propagator. Note the directions of the boson lines.

We derive $\mathcal{S}^{(2)}_{\Delta\mu^*,\Delta\mu}$, $\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta\mu^*,\Delta\mu}$, and $\mathcal{S}^{(2+1)}_{j_\alpha;\Delta\mu^*,\Delta\mu}$ in Appendix B. We can then integrate out the fluctuations. $\Delta\mu$ is a complex field so that the contraction of a pair yields

$$\frac{1}{Z}\int D^2 z_\ell z^*_{\ell_1}z_{\ell_2}\exp\left(-\frac{N}{2}\mathcal{S}^{(2)}_{\ell_1'\ell_2'}z^*_{\ell_1'}z_{\ell_2'}\right)=\frac{2}{N}(\mathcal{S}^{(2)})^{-1}_{\ell_1\ell_2}. \tag{108}$$

Consequently, the diagrams of Figs. 4(a) and 7(a) added together, and Figs. 4(b) plus 7(b), respectively, are

$$\mathcal{D}_1^{(0)}=\frac{1}{2\mathcal{N}}\sum_{\mathbf{q},i\nu_n}\left(\frac{\sigma_0^{\alpha\prime}}{\Sigma_\beta\sigma_0^\beta}+\frac{\sigma_\perp^{\alpha\prime}}{\mathcal{N}N/6\tilde{J}+\Sigma_\beta\sigma_\perp^\beta}+2\frac{\sigma_\star^{\alpha\prime}}{\Sigma_\beta\sigma_\star^\beta}\right), \tag{109}$$

$$\mathcal{D}_2^{(0)}=-\frac{3}{2\mathcal{N}N}\frac{n_B(\overline{\Lambda}-\tilde{B}\hat{h}_\alpha^\alpha)}{n_B(\overline{\Lambda}-\tilde{B})+n_B(\overline{\Lambda})+n_B(\overline{\Lambda}+\tilde{B})}\times\sum_{\mathbf{q},i\nu_n}\left(\frac{\Sigma_\beta\sigma_0^{\beta\prime}}{\Sigma_\beta\sigma_0^\beta}+\frac{\Sigma_\beta\sigma_\perp^{\beta\prime}}{\mathcal{N}N/6\tilde{J}+\Sigma_\beta\sigma_\perp^\beta}+2\frac{\Sigma_\beta\sigma_\star^{\beta\prime}}{\Sigma_\beta\sigma_\star^\beta}\right), \tag{110}$$

where

$$\sigma_\star^\alpha\equiv\sum_{\mathbf{k}}\frac{1+n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\overline{\Lambda}}, \tag{111}$$

$$\sigma_\star^{\alpha\prime}\equiv\sum_{\mathbf{k}}\left(\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B^{(1)}(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\overline{\Lambda}}-2\frac{1+n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{(-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\overline{\Lambda})^2}\right), \tag{112}$$

the other symbols are identical to the SU(N) case if h_α^α is replaced by $\hat{h}_\alpha^\alpha$.

With Eq. (97) the $1/N$ contribution to the magnetization reads

$$M=M_0-\frac{1}{N}\frac{3a^2}{8\pi^2}\int d^2q\sum_{i\nu_n}\left(\frac{\Sigma_\alpha(c_1-\hat{h}_\alpha^\alpha)\sigma_0^{\alpha\prime}}{\Sigma_\beta\sigma_0^\beta}+\frac{\Sigma_\alpha(c_1-\hat{h}_\alpha^\alpha)\sigma_\perp^{\alpha\prime}}{\mathcal{N}N/6\tilde{J}+\Sigma_\beta\sigma_\perp^\beta}+2\frac{\Sigma_\alpha(c_1-\hat{h}_\alpha^\alpha)\sigma_\star^{\alpha\prime}}{\Sigma_\beta\sigma_\star^\beta}\right) \tag{113}$$

with

$$c_1\equiv\frac{n_B(\overline{\Lambda}-\tilde{B})-n_B(\overline{\Lambda}+\tilde{B})}{n_B(\overline{\Lambda}-\tilde{B})+n_B(\overline{\Lambda})+n_B(\overline{\Lambda}+\tilde{B})}. \tag{114}$$

Evaluation of the $\Delta\lambda$ and ΔQ_2 contributions is analogous to the SU(N) case. In particular, naive summation over $i\nu_n$ results in a strong divergence. The constant term in the integrand for large momenta is $3(c_1-M_0/S)$. Numerical calculations confirm this result. Again, the spurious divergence is removed by taking operator ordering into account.

