Density of states in the non-Hermitian Lloyd model

Christopher Mudry, P. W. Brouwer, and B. I. Halperin
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

V. Gurarie and A. Zee
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030
(Received 30 July 1998)

We reconsider the recently proposed connection between density of states in the so-called ''non-Hermitian quantum mechanics'' and the localization length for a particle moving in random potential. We argue that it is indeed possible to find the localization length from the density of states of a non-Hermitian random ''Hamiltonian.'' However, finding the density of states of a non-Hermitian random Hamiltonian remains an open problem, contrary to previous findings in the literature. [S0163-1829(98)05144-3]

I. INTRODUCTION

There are situations in physics in which observables can be obtained from properties of non-Hermitian operators. In this context, the recent work of Hatano and Nelson[1] on random ''Hamiltonians'' with an imaginary vector potential has caused considerable interest in so-called ''non-Hermitian quantum mechanics.''[2–12] Motivated by the study of the pinning of vortices by columnar defects in a superconductor, attention has focused on two main questions: What is the spectrum of eigenvalues of a non-Hermitian Hamiltonian, and are the corresponding eigenfunctions localized or extended in space?

In the model introduced by Hatano and Nelson, particles are hopping on a lattice with a non-Hermitian dynamics governed by the Hamiltonian[1]

$$\mathcal{H}_h = \mathcal{K}_h + \sum_{\mathbf{j}} w_{\mathbf{j}} \psi_{\mathbf{j}}^\dagger \psi_{\mathbf{j}}, \tag{1a}$$

$$\mathcal{K}_h = -\frac{t}{2} \sum_{\mathbf{j},\mathbf{a}} (e^{\mathbf{h}\cdot\mathbf{a}} \psi_{\mathbf{j}}^\dagger \psi_{\mathbf{j}+\mathbf{a}} + e^{-\mathbf{h}\cdot\mathbf{a}} \psi_{\mathbf{j}+\mathbf{a}}^\dagger \psi_{\mathbf{j}}). \tag{1b}$$

Here, $\psi_{\mathbf{j}}^\dagger$ creates the state at lattice site $\mathbf{j}$, $\mathbf{a}$ is a directed nearest-neighbor vector, t is the bandwidth, and $w_{\mathbf{j}}$ is the random (real) on-site potential. Periodic boundary conditions are assumed. The ''time evolution'' induced by $\mathcal{H}_h$ is nonunitary because of the imaginary vector potential $i\mathbf{h}$. Numerical simulations by Hatano and Nelson support their conjecture that in the thermodynamic limit, the spectrum of the non-Hermitian operator (1) is concentrated on the real axis for energies $\mathrm{Re}\,\varepsilon \geqslant \varepsilon_0$ and extends into the complex plane near the center of the band, $-\varepsilon_0 < \mathrm{Re}\,\varepsilon < \varepsilon_0$. They showed that the eigenstates with real eigenvalues in the region $|\mathrm{Re}\,\varepsilon| > \varepsilon_0$ are localized, while the eigenstates corresponding to complex eigenvalues are extended in space.[1] The picture that emerges from their analysis is that the energy ε_0, which separates the real and complex eigenvalues, serves as a ''mobility edge'' for the non-Hermitian problem (barring some unforseen ''conspiracy'' in which all extended eigenfunctions in some energy range $\varepsilon_0 - \epsilon < |\mathrm{Re}\,\varepsilon| < \varepsilon_0$ have real energy eigenvalues). As was shown by Hatano and Nelson, the value of the imaginary vector potential $\mathbf{h}$ where the eigenvalues of the non-Hermitian Hamiltonian start to pop out into the complex plane is related to the localization length $\xi(\varepsilon)$ of the problem in the absence of the imaginary vector potential,

$$|\mathbf{h}| = \xi(\varepsilon_0)^{-1}. \tag{2}$$

It was recently suggested by Hatano[10] and Gurarie and Zee[11] that the relationship (2) can be inverted, to use it as a method to extract the localization length of the Hermitian problem from the support of the spectrum of the non-Hermitian problem. In this way, knowledge of the support of the density of states (DOS) of the random Hamiltonian (1) as a function of the imaginary vector potential permits the calculation of the localization length as a function of energy in the Hermitian case $h \equiv |\mathbf{h}| = 0$.

