Faraday effect in composites

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We study the Faraday effect in a binary composite consisting of a dielectric matrix with metallic inclusions. We first use the replica trick together with a variational method in order to compute the effective permittivity tensor in the quasistatic limit of this composite in a static magnetic field. In order to find scaling exponents near the percolation threshold p_c, we use a high contrast or low-frequency expansion combined with scaling. The results of the two methods are in agreement, and predict that near p_c (and below, that is, in the dielectric region), the Faraday effect is greatly enhanced. [S0163-1829(98)01644-0]

I. INTRODUCTION

We will concentrate on the problem of the effective permittivity tensor of a binary mixture in a static magnetic field. Since the Faraday effect is usually very weak in dielectrics, we study a mixture made of a dielectric host matrix with metallic inclusions. In such a system, the Hall effect in the inclusions is expected to induce a measurable Faraday effect in the composite dielectric even when this effect is totally negligible in the pure dielectric host. We will study the quasistatic limit, the wavelength of the incoming wave being much larger than the typical inhomogeneity length, which is, for example, the size of the metallic grains. Therefore, the composite medium can be seen as quasihomogeneous, and the equivalent homogeneous material is called the effective-medium. Determining the effective-medium properties of disordered materials such as composites or suspensions is a difficult problem, and one has often to resort to perturbative methods (low field, low density, or low contrast) which cannot be applied for a high magnetic field, for instance. Here we propose two different approaches to this problem. The first is based on the use of the replica method together with a variational principle. This treatment possesses the advantage that it is nonperturbative, and may thus be useful for strong disorder or strong fields. This method was successfully applied to different problems such as the random resistor network problem,[1] and the Hall effect in composites.[2] Moreover, it has been shown that it can give reasonable values for the permittivity tensor if the system is not too close to the percolation threshold p_c. However, the critical exponents are not reproduced correctly, and one usually obtains mean-field or effective-medium-approximation exponents. In order to present an alternative discussion of this problem, and obtain correct exponents near p_c, we introduce a high contrast expansion. This is essentially an expansion in powers of the ratio of resistivities or permittivities of the two components, which can be made very small by making the frequency of the incoming wave very small. This expansion can be used for weak magnetic fields as well as strong magnetic fields. In order to discuss the critical properties near p_c, we apply some scaling *Ansätze* to that expansion, which are based upon previous discussions of dc magnetotransport.[3,4]

The Hall effect in percolating composites has been studied using a number of different methods which enabled the critical behavior to be determined quite reliably.[5–7,3,2] However, until now the Faraday effect in metal-dielectric composites was only discussed using a Clausius-Mossotti-type approximation,[8] which is good for dilute systems, and a Bruggeman-type self-consistent effective-medium approximation (SEMA),[9] which exhibits a percolation threshold but with incorrect values of the critical exponents. In the present study we employ different approaches (see above). Both of our approaches are not limited to dilute systems, and one, the high contrast expansion together with scaling *Ansätze*, is expected to lead to reliable results for the critical behavior near p_c.

Let us first recall some facts about the Faraday effect. When an isotropic material is subjected to a static and uniform magnetic field **B** directed along the z axis, it can be described by the permittivity tensor

$$\hat{\varepsilon}=\begin{pmatrix} \varepsilon & i\tilde{\varepsilon} & 0 \\ -i\tilde{\varepsilon} & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}, \tag{1.1}$$

where $\tilde{\varepsilon}$ must depend upon **B** (see, e.g., Ref. 10). In a homogeneous medium, the dispersion equation gives rise to two solutions, which are the left and right circularly polarized waves with two different refractive indices,

$$n_{\pm}=\sqrt{\varepsilon\pm\tilde{\varepsilon}}, \tag{1.2}$$

where ε and $\tilde{\varepsilon}$ are positive and real and $\varepsilon>\tilde{\varepsilon}$, which ensures that the wave is undamped. If a linearly polarized wave of frequency ω propagates over a distance L through this medium, the polarization plane will rotate (the so-called Faraday effect) by an angle

$$\theta=\frac{\omega}{c}L(n_+-n_-), \tag{1.3}$$

where c is the light speed in vacuum. Usually the Faraday effect is weak and $\tilde{\varepsilon}$ is very small compared to ε, and the rotation angle is therefore approximately given by

$$\theta \simeq \frac{\omega}{c} L \frac{\tilde{\varepsilon}}{\sqrt{\varepsilon}}. \tag{1.4}$$

We will also use the Faraday coefficient, which is defined by $\mathcal{F} \equiv \tilde{\varepsilon}/\sqrt{\varepsilon}$. We note here that the Faraday effect is usually weak, and $\mathcal{F}$ ranges from 10^{-6} in dielectrics to 10^{-2} for thin films of metallic iron.