We now turn to the $\Delta\mu$ contribution. From Eq. (111) we see that $\sigma_\star^\alpha$ diverges logarithmically at large momentum $\mathbf{k}$ because of the summand 1 in the numerator. However, the $\Delta\mu$ contribution to the magnetization is finite. To see this, we use a finite cutoff K and let $K\to\infty$ in the result. $\sigma_\star^\alpha$ is dominated by

$$\sigma_\star^\alpha\cong\frac{\mathcal{N}a^2}{4\pi^2}\int_{k\leq K}d^2k\,\frac{1}{-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\overline{\Lambda}}$$

$$=\frac{\mathcal{N}}{8\pi\tilde{J}S}[\ln(-i\beta\hbar\nu_n+2\tilde{J}SK^2a^2+\tilde{J}Sq^2a^2/2+2\overline{\Lambda})$$

$$-\ln(-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2/2+2\bar{\Lambda})]. \tag{115}$$

For $\sigma_\star^{\alpha\prime}$ the corresponding contribution is

$$\sigma_\star^{\alpha\prime}\cong\frac{-2\mathcal{N}}{8\pi\tilde{J}S}\left(\frac{1}{-i\beta\hbar\nu_n+2\tilde{J}SK^2a^2+\tilde{J}Sq^2a^2/2+2\bar{\Lambda}}-\frac{1}{-i\beta\hbar\nu_n+\tilde{J}Sq^2a^2/2+2\bar{\Lambda}}\right). \tag{116}$$

Note that these two expressions do not depend on α. The frequency sum over the $\Delta\mu$ contribution is

$$\sum_{i\nu_n}\Delta M_\star(\mathbf{q},i\nu_n)\equiv\sum_{i\nu_n}\frac{\Sigma_\alpha(c_1-\hat{h}_\alpha^\alpha)\sigma_\star^{\alpha\prime}}{\Sigma_\alpha\sigma_\star^\alpha}=-2c_1\sum_{i\nu_n}\frac{\dfrac{1}{-i\beta\hbar\nu_n+\epsilon_K}-\dfrac{1}{-i\beta\hbar\nu_n+\epsilon_0}}{\ln(-i\beta\hbar\nu_n+\epsilon_K)-\ln(-i\beta\hbar\nu_n+\epsilon_0)}, \tag{117}$$

where $\epsilon_k\equiv2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\bar{\Lambda}$. Since this contribution is proportional to c_1 it comes only from the diagram Fig. 7(b).

The sum over $i\nu_n$ can be evaluated by contour integration. As noted in Appendix B, splitting the time to enforce correct operator ordering results in an overall factor of $\exp(i\nu_n\eta)$, which removes any ambiguity in the $i\nu_n$ sum. In the complex ν plane, $\Delta M_\star$ has a branch cut along the real axis between the points $\epsilon_0/\hbar\beta$ and $\epsilon_K/\hbar\beta$ and two poles on top of the branch points. The contour integral contains three terms: Two from integrating around the branch points in small semicircles and one from integrating the spectral function of $\Delta M_\star$ along the branch cut. The two semicircles contribute $-2c_1[n_B(\epsilon_0)+n_B(\epsilon_K)]$. For $K\to\infty$ this expression becomes $-2c_1n_B(\epsilon_0)$. The spectral function is

$$\mathcal{A}_\star=-\frac{4\pi c_1}{\hbar\beta}\frac{P\dfrac{1}{\epsilon_K/\hbar\beta-\nu}+P\dfrac{1}{\nu-\epsilon_0/\hbar\beta}}{[\ln(\epsilon_K/\hbar\beta-\nu)-\ln(\nu-\epsilon_0/\hbar\beta)]^2+\pi^2} \tag{118}$$

and the integral over it can be shown to vanish for $K\to\infty$. Thus

$$\sum_{i\nu_n}\Delta M_\star=-2c_1n_B(\tilde{J}Sq^2a^2/2+2\bar{\Lambda}) \tag{119}$$

for $K\to\infty$. With Eq. (113) the full contribution from $\Delta\mu$ to the magnetization is

$$-\frac{1}{N}\frac{3a^2}{8\pi^2}\int d^2q\sum_{i\nu_n}2\Delta M_\star=-\frac{1}{N}\frac{3}{\pi\tilde{J}S}\ln(1-e^{-2\bar{\Lambda}})\times\frac{n_B(\bar{\Lambda}-\tilde{B})-n_B(\bar{\Lambda}+\tilde{B})}{n_B(\bar{\Lambda}-\tilde{B})+n_B(\bar{\Lambda})+n_B(\bar{\Lambda}+\tilde{B})}. \tag{120}$$

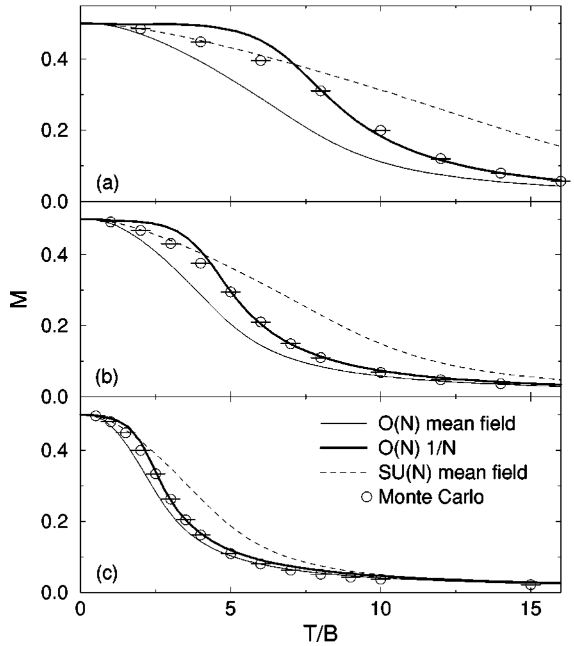

FIG. 8. O(N) magnetization for magnetic fields (a) $B/J=0.05$, (b) $B/J=0.1$, and (c) $B/J=0.25$. The thin solid line is the MF magnetization, the thick solid curve includes $1/N$ corrections, the circles with error bars are quantum Monte Carlo results for a 32×32 lattice (Refs. 12, 13, and Appendix C), and the dashed curve shows the SU(N) MF magnetization for comparison.