To our knowledge, the (ensemble-averaged) non-Hermitian DOS for the random operator (1) is known only in zero[2] and one dimension.[3,7] In this paper we consider the DOS and its relation to the localization length for the so-called Lloyd model,[13] in which the random potentials $w_{\mathbf{j}}$ are independently distributed with the Cauchy distribution

$$P(w) = \frac{\gamma}{\pi} \frac{1}{\gamma^2 + w^2}. \tag{3}$$

It is believed that the choice of the Cauchy distribution (3) (instead of, say, a Gaussian one) does not modify the universal properties of the Anderson metal-insulator transition in dimension $d > 2$.[14]

As shown by Lloyd,[13] the advantage of this choice of the probability distribution is that the ensemble-averaged DOS of the Hermitian problem can be found exactly in any dimension d. It has been proposed in the literature that the DOS can also be obtained for arbitrary d in the case of the *non-Hermitian* Lloyd model.[6] According to relation (2), such a result would permit us to find the localization length of the Hermitian Lloyd model for arbitrary d. We show in this paper that the calculation in Ref. 6 does not give the correct DOS when applied to dimensions $d > 1$. We also discuss the difference between the Lloyd model in $d = 1$ and $d > 1$ and

illustrate for the one-dimensional Lloyd model how the localization length for the hermitian problem can be extracted from the support of the non-Hermitian DOS. Calculation of the localization length in the Lloyd model for $d \geqslant 2$, however, remains an open problem.

II. LOCALIZATION LENGTH FROM NON-HERMITIAN DOS

We first discuss how one arrives at relation (2) between the mobility edge ε_0, the imaginary vector potential $\mathbf{h}$, and the localization length $\xi(\varepsilon)$ of the Hermitian problem.

Hereto we consider, for a given realization of the random potential $w_{\mathbf{j}}$, a nondegenerate eigenvalue ε of the Hermitian operator $\mathcal{H}_0$ with periodic boundary conditions. (The subscript 0 indicates that the imaginary vector potential h is set to zero, i.e., that the Hamiltonian $\mathcal{H}_0$ is Hermitian.) Following Ref. 1, we assume that the corresponding eigenstate $\Psi_0(\mathbf{j})$ is localized by the random potential $w_{\mathbf{j}}$, i.e., $\Phi_0(\mathbf{j})$ is maximum at a site $\mathbf{m}$ and decays exponentially far away from $\mathbf{m}$:

$$\Phi_0(\mathbf{j}) \sim \exp\left[-\frac{|\mathbf{j}-\mathbf{m}|}{\xi(\varepsilon)} \right]. \tag{4}$$

By definition, the exponential decay length $\xi(\varepsilon)$ in Eq. (4) is the localization length. Let us now switch on an imaginary vector potential $i\mathbf{h}$. As long as $\mathbf{h}$ is sufficiently small, the wave function

$$\Psi(\mathbf{j}) = e^{\mathbf{h}\cdot\mathbf{j}} \Phi_0(\mathbf{j}) \tag{5}$$

is a very good approximation to the exact eigenfunction $\Phi_h(\mathbf{j})$ of $\mathcal{H}_h$, which adiabatically evolves from $\Phi_0(\mathbf{j})$ as $|\mathbf{h}|$ is increased. Although Ψ satisfies $\mathcal{H}_h \Psi = \varepsilon \Psi$, it is not an exact eigenfunction, because it violates the periodic boundary conditions. The error that one makes is of order $\exp\{[|\mathbf{h}| - 1/\xi(\varepsilon)]L\}$.

Hence, as long as

$$|\mathbf{h}| < \frac{1}{\xi(\varepsilon)} \tag{6}$$

the wave function Ψ will be a good approximation, and its energy ε will remain real and unshifted (up to an exponentially small correction, in principle).