We will study the case where the material is a random binary composite medium made of a dielectric host with metallic inclusions, and where a static uniform magnetic field **B** is applied along the z axis. We suppose that the medium has a position-dependent permittivity tensor $\hat{\varepsilon}(\mathbf{r})$ which is an independent random variable at each point **r**, distributed according to the probability density

$$p(\hat{\varepsilon}) = p\,\delta(\hat{\varepsilon} - \hat{\varepsilon}_M) + (1-p)\,\delta(\hat{\varepsilon} - \hat{\varepsilon}_I). \tag{1.5}$$

Let us note that, in real materials, the grains have finite sizes and that in a finite-frequency calculation one should take this into account. However, in the quasistatic limit, the grain sizes are irrelevant and this simplified characterization of the disorder [Eq. (1.5)] is justified.

Equivalently, the local resistivity tensor $\hat{\rho}(\mathbf{r})$ (related to the permittivity tensor $\hat{\varepsilon}$ by $\hat{\rho} = 4\pi/i\omega\hat{\varepsilon}$) is a step function that is equal to $\hat{\rho}_M$ inside the metal and to $\hat{\rho}_I$ inside the dielectric component. It will be represented, with the help of the appropriate characteristic functions $\theta_M(\mathbf{r})$ and $\theta_I(\mathbf{r})$, as

$$\hat{\rho}(\mathbf{r}) = \hat{\rho}_M \theta_M(\mathbf{r}) + \hat{\rho}_I \theta_I(\mathbf{r}), \tag{1.6}$$

$$\theta_M(\mathbf{r}) = 1 - \theta_I(\mathbf{r}) = \begin{cases} 1 & \text{for } \mathbf{r} \text{ inside the metal} \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

In the dielectric component, the permittivity is taken to be

$$\hat{\varepsilon}_I = \varepsilon_I \hat{I}, \tag{1.8}$$

where ε_I, the dielectric constant of the host, is a real scalar quantity and is independent of **B**. The metallic component is nonpercolating, and is characterized by a free-electron-like resistivity tensor in the presence of a magnetic field $\mathbf{B} \| z$ or, equivalently, by its permittivity tensor

$$\hat{\varepsilon}_M = \varepsilon_M \begin{pmatrix} \frac{1}{1+H^2} & \frac{H}{1+H^2} & 0 \\ -\frac{H}{1+H^2} & \frac{1}{1+H^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H \equiv \omega_c \tau \propto |\mathbf{B}|, \tag{1.9}$$

where ω_c is the cyclotron frequency, τ is the conductivity relaxation time, and $\varepsilon_M = 4\pi\sigma_M/i\omega$ is purely imaginary and independent of **B** (σ_M is the conductivity of the metallic component). We assume that this form continues to be valid even at finite frequencies. This probably means that the entire subsequent discussion will not be valid for optical frequencies in the visible range. But it will be relevant for frequencies up to, and including, the microwave regime.

These assumptions mean that the host exhibits no intrinsic Faraday effect, and the metallic component has no intrinsic magnetoresistance, only a Hall effect. In terms of resistivity, the metallic component is characterized by a free-electron-like resistivity tensor obtained by inverting $\hat{\varepsilon}_M$,

$$\hat{\rho}_M = \rho_M \begin{pmatrix} 1 & H & 0 \\ -H & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.10}$$

and the impedance of the dielectric host is an imaginary scalar tensor

$$\hat{\rho}_I = \rho_I \hat{I}, \quad \rho_I = \frac{4\pi}{i\omega\varepsilon_I}. \tag{1.11}$$

We will assume that the quasistatic approximation can be used, that is, both wavelength and skin depth are large compared to the sizes of metallic inclusions.

We define the bulk effective permittivity tensor $\hat{\varepsilon}_e$ of the medium by the relation

$$\langle \hat{\varepsilon}(r)\mathbf{E}(r) \rangle = \hat{\varepsilon}_e \langle \mathbf{E}(r) \rangle, \tag{1.12}$$

where **E** is the electric field, and where the brackets denote a quenched average over the probability distribution given by Eq. (1.5), or equivalently a spatial average over the volume of the sample. The effective medium will be homogeneous and isotropic, and we expect an effective permittivity tensor of the form

$$\hat{\varepsilon}_e = \begin{pmatrix} \varepsilon_e & i\tilde{\varepsilon}_e & 0 \\ -i\tilde{\varepsilon}_e & \varepsilon_e & 0 \\ 0 & 0 & \varepsilon_{ze} \end{pmatrix}. \tag{1.13}$$

We can also define the bulk effective complex resistivity tensor by

$$\hat{\rho}_e \cdot \langle \mathbf{J} \rangle \equiv \langle \hat{\rho}\mathbf{J} \rangle, \tag{1.14}$$

where **J** is the local current density.