A few remarks are in order: (i) This contribution *increases* the magnetization, whereas fluctuations in λ and **Q** decrease it. The physical explanation is that the MF approximation, which enforces the second constraint $b_\alpha^\dagger b_\alpha^\dagger=0$ only on average, *under*estimates the magnetization because it contains contributions from spurious multiplets of lower total spin. (ii) The $\Delta\mu$ contribution has a typical energy scale of $2\bar{\Lambda}$ since excitations of energy $2\bar{\Lambda}$ (and higher) are removed by the second constraint (75). (iii) The **q** integral over $\Delta M_\star$ is well behaved for large **q** so that a cutoff, which is necessary for the Δλ and Δ**Q** contributions, does not change the result appreciably but would complicate the calculations.

Figure 8 shows the magnetization for $S=1/2$ and $B/J=0.05,0.1,0.25$ as a function of T/B. The $\Delta\mu$ fluctuations win over the other contributions; the magnetization is larger than the MF result. We see that the O(N) $1/N$ expansion gives much better results than the SU(N) model except at low temperatures. At small T/B the magnetization seems to be unphysically large, especially for smaller fields. At moderate temperatures the O(N) $1/N$ magnetization is better than both MF results. At high temperatures, the Monte Carlo data consistently fall slightly below the O(N) $1/N$ and, for $B/J=0.25$, even below the O(N) MF results.

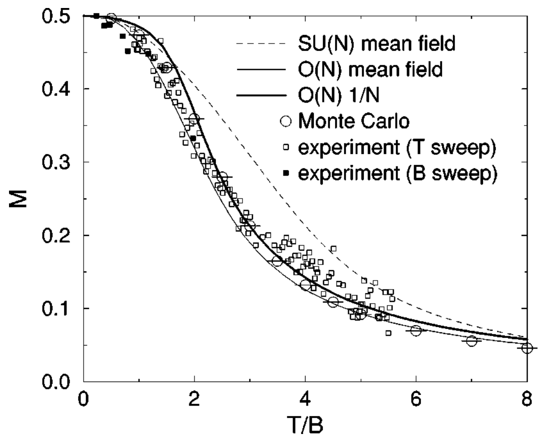

FIG. 9. Comparison of SU(N) and O(N) results, as well as quantum Monte Carlo results for a 16×16 lattice, with experimental data from Manfra *et al.* (Ref. 9). The open squares were obtained sweeping the temperature at fixed field and the filled squares by sweeping the field at fixed temperature.

Recall that the SU(N) MF approximation works well at low temperatures because it coincides with the noninteracting magnon approximation for the Heisenberg model, whereas the O(N) MF magnetization does not. The $1/N$ corrections for the O(N) model (predominantly from $\Delta\mu$ at low T) are large and in fact overcompensate for the error made at the MF level.

There is a distinct crossover to the moderate T regime, where SU(N) MF becomes too large, SU(N) $1/N$ becomes quite wrong, and O(N) $1/N$ is rather good. In fact it is surprisingly good considering that $1/N$ is not really small. It is not fully clear why the O(N) model works better than the SU(N) model at moderate and high temperatures. The reason may lie in the different behavior of gauge ($\Delta\mathbf{Q}$) fluctuations in the two models.[25] In the SU(N) model they are massless in general, whereas for O(N) they are massless only because the MF value of μ happens to vanish for this particular system. In both cases the zero mass leads to an overestimate of fluctuations at the $1/N$ level. However, in the O(N) model fluctuations in μ are available to compensate for this, thereby partly restoring the effect a massive gauge field would have.

One might think that O(N) should be worse since the O(N) model for $N>3$ does not have skyrmions, whereas the SU(N) model has them for all N.[6] However, the $1/N$ expansion does not contain these nonperturbative effects anyway. On the other hand, they are, in principle, captured by the Monte Carlo simulations.[13]

The deviations between O(N) results and Monte Carlo data at high temperatures and $B/J=0.25$ or larger (see Fig. 9) are probably due to thermally created skyrmions or to the fact that the simulations are done on a lattice, whereas the $1/N$ calculations use a continuum approximation. The dispersion of the former is a cosine band if bosonized, whereas the latter has parabolic dispersion. Both effects should become important for temperatures $T\geqslant J$ since both the bandwidth and the typical skyrmion energy are of the order of J. Indeed, the deviations start at $T\sim J$. (In the same region higher-order gradient terms not included in the Heisenberg model should become important.)

We have also investigated the universal dependence on $\tilde{J}S^2$ and $\tilde{B}$. Whereas the MF results exhibit this universality, it is violated by small logarithmic corrections at the $1/N$ level for both models, as expected.[11]

Our results can be compared with the microscopic approach of Kasner and MacDonald,[14] which includes spin-wave corrections to the electronic self-energy. This approach is microscopically better justified than the Heisenberg model. However, the magnetization from Ref. 14 is consistently too large and even MF SU(N) and O(N) results agree better with Monte Carlo data.