When the magnitude of the imaginary vector potential is larger than the inverse localization length $1/\xi(\varepsilon)$, the wave function (4) will no longer be a good approximation. Both the eigenvalue and eigenfunction undergo a qualitative change reflecting the non-Hermiticity of the Hamiltonian. Hence, in the limit of an infinite system size, at $|\mathbf{h}| = 1/\xi(\varepsilon)$, a generic eigenvalue ε enters the complex plane with unit probability, resulting in relation (2).

To justify the inversion of Eq. (2) to find the localization length $\xi(\varepsilon)$ from the support of the spectrum of $\mathcal{H}_h$, we note that for $|\mathbf{h}| \approx 1/\xi(\varepsilon)$ eigenfunctions are strongly sensitive to the boundary conditions. This sensitivity to the boundary conditions causes the phenomenon of level attraction[4] with complex eigenvalues coalescing along curves in $d=1$, or in compact sets in $d \geqslant 2$ as the system size increases. The support of the DOS of $\mathcal{H}_h$ appears to be self-averaging in the thermodynamic limit, i.e., subject to decreasing fluctuations as the system size increases. Therefore, the mobility edge ε_0 is well defined for the non-Hermitian problem. It has thus been proposed in Refs. 10 and 11 to relate the real part ε_0 of the energy at which the first complex eigenvalue appears in the spectrum of $\mathcal{H}_h$ to the localization length defined by Eq. (2).

III. SINGLE-PARTICLE GREEN FUNCTION FOR THE NON-HERMITIAN HAMILTONIAN

The advantage of the Cauchy distributed disorder is that it allows the exact calculation of the (ensemble averaged) single-particle Green function. In this section we discuss whether a similar property exists for a non-Hermitian system with Cauchy disorder.

The DOS of the non-Hermitian Hamiltonian (1) is computed from the Green function or resolvent

$$G_h(z) = \frac{1}{N} \mathrm{Tr} \frac{1}{z - \mathcal{H}_h}, \tag{7}$$

where N is the total number of lattice sites (periodic boundary conditions are assumed). The DOS $\rho_h(z)$ in the complex plane reads[4]

$$\rho_h(z) = \frac{1}{\pi} \frac{\partial}{\partial z^*} G_h(z). \tag{8}$$

For a Hermitian system, when $G_0(z)$ is analytic for $\mathrm{Im}\, z \neq 0$ Eq. (8) reproduces the usual DOS concentrated on the real axis.

Let us now consider the ensemble average of the Green function G_0 and the DOS ρ_0. Lloyd[13] has shown that the average Green function $\langle G_0 \rangle$ of the Hermitian Hamiltonian $\mathcal{H}_0$ is related to the Green function K_0 of the nonrandom Hamiltonian $\mathcal{K}_0$ [see Eq. (1)],

$$\langle G_0(z) \rangle = \sum_{\pm} K_0(z \pm i\gamma)\, \theta(\pm \mathrm{Im}\, z), \tag{9a}$$

$$K_0(z) = \frac{1}{N} \mathrm{Tr} \frac{1}{z - \mathcal{K}_0}. \tag{9b}$$

The angular brackets denote an average over the random disorder potential $w_{\mathbf{j}}$, γ is the width of the distribution of $w_{\mathbf{j}}$ [see Eq. (3)], and $\theta(x) = 1$ (0) for $x > 0$ ($x < 0$). It follows that the average DOS can be expressed in terms of the nonrandom operator $\mathcal{K}_0$ only:

$$\langle \rho_0(z) \rangle = \frac{1}{2\pi i} [K_0(z - i\gamma) - K_0(z + i\gamma)]\, \delta(\mathrm{Im}\, z). \tag{10}$$

Does the Green function relation, Eq. (9), also hold for the Green function G_h of the non-Hermitian Hamiltonian $\mathcal{H}_h$? The answer is positive, provided

$$|\mathrm{Im}\, z| > \lambda, \tag{11}$$

where λ is the imaginary part of the eigenvalue of the non-Hermitian Hamiltonian $\mathcal{H}_h$ with the largest imaginary part,

$$\lambda = \sup \{\max_k \mathrm{Im}\, \varepsilon_k\}$$

$$= \max_k \mathrm{Im}\, \varepsilon_k'$$

$$= t\, \sinh(|\mathbf{h}||\mathbf{a}|) \quad \text{for } N \gg 1. \tag{12}$$

Here ε_k (ε_k'), $k=1,\ldots,N$ are the N eigenvalues of $\mathcal{H}_h$ ($\mathcal{K}_h$) for a given realization of the disorder and the supremum in the first line is taken with respect to all possible disorder realizations.