In order to evaluate the effective properties of the heterogeneous medium, we can proceed in different ways. The first one (Sec. II) relies on the observation that the effective permittivity tensor can be exactly related to the inverse of a random operator $\hat{M}$. The problem is thus reduced to the calculation of $\langle \hat{M}^{-1} \rangle$, and we will use replicas together with a variational approximation to evaluate this quantity. Let us note here that this is in principle equivalent to computing $\hat{\varepsilon}_e$ using Eq. (1.12) or $\hat{\rho}_e$ using Eq. (1.14), since the product of these tensors is proportional to the unit tensor $\hat{I}$. However, since we use approximations, the two procedures are not necessarily equivalent. In fact, it has been shown[11,12] that, for the variational approximation, computing $\hat{\varepsilon}_e$ or $\hat{\rho}_e$ leads to different results, and that it is more reliable to calculate the permittivity or the conductivity than it is to calculate the resistivity.

Since the variational method is nonperturbative, it possesses the advantage that it is reliable in the whole range of concentration and for any strength of the disorder. However, near the percolation threshold p_c, this method leads to mean-field exponents which are usually not accurate especially in three dimensions ($d=3$). Moreover, it is difficult to control the quality of the variational approximation.

In order to describe the Faraday effect near p_c, we will use a second approach, which relies on the fact that if ω is small enough, we have two small parameters ρ_M/ρ_I and $H\rho_M/\rho_I$. The local electric field $\mathbf{E}(\mathbf{r})$ and current density $\mathbf{J}(\mathbf{r})$ can be found by defining a vector potential $\mathbf{A}(\mathbf{r})$ such that

$$\mathbf{J}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r}), \tag{1.15}$$

and then solving the equation

$$\nabla\times\mathbf{E}=0 \tag{1.16}$$

using the constitutive relation $\mathbf{E}(\mathbf{r})=\hat{\rho}(\mathbf{r})\mathbf{J}(\mathbf{r})$. It is of course out of the question to solve this equation exactly; therefore we will expand $\mathbf{A}$ in powers of the contrast between the two components, i.e., the resistivity ratio ρ_M/ρ_I. This approach is valid only in the quasistatic regime. Besides confirming results obtained by means of the replicas, when combined with scaling this allows us to predict the behavior of the medium near the percolation threshold.

One of the conclusions of the present work is that the Faraday effect can be greatly enhanced in a metal-dielectric composite just below the percolation threshold. This differs from the enhancement found in ferrofluids:[13,14] In those systems it arises due to the presence of small ferromagnetic particles in suspension. By contrast, in the system we are considering none of the components are assumed to have magnetic properties. The magnetic field influences the response only through the Hall effect in the metal component.

The rest of this paper is organized as follows. In Sec. II we apply the replica method to the Faraday effect in random composites. Results are obtained for the scaling behaviors near the percolation threshold. In Sec. III we present the high-contrast or low-frequency expansion. Section IV presents a scaling theory based upon the approaches described in the two previous sections. Section V summarizes the main conclusions from this work.

II. REPLICA APPROACH

In this section we present the replica approach. Here we compute the effective permittivity tensor of a binary mixture, where the tensor $\hat{\varepsilon}$ is a random variable equal to $\hat{\varepsilon}_1$ with probability p and to $\hat{\varepsilon}_2$ with probability $q=1-p$, each component having a permittivity of the form given in Eq. (1.1). The calculation is essentially the same as for the Hall effect,[2] and we give the main steps of the derivation in Appendix A. We obtain the following equations: The first one gives the longitudinal effective permittivity

$$\varepsilon_{ze}=\int_0^\infty du\, e^{-u}\frac{\langle\varepsilon_z e^{-u\varepsilon_z/3\varepsilon_{ze}}\rangle}{\langle e^{-u\varepsilon_z/3\varepsilon_{ze}}\rangle}, \tag{2.1}$$

where the brackets still denote an average over Eq. (1.5), and we also obtain two coupled equations for the transverse permittivities $\varepsilon_e,\tilde{\varepsilon}_e$:

$$1=-\tfrac{3}{2}\int_0^\infty du\, e^{-u}\{\ln\langle e^{-uX/3\varepsilon_e}\rangle+\ln\langle e^{-uY/3\varepsilon_e}\rangle\}, \tag{2.2}$$

$$\tilde{\varepsilon}_e=\tfrac{3}{2}\varepsilon_e\int_0^\infty du\, e^{-u}\{\ln\langle e^{-uX/3\varepsilon_e}\rangle-\ln\langle e^{-uY/3\varepsilon_e}\rangle\}, \tag{2.3}$$

where $X\equiv\varepsilon-\tilde{\varepsilon}$ and $Y\equiv\varepsilon+\tilde{\varepsilon}$. Equation (2.2) determines ε_e self-consistently, while Eq. (2.3) determines $\tilde{\varepsilon}_e$ after ε_e is known.

Equation (2.1) is the same as that obtained in Refs. 1 and 13 for the bulk effective conductivity of a binary mixture of zero-field conductivities σ_{z1} with concentration p, and σ_{z2} with concentration $1-p$. That equation was studied thoroughly in Ref. 14; it displays a percolation threshold at $p_c=1-e^{-1/3}\simeq 0.28$ for $d=3$.

The Faraday effect is contained in Eqs. (2.2) and (2.3). We first note that the percolation threshold is independent of the magnetic field. Let us note here that the percolation threshold, which is a geometrical quantity, is still meaningful here since we are working in the quasistatic limit.

By expanding the logarithms (these expansions are valid for $p<\frac{1}{2}$), we obtain, from Eq. (2.2),

$$1=-3\ln q+\frac{\varepsilon_2}{\varepsilon_e}-\frac{3}{2}\sum_{n\geqslant 1}\frac{\lambda^n}{n}(-)^{n+1}\left(\frac{1}{1+n\dfrac{(X_1-X_2)}{3\varepsilon_e}}+\frac{1}{1+n\dfrac{(Y_1-Y_2)}{3\varepsilon_e}}\right), \tag{2.4}$$

and, from Eq. (2.3),

$$\tilde{\varepsilon}_e=\tilde{\varepsilon}_2+(\tilde{\varepsilon}_1-\tilde{\varepsilon}_2)\sum_{n\geqslant 1}\lambda^n(-)^{n+1}\frac{1}{\left[1+\dfrac{n}{3\varepsilon_e}(X_1-X_2)\right]\left[1+\dfrac{n}{3\varepsilon_e}(Y_1-Y_2)\right]}, \tag{2.5}$$

where $\lambda=p/q$, with p the fraction of component 1 and $q=1-p$ the fraction of component 2. By some simple algebraic manipulations, these equations can be rewritten as

$$0=\frac{1}{3}+\ln q-\frac{\alpha_2}{3\alpha_e}+\sum_{n\geqslant 1}\frac{\lambda^n}{n}(-)^{n+1}\frac{1+\dfrac{n(\alpha_1-\alpha_2)}{3\alpha_e}}{\left(1+\dfrac{n(\alpha_1-\alpha_2)}{3\alpha_e}\right)^2+\left(\dfrac{n(\beta_1-\beta_2)}{3\alpha_e}\right)^2} \tag{2.6}$$

and

$$\beta_e=\beta_2+(\beta_1-\beta_2)\sum_{n\geqslant 1}\frac{\lambda^n}{n}(-)^{n+1}\frac{1}{\left(1+\dfrac{n(\alpha_1-\alpha_2)}{3\alpha_e}\right)^2+\left(\dfrac{n(\beta_1-\beta_2)}{3\alpha_e}\right)^2}, \tag{2.7}$$

where α_i and β_i describe, respectively, the diagonal and the off-diagonal transverse elements of the conductivity tensor of component i $[\alpha_i=\sigma_i/(1+H_i^2),\ \beta_i=\sigma_i H_i/(1+H_i^2)]$, and where σ_i is the zero-field conductivity of component i. The quantity H_i is the Hall-to-Ohmic resistivity ratio in component i: it is proportional to the magnetic-field strength $|\mathbf{B}|$—see Eq. (1.9). The quantities α_e and β_e are the effective coefficients of the composite. Equations (2.6) and (2.7) are identical to the equations obtained in the case of magnetotransport.[2] We have thus shown that the results obtained in that case can be continued analytically to complex values of α and β: We can go from the Hall effect to the Faraday effect by changing α to ε and β into $i\tilde{\varepsilon}$.

We now study these equations in the case of a metal-dielectric mixture (the metal will be component 1 and the dielectric component 2). We assume here that the Hall effect in the dielectric is very weak, so that $\alpha_2/\alpha_1\ll 1$ or $H_2\simeq 0$, and we use H instead of H_1. We also assume that $\alpha_e\ll\alpha_1$.