Comparison to NMR experiments by Barrett *et al.*[8] shows a number of discrepancies. At low temperature, the experimental data look flat, whereas at high T they drop well below the theoretical results. These discrepancies are mainly due to the scaling of the data, which is done by setting the measured magnetization, which is reduced by disorder, to S in the limit $T\to0$.

Recent magnetoabsorption measurements by Manfra *et al.*[9] show better agreement with our results. In Fig. 9 we compare data of Ref. 9 with SU(N), O(N), and quantum Monte Carlo results. In the calculations we have used the exchange constant corrected for finite width of the quantum well, which yields $B/J\approx0.32$. The Monte Carlo data fall below the O(N) $1/N$ results above $T\sim J$, as discussed above. The experimental data agree quite well with SU(N) theory at low temperatures and with O(N) $1/N$ (and Monte Carlo) results at moderate temperatures, as expected. At higher T the experimental data show more noise but lie mostly above the O(N) $1/N$ curve. This discrepancy for $T>J$ is probably due to neglected higher gradient terms in Eq. (1). The experimental system is a continuous itinerant magnet, which probably explains the deviations from Monte Carlo lattice simulations.

IV. SUMMARY AND CONCLUSIONS

We have calculated $1/N$ corrections to large N Schwinger boson mean field theories for the two-dimensional ferromagnetic Heisenberg model, meant to describe a quantum Hall system at filling factor $\nu=1$. Normal ordering of operators has to be carefully taken into account to obtain the corrections. Using a O(N) model, we find reasonable agreement of the $1/N$ corrected magnetization with both quantum Monte Carlo simulations[12,13] and experiments[9] at moderate and higher temperatures. At low temperatures, the SU(N) model works better since it reproduces the correct low-energy physics. However, the SU(N) model does not describe the data anywhere else, confirming Auerbach's remark that large N methods are ''either surprisingly successful or completely wrong.''[17] Effects of thermally created skyrmions, which are not included in our approach, are small. Away from filling factor $\nu=1$, skyrmions are present in the ground state and should be important. The natural next step leading on from this work would be to incorporate these skyrmions. In addition higher derivative terms due to the long-range Coulomb interaction should be investigated.

Details of the numerical techniques (briefly outlined in Appendix C) as well as Monte Carlo results for the NMR relaxation rate $1/T_1$ will be presented elsewhere (Ref. 13).

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION OF DIAGRAMS FOR SU(*N*)

Here, we derive explicit expressions for $\mathcal{S}^{(2)}$, $\mathcal{S}^{(3)}$, $\mathcal{S}^{(1+1)}$, and $\mathcal{S}^{(2+1)}$. We first calculate the Gaussian part $\mathcal{S}^{(2)}_{\ell_1\ell_2} r_{\ell_1} r_{\ell_2}/2$ in the action. It consists of two contributions, see Figs. 1(b) and 2(b). The first term is easily read off from $\mathcal{S}_{\rm dir}$ in Eq. (32),

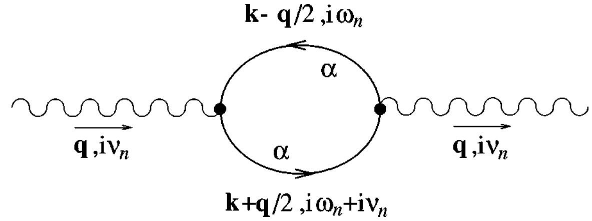

FIG. 10. Notation of momenta and frequencies for $\mathcal{S}^{(2)}$ with external $\Delta\lambda$ or $\Delta\mathbf{Q}$ legs.

$$\tfrac{1}{2}\mathcal{S}^{(2)}_{\ell_1\ell_2}\big|_{\rm dir} r_{\ell_1} r_{\ell_2} = \frac{4\pi^2}{\mathcal{N}} \times \sum_{\mathbf{q},i\nu_n} \tilde{J}\Delta\mathbf{Q}(-\mathbf{q},-i\nu_n)\cdot\Delta\mathbf{Q}(\mathbf{q},i\nu_n)a^2. \tag{A1}$$

Note that $\Delta Q_j(-\mathbf{q},-i\nu_n)=\Delta Q_j^*(\mathbf{q},i\nu_n)$ since $\Delta Q_j(\mathbf{r},\tau)$ is real. The same holds for $\Delta\lambda$, below.

The loop part follows from Eq. (35). The notation is shown in Fig. 10. Inserting Eqs. (34), (37), and (38) we obtain

$$\tfrac{1}{2}\mathcal{S}^{(2)}_{\ell_1\ell_2}\big|_{\rm loop} r_{\ell_1} r_{\ell_2} = -\frac{1}{2N}\sum_{\mathbf{q},i\nu_n}\sum_{\mathbf{k},i\omega_n}\sum_{\alpha} \frac{1}{-i\beta\hbar\omega_n - i\beta\hbar\nu_n + \tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2 - \tilde{B}h^\alpha_\alpha + \overline{\Lambda}}\,\frac{2\pi}{\mathcal{N}}$$
$$\times[i\beta a^2\Delta\lambda(\mathbf{q},i\nu_n) - 2\tilde{J}a^2(\mathbf{k}+\mathbf{q}/2)\cdot\Delta\mathbf{Q}(\mathbf{q},i\nu_n)]\,\frac{1}{-i\beta\hbar\omega_n + \tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2 - \tilde{B}h^\alpha_\alpha + \overline{\Lambda}}\,\frac{2\pi}{\mathcal{N}}$$
$$\times[i\beta a^2\Delta\lambda(-\mathbf{q},-i\nu_n) - 2\tilde{J}a^2(\mathbf{k}-\mathbf{q}/2)\cdot\Delta\mathbf{Q}(-\mathbf{q},-i\nu_n)]. \tag{A2}$$