To see why this is so, we choose to express the Green function in terms of a replicated bosonic path integral,

$$G_h(z) = \frac{1}{N}\mathrm{Tr} \int \mathcal{D}[\phi_\alpha, \phi_\alpha^*] \phi_1^* \phi_1 e^{\pm i \int \phi_\alpha^* (z - H_h) \phi_\alpha}. \tag{13}$$

Here, α is a replica index. The sign in the exponent is fixed by the condition that the path integral be convergent. It is $+$ if $\mathrm{Im}\, z > \lambda$ and $-$ if $\mathrm{Im}\, z < -\lambda$. If neither of these two inequalities holds, i.e., if z lies inside the strip $|\mathrm{Im}\, z| < \lambda$, the path integral (13) cannot be constructed. Averaging Eq. (13) over the random potential $w_{\mathbf{j}}$ is easily done with the Cauchy distribution of Eq. (3) if $|\mathrm{Im}\, z| > \lambda$. In that case, the replicated integrand satisfies the condition of applicability of the Cauchy theorem after closing the contour of integration over w either in the upper half-plane or lower half-plane, depending on whether the sign $+$ or $-$ is chosen in Eq. (13). It is thus found that, for $|\mathrm{Im}\, z| > \lambda$,

$$\langle G_h(z) \rangle = \sum_{\pm} K_h(z \pm i\gamma)\, \theta(\pm \mathrm{Im}\, z), \quad |\mathrm{Im}\, z| > \lambda, \tag{14a}$$

$$K_h(z) = \frac{1}{N}\mathrm{Tr} \frac{1}{z - \mathcal{K}_h}. \tag{14b}$$

As in the Hermitian case, the right-hand side is expressed solely in terms of the nonrandom resolvent $\mathcal{K}_h(z)$.

Equation (14) first appeared in Ref. 6 but without the restriction $|\mathrm{Im}\, z| > \lambda$. The authors of Ref. 6 applied Eq. (14) to the strip $|\mathrm{Im}\, z| < \lambda$ to obtain the non-Hermitian DOS in the complex plane,

$$\langle \rho_h(z) \rangle = \sigma_h(z + i\gamma)\, \theta(\mathrm{Im}\, z) + \sigma_h(z - i\gamma)\, \theta(-\mathrm{Im}\, z) + \frac{1}{2\pi i}[K_h(z - i\gamma) - K_h(z + i\gamma)]\, \delta(\mathrm{Im}\, z), \tag{15}$$

where $\sigma_h(z) = \pi^{-1} \partial_{z^*} K_h(z)$ is the DOS of the non-Hermitian problem in the absence of disorder. The DOS (15) corresponds to a non-Hermitian DOS coalescing both on the real axis (second line) and on a compact set in the complex plane (first line).

The analytical continuation of Eq. (14) to the strip $|\mathrm{Im}\, z| < \lambda$ in order to find the DOS can be problematic since the Green function G_h is a nonanalytic function of z where the DOS is nonzero [compare with Eq. (8)]. It can only be justified in the thermodynamic limit in one dimension where the non-Hermitian spectrum collapses to a $1d$ curve. We return to this case in the next section. In all other cases Eq. (15) is incorrect. To illustrate where it might lead to, we consider Eq. (15) in the thermodynamic limit $N \to \infty$ and extract the length scale $l(\varepsilon) = 1/h$ by locating the edge $\varepsilon(h)$ at which the wings of the spectrum fork into the complex energy plane. Using the arguments leading to Eq. (2), one would identify $l(\varepsilon)$ with the localization length. The dependence of l on energy ε and dimensionality d is then given by