We will first study the weak-field regime $H\ll 1$; in this case $\beta_i\ll\alpha_i$. Near the percolation threshold $p_c=1-e^{-1/3}$ of component 1 ($\Delta p=p-p_c\ll 1$), we find that α_e has the scaling behavior $\alpha_e\simeq\alpha_1|\Delta p|\phi[(\alpha_2/\alpha_1)/\Delta p^2]$, where the scaling function $\phi(z)$ satisfies

$$0\simeq-\frac{\Delta p}{q_c}-\frac{\alpha_2}{3\alpha_1|\Delta p|\phi}+|\Delta p|\phi A, \tag{2.8}$$

where $A=3\Sigma_{n\geqslant 1}(\lambda_c^n/n^2)(-)^{n+1}$ (with $\lambda_c=p_c/1-p_c$). We thus obtain the following equation for ϕ

$$A\phi^2-\frac{\epsilon\phi}{q_c}-\frac{z}{3}=0, \tag{2.9}$$

where $\epsilon=+1$ if $p>p_c$ and $\epsilon=-1$ for $p<p_c$, and where $z=\alpha_2/\alpha_1\Delta p^2$. The solution of this equation is $\phi=(1/2A)[(\epsilon/q_c)+\sqrt{(1/q_c^2)+(4Az/3)}]$, which for small z becomes (up to a constant factor $q_c/3$)

$$\phi(z)\propto\begin{cases} z-\dfrac{A}{3}q_c^2z^2, & p<p_c \\ \text{const}, & p>p_c. \end{cases} \tag{2.10}$$

We can now easily obtain the behavior of β_e from Eq. (2.7) (for $p<p_c$),

$$\beta_e\propto\frac{\beta_1}{\alpha_1^2}\alpha_e^2\propto\beta_1\left(\frac{\alpha_2/\alpha_1}{|\Delta p|}\right)^2, \tag{2.11}$$

which is proportional to H.

We now consider the regime $H\gg 1$ for p below p_c which is the interesting one for the following. In this regime $\beta_i\gg\alpha_i$, and Eq. (2.6) then reads

$$0\simeq-\frac{\Delta p}{q_c}-\frac{\alpha_2}{3\alpha_e}+\frac{\alpha_1\alpha_e}{\beta_1^2}A, \tag{2.12}$$

from which we can deduce that the scaling behavior of α_e is of the form $\alpha_e\simeq(\beta_1^2/\alpha_1)|\Delta p|\tilde{\phi}(\alpha_2\alpha_1/\beta_1^2\Delta p^2)$, where the scaling function $\tilde{\phi}(z)$ has a behavior like that of $\phi(z)$, up to the same constant factor $q_c/3$ [see (2.10)]:

$$\tilde{\phi}(z)\propto\begin{cases} z-\dfrac{A}{3}q_c^2z^2, & p<p_c \\ \text{const}, & p>p_c. \end{cases} \tag{2.13}$$

The behavior of β_e is *a priori* different, since now $\beta_1\propto 1/H\gg\alpha_1\propto 1/H^2$. The equation for β_e then becomes

$$\beta_e\simeq\beta_1\sum_{n\geqslant 1}\frac{\lambda_c^n}{n^3}(-)^{n+1}\frac{1}{(\beta_1/3\alpha_e)^2}, \tag{2.14}$$

which for $p<p_c$ leads to the behavior

$$\beta_e\propto\frac{\alpha_2^2}{\Delta p^2}\frac{1}{\beta_1}, \tag{2.15}$$

which is again proportional to H. We will discuss the physical consequences relevant to the Faraday effect in Sec. IV.

III. LOW FREQUENCY OR HIGH CONTRAST EXPANSION

In this section we derive an expansion for $\hat{\varepsilon}_e$ in powers of the complex resistivity ratio ρ_M/ρ_I. While expansions in powers of volume fraction or low contrast expansions [i.e., expansion in powers of $(\rho_I-\rho_M)/\rho_I$], abound in the literature,[15] we believe this is the first example of a high contrast expansion. Throughout this section we will assume that ω is small enough so that both $\rho_I\gg\rho_M$ and $\rho_I\gg H\rho_M$, and that the quasistatic limit is valid.