Performing the Matsubara sum over $i\omega_n$ and utilizing the periodicity of the Bose function n_B we get

$$\tfrac{1}{2}\mathcal{S}^{(2)}_{\ell_1\ell_2}\big|_{\rm loop} r_{\ell_1} r_{\ell_2} = \frac{1}{2N}\,\frac{4\pi^2}{\mathcal{N}^2}\sum_{\mathbf{q},i\nu_n}\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}S\mathbf{q}\cdot\mathbf{k}a^2}[i\beta a^2\Delta\lambda(-\mathbf{q},-i\nu_n) - 2\tilde{J}a^2(\mathbf{k}-\mathbf{q}/2)\cdot\Delta\mathbf{Q}(-\mathbf{q},-i\nu_n)]$$
$$\times[i\beta a^2\Delta\lambda(\mathbf{q},i\nu_n) - 2\tilde{J}a^2(\mathbf{k}+\mathbf{q}/2)\cdot\Delta\mathbf{Q}(\mathbf{q},i\nu_n)], \tag{A3}$$

where $\epsilon^\alpha_{\mathbf{k}} \equiv \tilde{J}Sk^2a^2 - \tilde{B}h^\alpha_\alpha + \overline{\Lambda}$. At this point we use the transverse gauge, $\mathbf{q}\cdot\Delta\mathbf{Q}(\mathbf{q},i\nu_n)=0$. We choose coordinates in such a way that k_1 and ΔQ_1 are parallel to $\mathbf{q}$. Then $\Delta Q_1(\mathbf{q},i\nu_n)=0$ and the last expression simplifies,

$$\frac{1}{2}\,\mathcal{S}^{(2)}_{\ell_1\ell_2}\big|_{\rm loop} r_{\ell_1} r_{\ell_2} = \frac{1}{2N}\,\frac{4\pi^2}{\mathcal{N}^2}\sum_{\mathbf{q},i\nu_n}\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}$$
$$\times[-\beta^2a^4\Delta\lambda(-\mathbf{q},-i\nu_n)\Delta\lambda(\mathbf{q},i\nu_n) + 4\tilde{J}^2a^4k_2^2\Delta Q_2(-\mathbf{q},-i\nu_n)\Delta Q_2(\mathbf{q},i\nu_n)]. \tag{A4}$$

Terms mixing $\Delta\lambda$ and $\Delta\mathbf{Q}$ vanish since their coefficient is odd in k_2. Adding Eq. (A1) to Eq. (A4) yields

$$\mathcal{S}^{(2)}_{\Delta\lambda(-\mathbf{q},-i\nu_n),\Delta\lambda(\mathbf{q},i\nu_n)} = \frac{4\pi^2}{\mathcal{N}^2N}\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}(-\beta^2a^4), \tag{A5}$$

$$\mathcal{S}^{(2)}_{\Delta Q_2(-\mathbf{q},-i\nu_n),\Delta Q_2(\mathbf{q},i\nu_n)} = \frac{8\pi^2}{\mathcal{N}}\tilde{J}a^2 + \frac{4\pi^2}{\mathcal{N}^2N}\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) - n_B(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sqk_1a^2}4\tilde{J}^2a^4k_2^2, \tag{A6}$$

all other components vanish. The fact that $\mathcal{S}^{(2)}$ only connects fluctuations at $(\mathbf{q},i\nu_n)$ and $(-\mathbf{q},-i\nu_n)$ just means that the RPA propagator conserves energy and momentum. The real part of $\mathcal{S}^{(2)}$ is always positive except for $\mathcal{S}^{(2)}_{\Delta Q_2(0,0),\Delta Q_2(0,0)}=0$. Thus there is one zero mode, which results in an additional factor in the partition function, which, however, does not depend on field and is thus irrelevant for the magnetization. The zero mode at $\mathbf{q}=0$, $i\nu_n=0$ shows that gauge ($\Delta\mathbf{Q}$) fluctuations are massless. For the remaining modes, $\mathcal{S}^{(2)}$ can be inverted to get the RPA propagator D, which is also positive. The saddle point is thus stable.