$$\cosh\left[\frac{1}{l(\varepsilon)}\right] = \begin{cases} \dfrac{1}{2t}[A_0(\varepsilon) + A_2(\varepsilon)], & |\varepsilon| > (d-1)t, \\ \dfrac{1}{t}\sqrt{\gamma^2 + t^2}, & |\varepsilon| \leqslant (d-1)t, \end{cases}$$

$$A_n(\varepsilon) = \sqrt{(|\varepsilon| - (d-n)t)^2 + \gamma^2}. \tag{16}$$

Note that the length scale $l(\varepsilon)$ given by Eq. (16) is finite for all energies and all dimensions. Hence, if Eq. (15) were true, one would conclude that, irrespective of dimensionality, all states are localized in the Lloyd model.[11] This conclusion is not surprising in $1d$ or $2d$. In fact, the length $l(\varepsilon)$ agrees with the localization length of the Lloyd model in $1d$.[15–17] In $2d$, however, $l(\varepsilon)$ is much smaller than the weak disorder estimate for a Gaussian disorder.[18] Moreover, in $d > 2$, such a conclusion contradicts the belief that the existence of large tails in the Cauchy distribution does not modify the universal properties of the Anderson metal-insulator transition.[14] We return to the issue of dimensions $d \geqslant 2$ and the interpretation of $l(\varepsilon)$ in Sec. V. The reason why the length scale $l(\varepsilon)$ cannot be interpreted as the localization length for $d > 1$ is that analytical continuation of Eq. (14) to the strip $\mathrm{Im}\, z < \lambda$ is in general invalid unless the DOS is supported on a one-dimensional curve, as in $1d$ or in the hermitian case $\mathbf{h} = 0$. In particular, we conclude that Eq. (15) does not yield the average DOS of the non-Hermitian extension of the Lloyd model in $d > 1$.

IV. NON-HERMITIAN DOS FOR ONE CHAIN

In view of the unreliability of analytic continuation of Eq. (14), it is important to compare Eq. (14) with what is known about the spectral properties of the non-Hermitian Hamiltonian $\mathcal{H}_h$ from other methods.

First, we note that in any dimension, analytic continuation of Eq. (14) is certainly wrong in a system of finite size: For any finite system and for any dimension the support of the averaged DOS $\langle \rho_h(z) \rangle$ occupies the entire strip in the complex energy plane that is excluded in Eq. (11). To see this, choose the realization $w_1 = \cdots = w_N = V$ with V an arbitrary real number. Equation (14), however, results in a DOS with a significantly smaller support: it is the DOS of the system without disorder shifted by an amount $\pm\gamma$ towards the real axis.[6]

What about the DOS in the thermodynamic limit $N \to \infty$? Let us first discuss the one-dimensional Lloyd model. A discussion of the case $d > 1$ is postponed to the next section. In one dimension, several independent approaches have been taken in the literature.[3,5,7] For the Lloyd model, Goldsheid and Khoruzhenko[7] have shown that the support of the spectrum of $\mathcal{H}_h$ is self-averaging in the thermodynamic limit and found a DOS that coincides with Eq. (15). Hence, the DOS obtained from Eq. (15) is correct in an infinite one-dimensional system, despite the flaws in its derivation. Start-

ing from this non-Hermitian DOS, one can use the arguments of Sec. II to identify $l(\varepsilon)$ with the localization length $\xi(\varepsilon)$ of the Lloyd model in one dimension.