Under these assumptions, the local electric field $\mathbf{E}(\mathbf{r})$ and current density $\mathbf{J}(\mathbf{r})$ can be found by defining a vector potential $\mathbf{A}(\mathbf{r})$, such that

$$\mathbf{J}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r}), \tag{3.1}$$

and solving the equation

$$0=\nabla\times\mathbf{E}=\nabla\times\{\hat{\rho}(\mathbf{r})\cdot[\nabla\times\mathbf{A}(\mathbf{r})]\} \tag{3.2}$$

along with appropriate boundary conditions on $\mathbf{n}\times\mathbf{A}(\mathbf{r})$ at the system surface ($\mathbf{n}$ is the unit normal vector to that surface).

We recall that the local resistivity tensor $\hat{\rho}(\mathbf{r})$ is a step function equal to $\hat{\rho}_M$ inside the metal and to $\hat{\rho}_I$ inside the dielectric component, and that it can be represented with the help of the appropriate characteristic functions as in Eq. (1.6).

In connection with Eq. (3.2) it is useful to define a Green tensor $\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')$ by the equations

$$\{\nabla\times[\hat{\rho}\cdot(\nabla\times G^{(\rho)}_{\cdot\beta})]\}_\alpha-k^2G^{(\rho)}_{\alpha\beta}=\delta_{\alpha\beta}\delta^3(\mathbf{r}-\mathbf{r}') \quad \text{for any } \mathbf{r},\mathbf{r}', \tag{3.3}$$

$$\mathbf{n}\times G^{(\rho)}_{\cdot\beta}=0 \quad \text{for } \mathbf{r} \text{ at the system surface.} \tag{3.4}$$

This tensor can be used to solve Eq. (3.2):

$$\mathbf{A}(\mathbf{r})=\mathbf{A}^{(0)}(\mathbf{r})-\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')]\times\hat{\rho}(\mathbf{r}')\cdot[\nabla'\times\mathbf{A}^{(0)}(\mathbf{r}')], \tag{3.5}$$

where $\mathbf{A}^{(0)}(\mathbf{r})$ is a vector field that satisfies the same boundary conditions as $\mathbf{A}(\mathbf{r})$, but is otherwise arbitrary. Note that we need to use the limit $k\to0$ of $\nabla'\times\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')$ here: We could not take that limit in Eq. (3.3), because then the equations for $\hat{G}^{(\rho)}$ would have no solution (see Appendix B for a discussion of this point).

Since we intend to expand $\mathbf{A}$ in powers of ρ_M/ρ_I, we define

$$\hat{G}^{(I)}\equiv\lim_{\rho_M\to0}\hat{G}^{(\rho)}. \tag{3.6}$$

It is then possible to transform Eqs. (3.3) and (3.4) into an integrodifferential equation that relates $\hat{G}^{(I)}$ and $\hat{G}^{(\rho)}$:

$$G^{(\rho)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')=G^{(I)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')-\int dr''[\nabla''\times G^{(I)}_{\alpha\cdot}(\mathbf{r},\mathbf{r}'')]\times\hat{\rho}_M\theta_M(\mathbf{r}'')\cdot[\nabla''\times G^{(\rho)}_{\cdot\beta}(\mathbf{r}'',\mathbf{r}')]. \tag{3.7}$$

Iteration of this equation leads, in the usual way, to an expansion of $\hat{G}^{(\rho)}$ in powers of $\hat{\rho}_M$ around $\hat{G}^{(I)}$. We note that, although $G^{(\rho)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is not a symmetric kernel [because $\hat{\rho}(\mathbf{r})$ is a nonsymmetric tensor], $G^{(I)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is symmetric because $\hat{\rho}_I$ is symmetric, and, in fact, $\hat{\rho}_I$ is a scalar tensor (see Appendix B for a discussion of this point):

$$G^{(I)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')=G^{(I)}_{\beta\alpha}(\mathbf{r}',\mathbf{r}). \tag{3.8}$$

A possible choice of $\mathbf{A}^{(0)}$ in Eq. (3.5) is $\lim_{\rho_M\to0}\mathbf{A}$—henceforth we adopt that choice. If we then take the limit $\rho_M\to0$ also in the functions $\mathbf{A}(\mathbf{r})$, $\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')$, we conclude that

$$\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(I)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_I\theta_I(\mathbf{r}')[\nabla'\times\mathbf{A}^{(0)}(\mathbf{r}')]=0. \tag{3.9}$$

Using the above-mentioned power series expansion for $\hat{G}^{(\rho)}$, this result can be extended to hold also when $\hat{G}^{(I)}$ is replaced by $\hat{G}^{(\rho)}$:

$$\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_I\theta_I(\mathbf{r}')[\nabla'\times\mathbf{A}^{(0)}(\mathbf{r}')]=0. \tag{3.10}$$