Looking at the diagrams in Fig. 4 we see that the horizontal propagator in the right diagram can only be at $\mathbf{q}=0$, $i\nu_n=0$ since the source j_α does not insert any frequency or momentum. In fact, it can only be $\Delta\lambda(0,0)$, as we will see. Keeping this is mind we calculate $\mathcal{S}^{(3)}_{\ell_1\ell_2\ell_3}$. Since $\mathcal{S}^{(3)}$ is symmetric in its indices we can assume that r_{ℓ_1} is $\Delta\lambda(0,0)$. Furthermore, r_{ℓ_2} determines r_{ℓ_3}. We start from the definition (35),

$$\mathcal{S}^{(3)}_{\ell_1\ell_2\ell_3}=\frac{1}{N}[\mathrm{Tr}(G_0v_{\ell_1}G_0v_{\ell_2}G_0v_{\ell_3})+\mathrm{Tr}(G_0v_{\ell_1}G_0v_{\ell_3}G_0v_{\ell_2})]. \tag{A7}$$

The first of the two summands is

$$\begin{aligned}&\frac{1}{N}\sum_{\mathbf{k},i\omega_n}\sum_{\alpha}\beta^3v_{\ell_1}v_{\ell_2}v_{\ell_3}\frac{1}{-i\beta\hbar\omega_n+\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}}\\&\quad\times\frac{1}{-i\beta\hbar\omega_n-i\beta\hbar\nu_n+\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}}\,\frac{1}{-i\beta\hbar\omega_n+\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}}\\&=\frac{1}{N}\sum_{\mathbf{k}}\sum_{\alpha}\beta^3v_{\ell_1}v_{\ell_2}v_{\ell_3}\Bigg[\frac{n_B[\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{(i\beta\hbar\nu_n-2\tilde{J}Sqk_1a^2)^2}\\&\quad-\frac{d}{dz}\,\frac{n_B(z)}{-z-i\beta\hbar\nu_n+\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}}\Bigg|_{z=\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}}\Bigg]\\&=\frac{1}{N}\sum_{\mathbf{k}}\sum_{\alpha}\beta^3v_{\ell_1}v_{\ell_2}v_{\ell_3}\Bigg[\frac{n_B[\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{(-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2)^2}-\frac{n_B[\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{(-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2)^2}\\&\quad-\frac{n_B^{(1)}[\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}\Bigg],\end{aligned} \tag{A8}$$

where $n_B^{(\nu)}(\epsilon)\equiv d^\nu n_B(\epsilon)/d\epsilon^\nu$ is the ν-th derivative of the Bose function. With the vertex factors the last expression becomes

$$\begin{aligned}\cdots=&\frac{(2\pi)^3}{\mathcal{N}^3N}\sum_{\mathbf{k}}\sum_{\alpha}i\beta a^2\begin{Bmatrix}-\beta^2a^4\\4\tilde{J}^2a^4k_2^2\end{Bmatrix}\\&\times\Bigg[\frac{n_B[\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{(-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2)^2}\\&-\frac{n_B(\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda})}{(-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2)^2}\\&-\frac{n_B^{(1)}[\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda}]}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}\Bigg],\end{aligned} \tag{A9}$$

where the upper [lower] term in the curly brackets is for $r_{\ell_2}=\Delta\lambda(\mathbf{q},i\nu_n)$ [$\Delta Q_2(\mathbf{q},i\nu_n)$]. For $r_{\ell_1}=\Delta Q_2(0,0)$ the integrand would be odd in k_2 so that this contribution vanishes.

The second term in Eq. (A7) just has r_{ℓ_2} and r_{ℓ_3} exchanged, which means $\mathbf{q}$ and $i\nu_n$ have opposite sign. Thus,

$$\begin{aligned}&\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta\lambda(-\mathbf{q},-i\nu_n),\Delta\lambda(\mathbf{q},i\nu_n)}\\&=\frac{(2\pi)^3}{\mathcal{N}^3N}i\beta a^2\\&\quad\times\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})-n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}(-\beta^2a^4),\end{aligned} \tag{A10}$$

$$\begin{aligned}&\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta Q_2(-\mathbf{q},-i\nu_n),\Delta Q_2(\mathbf{q},i\nu_n)}\\&=\frac{(2\pi)^3}{\mathcal{N}^3N}i\beta a^2\\&\quad\times\sum_{\mathbf{k}}\sum_{\alpha}\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})-n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}4\tilde{J}^2a^4k_2^2,\end{aligned} \tag{A11}$$

where the fractions in Eq. (A9) containing the denominator squared have cancelled upon adding the two terms.

We can now calculate $\mathcal{S}^{(1+1)}$ and $\mathcal{S}^{(2+1)}$. These expressions contain a vertex $v_{j_\alpha}=2\pi a^2/\mathcal{N}$ instead of $v_{\Delta\lambda}=2\pi i a^2/\mathcal{N}$. The source j_α inserts zero momentum and frequency. For $\mathcal{S}^{(1+1)}$ we are thus only interested in $\mathcal{S}^{(1+1)}_{j_\alpha;\Delta\lambda(0,0)}$ [the left loop in Fig. 4(b)]. By taking the limit to zero frequency and momentum, we obtain

$$\mathcal{S}^{(2)}_{\Delta\lambda(0,0),\Delta\lambda(0,0)}=\frac{4\pi^2}{\mathcal{N}^2N}\times\sum_{\mathbf{k}}\sum_{\alpha} n_B^{(1)}(\tilde{J}Sk^2a^2-\tilde{B}h^\alpha_\alpha+\bar{\Lambda})(-\beta^2a^4)$$
$$=\frac{\pi}{2\mathcal{N}\tilde{J}S}\beta^2a^4[n_B(\bar{\Lambda}-\tilde{B})+n_B(\bar{\Lambda}+\tilde{B})]. \tag{A12}$$