We find it instructive to present an alternative derivation of Eq. (15) for weak disorder, using the approach of Ref. 3, where the support of the DOS was calculated for weak Gaussian disorder. In this approach, knowledge of the localization length is required to calculate the non-Hermitian DOS. In the absence of disorder, the energy spectrum is parameterized in terms of the (complex valued) wave numbers $s=2\pi/N+ih, \ldots, 2\pi+ih$ of the plane-wave states diagonalizing $\mathcal{K}_h$,

$$\varepsilon'(s)=-t\cos s. \tag{17}$$

In Ref. 3, a transfer-matrix approach was used to calculate the spectrum of $\mathcal{H}_h$ for weak non-Hermiticity and weak disorder to leading order in $1/N$. Weak non-Hermiticity means $|\mathbf{h}||\mathbf{a}|\ll 1$, whereas weak disorder amounts to $|\sin \mathrm{Re}\, s|\,\xi \gg 1$, where ξ is the localization length of the $1d$ Hermitian Lloyd model, see Eq. (16). To leading order in $1/N$, it was found that[3]

$$|\mathrm{Im}\, s|=|h|-\xi^{-1} \tag{18a}$$

$$=|h|-\frac{\gamma}{\sqrt{t^2-(\mathrm{Re}\,\varepsilon)^2}}+\mathcal{O}(\gamma^2/t^2). \tag{18b}$$

Here, we have expanded Eq. (16) to lowest nontrivial order in ξ^{-1} and γ/t. Hence, we find that the support of the spectrum is given by

$$\mathrm{Im}\,\varepsilon=-t\cos(\mathrm{Re}\, s)\sinh(\mathrm{Im}\, s)$$

$$=\pm h\sqrt{t^2-(\mathrm{Re}\,\varepsilon)^2}\mp\gamma, \tag{19}$$

in agreement with Eq. (15) when $d=1$. The variance of $\mathrm{Im}\,\varepsilon$ vanishes like N^{-1} in the thermodynamic limit. Note that Eq. (19) relies on very general properties of one-dimensional disordered systems through Eq. (18a) and only on the specificities of the Hermitian Lloyd model through Eq. (18b). One dimension is very special in that localization length and DOS are closely related.[16] The situation is more intricate in higher dimensions where a new length scale, the mean free path, appears besides the localization length.

For small γ, the DOS in the Lloyd model is qualitatively different from the non-Hermitian DOS in the presence of Gaussian distributed disorder:[3,7] In the latter case, the uniform shift γ in Eq. (19) should be replaced by the nonuniform and much smaller shift $\gamma^2/\sqrt{t^2-(\mathrm{Re}\,\varepsilon)^2}$, where γ^2 is the variance of the Gaussian distribution. To explain the difference, we compare the γ dependences of the localization lengths at the center of the band for both disorder distributions. For the Cauchy distribution we have $\xi(0)=t/\gamma$, while $\xi(0)=t^2/\gamma^2$ for a Gaussian distribution.[19] Using the relation (18) between localization length and non-Hermitian DOS, such a difference is directly carried over to the DOS. In fact, the mechanism of localization in $1d$ is very different for Cauchy disorder compared to that of Gaussian disorder: For a Cauchy distribution, the dependence on the disorder strength γ of the inverse localization length $\xi^{-1}\sim\gamma/t$ follows immediately from estimating the probability that the disorder potential $w_{\mathbf{j}}$ at an arbitrary site be larger than the bandwidth t, in which case the chain is classically broken. Thus, localization in the one-dimensional Hermitian Lloyd model is not caused by quantum interferences effects, but rather by wave functions accomodating to large fluctuations in the disorder by vanishing locally. In contrast, in the case of Gaussian distributed disorder, localization is entirely due to quantum interference.

V. HIGHER DIMENSIONS: MEAN FREE PATH

In Sec. III we have shown that analytical continuation of Eq. (14) into the strip $|\mathrm{Im}\, z|\leqslant\lambda$ yields a length scale $l(\varepsilon)$ that remains finite irrespective of dimensionality. If this analytical continuation were justified, the arguments of Sec. II would allow us to identify $l(\varepsilon)$ as the localization length of the Lloyd model. Then the Lloyd model would not display a metal-insulator transition irrespective of dimensionality. However, as we have seen, in general, Eq. (14) cannot be applied inside the strip $|\mathrm{Im}\, z|\leqslant\lambda$. This does not mean that the length scale $l(\varepsilon)$ obtained from Eq. (14) is entirely meaningless. In this section, we compare the length scale $l(\varepsilon)$ defined by Eq. (16) with other length scales that appeared in previous studies of the Lloyd model,[20–23] and find that, in the limit of weak disorder, $l(\varepsilon)$ is to be interpreted as the mean free path, rather than the localization length. [For strong disorder there is no distinction between mean free path and localization length.] The fact that analytical continuation of Eq. (14) yields the correct DOS and localization length in one dimension is thus simply understood as the fact that localization length and mean free path coincide in $d=1$.