Using this result together with Eq. (3.7), (3.5) can be transformed into an expansion for $\mathbf{A}(\mathbf{r})$ in powers of ρ_M:

$$\begin{aligned}\mathbf{A}(\mathbf{r})&=\mathbf{A}^{(0)}(\mathbf{r})-\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_M\theta_M(\mathbf{r}')\cdot[\nabla'\times\mathbf{A}^{(0)}(\mathbf{r}')]\\&=\mathbf{A}^{(0)}(\mathbf{r})-\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(I)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_M\theta_M(\mathbf{r}')\cdot[\nabla'\times\mathbf{A}^{(0)}(\mathbf{r}')]+O(\rho_M^2).\end{aligned} \tag{3.11}$$

We also note that $\mathbf{A}^{(0)}(\mathbf{r})$ can be obtained by an expression that is the analog of Eq. (3.5), namely

$$\mathbf{A}^{(0)}(\mathbf{r})=\mathbf{A}^{(00)}(\mathbf{r})-\int d\mathbf{r}'\lim_{k\to0}[\nabla'\times\hat{G}^{(I)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_I\theta_I(\mathbf{r}')\cdot[\nabla'\times\mathbf{A}^{(00)}(\mathbf{r}')]. \tag{3.12}$$

This was obtained by replacing $\mathbf{A}^{(0)}(\mathbf{r})$ by $\mathbf{A}^{(00)}(\mathbf{r})$ in Eq. (3.5), and then taking the limit $\rho_M\to0$ in that equation. Equation (3.12) is especially useful if we assume the following boundary condition for $\mathbf{A}(\mathbf{r})$ and $\mathbf{A}^{(0)}(\mathbf{r})$:

$$\mathbf{A}(\mathbf{r})=\mathbf{A}^{(0)}(\mathbf{r})=\frac{1}{3}(\mathbf{e}\times\mathbf{r}) \quad \text{at the system surface,} \tag{3.13}$$

and choose

$$\mathbf{A}^{(00)}(\mathbf{r}) = \frac{1}{3}(\mathbf{e}\times\mathbf{r}) \quad \text{everywhere,} \tag{3.14}$$

where $\mathbf{e}$ is some unit vector. This last choice corresponds to a uniform current density

$$\nabla\times\left[\frac{1}{3}(\mathbf{e}\times\mathbf{r})\right] = \mathbf{e},$$

which is equal to the *volume averaged* current density for both $\mathbf{A}^{(\mathbf{e})}(\mathbf{r})$ and $\mathbf{A}^{(0\mathbf{e})}(\mathbf{r})$, which satisfy Eq. (3.13),

$$\langle \nabla\times\mathbf{A}^{(\mathbf{e})}\rangle = \langle \nabla\times\mathbf{A}^{(0\mathbf{e})}\rangle = \mathbf{e}. \tag{3.15}$$

We thus find that $\mathbf{A}^{(0\mathbf{e})}$ is given by

$$\mathbf{A}^{(0\mathbf{e})}(\mathbf{r}) = \frac{1}{3}(\mathbf{e}\times\mathbf{r}) - \int d\mathbf{r}' \lim_{k\to 0}[\nabla'\times\hat{G}^{(I)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_I\theta_I(\mathbf{r}')\cdot\mathbf{e}. \tag{3.16}$$

We recall that the bulk effective complex resistivity tensor of the system is defined by [see Eqs. (1.14) and 3.1)]

$$\hat{\rho}_e\cdot\langle\nabla\times\mathbf{A}\rangle \equiv \langle\hat{\rho}\cdot(\nabla\times\mathbf{A})\rangle, \tag{3.17}$$

for any $\mathbf{A}$ of form (3.5). Using Eqs. (1.6) and (3.11), we can expand an arbitrary element of the tensor $\hat{\rho}_e$ in powers of ρ_M ($\mathbf{f}$, $\mathbf{e}$ are arbitrary unit vectors)

$$\begin{aligned}(\mathbf{f}\cdot\hat{\rho}_e\cdot\mathbf{e}) &= \mathbf{f}\cdot\langle\hat{\rho}\cdot(\nabla\times\mathbf{A}^{(\mathbf{e})})\rangle \\ &= \mathbf{f}\cdot\hat{\rho}_I\cdot\langle\theta_I(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle + \mathbf{f}\cdot\hat{\rho}_M\cdot\langle\theta_M(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle \\ &\quad -\mathbf{f}\cdot\frac{1}{V}\int d\mathbf{r}\hat{\rho}_I\theta_I(\mathbf{r})\cdot\nabla\int d\mathbf{r}'\lim_{k\to 0}[\nabla'\times\hat{G}^{(I)}(\mathbf{r},\mathbf{r}')]\cdot\hat{\rho}_M\theta_M(\mathbf{r}')\cdot[\nabla'\times\mathbf{A}^{(0\mathbf{e})}(\mathbf{r}')] + O(\rho_M^2).\end{aligned} \tag{3.18}$$