Keeping in mind that j_α couples only to the boson of flavor α we find similarly

$$\mathcal{S}^{(1+1)}_{j_\alpha;\Delta\lambda(0,0)}=-\frac{i\pi}{\mathcal{N}N\tilde{J}S}\beta^2a^4n_B(\bar{\Lambda}-\tilde{B}h^\alpha_\alpha). \tag{A13}$$

From $\mathcal{S}^{(3)}$ we infer $\mathcal{S}^{(2+1)}$,

$$\mathcal{S}^{(2+1)}_{j_\alpha;\Delta\lambda(-\mathbf{q},-i\nu_n),\Delta\lambda(\mathbf{q},i\nu_n)}=\frac{(2\pi)^3}{\mathcal{N}^3N}\beta a^2\sum_{\mathbf{k}}\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})-n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}(-\beta^2a^4), \tag{A14}$$

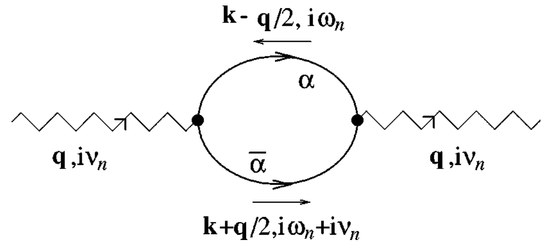

FIG. 11. Notation of momenta and frequencies for $\mathcal{S}^{(2)}$ with external $\Delta\mu$ legs. The momenta and frequencies are measured in the direction of the attached arrows.

$$\mathcal{S}^{(2+1)}_{j_\alpha;\Delta Q_2(-\mathbf{q},-i\nu_n),\Delta Q_2(\mathbf{q},i\nu_n)}=\frac{(2\pi)^3}{\mathcal{N}^3N}\beta a^2\sum_{\mathbf{k}}\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})-n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sqk_1a^2}4\tilde{J}^2a^4k_2^2. \tag{A15}$$

APPENDIX B: CALCULATION OF DIAGRAMS FOR O(N)

We start with $\mathcal{S}^{(2)}$, using the notation shown in Fig. 11. In analogy to Eq. (35),

$$\mathcal{S}^{(2)}_{\Delta\mu^*(\mathbf{q},i\nu_n),\Delta\mu(\mathbf{q},i\nu_n)}=-\frac{1}{N}\,\mathrm{Tr}(G_0v_{\Delta\mu^*}G_0^*v_{\Delta\mu}). \tag{B1}$$

Here, one of the Green functions is the complex conjugate since the line is traversed against the direction of the boson propagator. The momentum and frequency of G_0^* are measured counterclockwise. We find

$$\mathcal{S}^{(2)}_{\Delta\mu^*(\mathbf{q},i\nu_n),\Delta\mu(\mathbf{q},i\nu_n)}=-\frac{\pi^2}{\mathcal{N}^2N}\beta^2a^4\sum_{\mathbf{k},i\omega_n}\sum_{\alpha}\frac{1}{-i\beta\hbar\omega_n-i\beta\hbar\nu_n+\tilde{J}S(\mathbf{k}+\mathbf{q}/2)^2a^2-\tilde{B}\hat{h}^\alpha_\alpha+\bar{\Lambda}}$$
$$\times\frac{1}{-i\beta\hbar\omega_n-\tilde{J}S(\mathbf{k}-\mathbf{q}/2)^2a^2+\tilde{B}\hat{h}^{\bar{\alpha}}_{\bar{\alpha}}-\bar{\Lambda}}=\frac{\pi^2}{\mathcal{N}^2N}\beta^2a^4\sum_{\mathbf{k}}\sum_{\alpha}\frac{1+n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\bar{\Lambda}} \tag{B2}$$

with $\epsilon^\alpha_{\mathbf{k}}\equiv\tilde{J}Sk^2a^2-\tilde{B}\hat{h}^\alpha_\alpha+\bar{\Lambda}$. Here, we have used that $\hat{h}^\alpha_\alpha+\hat{h}^{\bar{\alpha}}_{\bar{\alpha}}=0$ and the identity $n_B(-\epsilon)=-n_B(\epsilon)-1$. The real part is positive so that the functional integral is well defined. This kind of expression is known from the theory of scattering processes.

$\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta\mu^*,\Delta\mu}$ can be derived similarly,

$$\mathcal{S}^{(3)}_{\Delta\lambda(0,0),\Delta\mu^*(\mathbf{q},i\nu_n),\Delta\mu(\mathbf{q},i\nu_n)}=\frac{(2\pi)^3}{\mathcal{N}^3N}\frac{i\beta^3a^6}{4}$$
$$\times\sum_{\mathbf{k}}\sum_{\alpha}\left(\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B^{(1)}(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\bar{\Lambda}}-2\frac{1+n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2})+n_B(\epsilon^{\bar{\alpha}}_{\mathbf{k}-\mathbf{q}/2})}{(-i\beta\hbar\nu_n+2\tilde{J}Sk^2a^2+\tilde{J}Sq^2a^2/2+2\bar{\Lambda})^2}\right). \tag{B3}$$