To see what is the correct interpretation of the length scale $l(\varepsilon)$ defined in Eq. (16), we go back to the work of Johnston and Kunz,[20] who considered the quantity

$$\frac{1}{l'_r(z)}=-r^{-1}\langle\ln|\mathcal{G}_{j,j+r}(z)|\rangle, \tag{20}$$

where $\mathcal{G}(z)=(z-\mathcal{H}_0)^{-1}$. They found an expression for Eq. (20) valid at least for sufficiently large $|\mathrm{Im}\, z|$, at fixed r and system size. Johnston and Kunz also found that if their expression for $l'_r(z)$ is analytically continued to $z\to\varepsilon$ on the real axis, and the limits of infinite system size and separation r are then taken, one obtains a result $l'(\varepsilon)$ which coincides with Eq. (16). Finally, Johnston and Kunz hypothesized that $l'(\varepsilon)$ should coincide with the localization length, which one may define by

$$\frac{1}{\xi(\varepsilon)}=-\Big\langle\lim_{r\to\infty} r^{-1}\lim_{z\to\varepsilon}\ln|\mathcal{G}_{j,j+r}(z)|\Big\rangle. \tag{21}$$

[We remark that the localization length $\xi(\varepsilon)$ is a self-averaging quantity, so that the ensemble average in Eq. (21) is not necessary.] However, as was noticed by Thouless,[21] MacKinnon,[22] and by Rodrigues, Pastawski, and Weisz,[23] this hypothesis is not correct. Moreover, Rodrigues, Pastawski, and Weisz showed that the result (16) for $l'(\varepsilon)$ coincides with the length scale $l''(\varepsilon)$ defined by

$$\frac{1}{l''(\varepsilon)} = -\lim_{r\to\infty} r^{-1} \ln \left| \lim_{z\to\varepsilon} \langle \mathcal{G}_{j,j+r}(z) \rangle \right|. \tag{22}$$

This length scale is naturally interpreted as the scale on which the phase of wave functions is randomized, rather than the length scale on which the amplitude decays exponentially. The length $l''(\varepsilon)$ can also be interpreted as the mean free path of a particle for weak scattering. Unfortunately, to find the true localization length $\xi(\varepsilon)$, defined by Eq. (21), remains an open problem in the Lloyd model in $d>1$.

A more intuitive picture of what is going on in higher dimensions can be obtained along the lines of the last paragraph of Sec. IV. The length scale $l(\varepsilon)$ of Eq. (16) behaves like

$$l(\varepsilon) = \frac{t}{\gamma} + O\left(\frac{\gamma^2}{t^2}\right), \quad |\varepsilon| < (d-1)t, \tag{23}$$

for all dimensions and weak disorder. This implies that the calculation scheme of Ref. 6, i.e., application of Eq. (14) to the strip $|\mathrm{Im}\, z| < \lambda$, is predicated on the same mechanism as in $1d$, namely, the removal of a site with probability γ/t. In contrast to $1d$, the removal of a site does not preclude propagation in $d \geqslant 2$. Instead, the removal of sites leads to the usual impurity scattering, and we see from Eq. (23) that $l(\varepsilon)$ has the semiclassical interpretation of an average length for free propagation between two impurities, i.e., a semiclassical mean free path.

If we are to interpret Eq. (16) for weak disorder as the mean free path, we have no reason to exclude that one parameter scaling[18] applies to $\mathcal{H}_h$ in the case $\mathbf{h}=0$. It is widely believed that propagation becomes diffusive for a window of length scales beyond the mean free path $l(\varepsilon)$. The upper length scale for diffusive propagation is, by definition, the localization length. Thus, for the Lloyd model in the weak disorder limit we are led to expect the same localization properties as for weak Gaussian distributed disorder: (i) localization at all energies if $d=2$, (ii) existence of a diffusive regime for a window of energies centered around $\varepsilon=0$ if $d=3$. This expectation is confirmed by the localization properties[24] of a caricature of the Lloyd model defined by a random hopping tight-binding Hamiltonian whereby the hopping amplitude takes the value t with probability $1-\gamma/t$ and vanishes with probability γ/t.

VI. CONCLUSION

In this paper we have outlined that the localization length of a particle moving in a random potential can be obtained from the averaged DOS of a non-Hermitian particle in a random potential. Non-Hermitian densities of states have only been calculated in zero and one dimensions. The calculation of the non-Hermitian DOS in closed form in more than one dimension thus remains an open problem. We expect that the non-Hermitian trick should provide an alternative to numerical calculations of the localization length relying on transfer-matrix approaches in strip geometries.

ACKNOWLEDGMENTS

We are indebted to J. T. Chalker and D. R. Nelson for useful discussions. We acknowledge the support by the NSF under Grant Nos. DMR 94-16910, DMR 96-30064, DMR 97-14725, and PHY 94-07194, and from the Swiss Nationalfonds (C.M.).

[1] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); Phys. Rev. B **56**, 8651 (1997).
[2] K. B. Efetov, Phys. Rev. Lett. **79**, 491 (1997); Phys. Rev. B **56**, 9630 (1997).
[3] P. W. Brouwer, P. G. Silvestrov, and C. W. J. Beenakker, Phys. Rev. B **56**, R4333 (1997).
[4] J. Feinberg and A. Zee, Nucl. Phys. B **504**, 579 (1997); cond-mat/9706218 (unpublished); cond-mat/9710040 (unpublished).
[5] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, cond-mat/9705098 (unpublished); Acta Phys. Pol. B **28**, 2949 (1997).
[6] E. Brézin and A. Zee, Nucl. Phys. B **509**, 599 (1998); A. Zee, Physica A **259**, 300 (1998).
[7] I. Ya. Goldsheid and B. A. Khoruzhenko, Phys. Rev. Lett. **80**, 2897 (1998).
[8] D. R. Nelson and N. M. Shnerb, Phys. Rev. E **58**, 1383 (1998); N. M. Shnerb and D. R. Nelson, Phys. Rev. Lett. **80**, 5172 (1998).
[9] C. Mudry, B. D. Simons, and A. Altland, Phys. Rev. Lett. **80**, 4257 (1998).
[10] N. Hatano, Physica A **254**, 317 (1998).
[11] V. Gurarie and A. Zee, cond-mat/9802042 (unpublished).
[12] P. G. Silvestrov, cond-mat/9802219 (unpublished).
[13] P. Lloyd, J. Phys. C **2**, 1717 (1969).
[14] J. T. Edwards and D. J. Thouless, J. Phys. C **4**, 453 (1971).
[15] T. Hirota and K. Ishii, Prog. Theor. Phys. **45**, 1713 (1971); K. Ishii, Prog. Theor. Phys. Suppl. **53**, 77 (1973).
[16] D. J. Thouless, J. Phys. C **5**, 77 (1972).
[17] T. P. Eggarter, J. Math. Phys. **15**, 7 (1974).
[18] For a review, see P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[19] O. N. Dorokhov, Phys. Rev. B **37**, 10 526 (1988).
[20] R. Johnston and H. Kunz, J. Phys. C **16**, 4565 (1983).
[21] D. J. Thouless, J. Phys. C **16**, L929 (1983).
[22] A. MacKinnon, J. Phys. C **17**, L289 (1984).
[23] D. E. Rodrigues, H. M. Pastawski, and J. F. Weisz, Phys. Rev. B **34**, 8545 (1986).
[24] R. Raghavan and D. C. Mattis, Phys. Rev. B **23**, 4791 (1981).