The term containing the denominator squared does not cancel in this case. To obtain this result we have summed over boson frequencies $i\omega_n$ using contour integration. One has to consider operator ordering to do this properly. Since the anomalous combinations $d^\dagger_{\bar\alpha} d^\dagger_\alpha$ and $d_{\bar\alpha} d_\alpha$ in the Hamiltonian contain two commuting operators time splitting is not necessary and no phase factors appear in $v_{\Delta\mu}$ and $v_{\Delta\mu^*}$. On the other hand, the other vertices obtain factors $\exp(i\omega_n\eta)$. It can be shown that the two terms in $\mathcal{S}^{(3)}$, coming from the symmetrization in Eq. (A7), obtain factors of $\exp(i\omega_n\eta)\exp(i\nu_n\eta)$ and $\exp(-i\omega_n\eta)\exp(i\nu_n\eta)$, respectively. The different factors in $i\omega_n$ are crucial in arriving at Eq. (B3). Furthermore, we obtain an overall factor of $\exp(i\nu_n\eta)$.

Immediately we find

$$\mathcal{S}^{(2+1)}_{j_\alpha;\Delta\mu^*(\mathbf{q},i\nu_n),\Delta\mu(\mathbf{q},i\nu_n)} = \frac{(2\pi)^3}{\mathcal{N}^3 N}\frac{\beta^3 a^6}{4} \times \sum_{\mathbf{k}} \left(\frac{n_B^{(1)}(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) + n_B^{(1)}(\epsilon^{\bar\alpha}_{\mathbf{k}-\mathbf{q}/2})}{-i\beta\hbar\nu_n + 2\tilde{J}Sk^2a^2 + \tilde{J}Sq^2a^2/2 + 2\bar\Lambda} - 2\,\frac{1 + n_B(\epsilon^\alpha_{\mathbf{k}+\mathbf{q}/2}) + n_B(\epsilon^{\bar\alpha}_{\mathbf{k}-\mathbf{q}/2})}{(-i\beta\hbar\nu_n + 2\tilde{J}Sk^2a^2 + \tilde{J}Sq^2a^2/2 + 2\bar\Lambda)^2} \right). \tag{B4}$$

Finally, we have to recalculate $\mathcal{S}^{(2)}_{\Delta\lambda(0,0),\Delta\lambda(0,0)}$. By replacing h^α_α by $\hat{h}^\alpha_\alpha$ in Eq. (A12) we get

$$\mathcal{S}^{(2)}_{\Delta\lambda(0,0),\Delta\lambda(0,0)} = \frac{\pi}{3\mathcal{N}\tilde{J}S}\beta^2 a^4 [n_B(\bar\Lambda - \tilde{B}) + n_B(\bar\Lambda) + n_B(\bar\Lambda + \tilde{B})]. \tag{B5}$$

APPENDIX C: THE QUANTUM MONTE CARLO TECHNIQUE

In order to test the accuracy of the analytic results, we have carried out quantum Monte Carlo simulations using the stochastic series expansion method,[26] which is ideally suited for the present calculation since it does not introduce any systematic errors. Sufficiently large lattices can be studied so that finite-size effects are completely negligible.

The method is based on a Taylor expansion of the density matrix $e^{-\beta H}$. Writing H in terms of its one- and two-body terms, $H = \Sigma_{i=1}^M H_i$, the partition function can be written as[26]

$$Z = \sum_\alpha \sum_{n=0}^\infty \sum_{S_n} \frac{(-\beta)^n}{n!} \left\langle \alpha \middle| A \prod_{i=1}^n H_{l_i} \middle| \alpha \right\rangle, \tag{C1}$$

where S_n denotes a sequence of indices $(l_1, l_2, \ldots, l_n)$, where $l_i \in 1,...,M$, and $|\alpha\rangle = |S_1^z, S_2^z, \ldots, S_N^z\rangle$ is an eigenstate of all the operators S_i^z. The sequences and the states are sampled using as the relative weight $(-\beta)^n/n!\langle\alpha|\Pi_{i=1}^n H_{l_i}|\alpha\rangle$, which for the present case can be made positive definite by adding a suitable constant to H. For a system of finite $\mathcal{N}$ and β only sequences of finite length contribute significantly and the limit $n\to\infty$ poses no problem (the average power $\langle n\rangle$ is given by $|E|\beta$, where E is the total internal energy).

We want to emphasize an important technical detail that makes the sampling particularly efficient: The external field is chosen in the $\hat{x}$ direction. This automatically causes the simulation to become grand-canonical and there are no longer any problems associated with a restricted winding number. If the transverse field is not too weak ($B/J \gtrsim 0.02$), it causes the autocorrelation times of all calculated quantities to become very short, even though only purely local updates are used. Furthermore, it enables easy access to observables involving both diagonal and off-diagonal operators. Details of the implementation will be presented elsewhere.[13]

For a 4×4 system we have compared our QMC data with exact diagonalization results, and they agree to within statistical errors. Relative errors are typically of the order 10^{-4} for all system sizes considered. For all the field strengths presented in this paper, the results for 16×16 and 32×32 sites agree to this precision (finite-size effects increase with decreasing B), and we have presented magnetization results only for the larger size in this paper.

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