[Note that $\langle\theta_I(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle$ is just the spatial average of $\nabla\times\mathbf{A}^{(0\mathbf{e})}$ over the subvolume of the dielectric component, while $\langle\theta_M(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle$ is the average of the same quantity over the metallic subvolume.] The integration over $\mathbf{r}$ can be performed using Eqs. (3.8) and (3.16), leading to the following result for the double integral of (3.18)

$$\int d\mathbf{r}'[\nabla'\times\mathbf{A}^{(0\mathbf{f})}(\mathbf{r}') - \mathbf{f}]\cdot\hat{\rho}_M\theta_M(\mathbf{r}')[\nabla'\times\mathbf{A}^{(0\mathbf{e})}(\mathbf{r}')]. \tag{3.19}$$

Part of this cancels the second term of Eq. (3.18), and we finally obtain

$$\begin{aligned}\mathbf{f}\cdot\hat{\rho}_e(\hat{\rho}_I,\hat{\rho}_M)\cdot\mathbf{e} &= \mathbf{f}\cdot\hat{\rho}_I\cdot\langle\theta_I(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle \\ &\quad + \langle\theta_M(\nabla\times\mathbf{A}^{(0\mathbf{f})})\cdot\hat{\rho}_M(\nabla\times\mathbf{A}^{(0\mathbf{e})})\rangle \\ &\quad + O(\rho_M^2).\end{aligned} \tag{3.20}$$

If the microstructure is isotropic, then since $\hat{\rho}_I$ is a scalar tensor, $\hat{\rho}_e(\hat{\rho}_I,0)$ [the first term on the right-hand side of Eq. (3.20)] is also a scalar tensor, and it is clearly independent of $\hat{\rho}_M$ and hence of H. The vector potentials $\mathbf{A}^{(0\mathbf{e})}$ and $\mathbf{A}^{(0\mathbf{f})}$, which appear in the second term on the right-hand side of Eq. (3.20), satisfy different boundary conditions *at the system surface* [see Eq. (3.15)]. Inside the metallic subvolume, those potentials can also be viewed as resulting from boundary conditions on $\mathbf{n}\times\mathbf{A}^{(0)}$ *at the interface between the two components*. Those latter boundary values are entirely determined by the microstructure when we impose the requirement that the electric potential must be constant over every connected subvolume of the metallic component, but the precise local values of $\mathbf{A}^{(0)}(\mathbf{r})$ inside those subvolumes also depend upon the Hall-to-Ohmic resistivity ratio of the metal H. Nevertheless, we now argue that even the second volume average which appears in Eq. (3.20) is independent of H in the two limits $H\ll 1$ and $H\gg 1$. The only H dependence in those limits arises from the explicit $\hat{\rho}_M$ factor in that term.

In order to prove this, we note that if the resistivity ratio ρ_M/ρ_I is small enough, then the current distribution inside the metallic subvolumes, though different for $H\ll 1$ and for $H\gg 1$, will be saturated in both limits: In the weak-field limit this is obvious, while in the strong-field limit this holds in a percolating system whenever the magnetic-field-dependent correlation length ξ_H, which diverges as $H\to\infty$, is greater than the percolation correlation length[4] ξ_p.

Recalling that $\mathbf{B}\|z$, and assuming that the microstructure is either isotropic or cubic, we now obtain that the diagonal elements of $\hat{\rho}_e$ are given by

$$\begin{aligned}\rho^{(e)}_{\alpha\alpha}(\hat{\rho}_I,\hat{\rho}_M) &= \rho_I\langle\theta_I(\nabla\times\mathbf{A}^{(0\alpha)})_\alpha\rangle + \rho_M\langle\theta_M(\nabla\times\mathbf{A}^{(0\alpha)})^2\rangle \\ &\quad + O(\rho_M^2),\end{aligned} \tag{3.21}$$

while the nonzero off-diagonal elements are

$$\rho_{xy}^{(e)}(\hat{\rho}_I,\hat{\rho}_M)=-\rho_{yx}^{(e)}(\hat{\rho}_I,\hat{\rho}_M)=H\rho_M\langle\theta_M[(\nabla\times\mathbf{A}^{(0x)})\times(\nabla\times\mathbf{A}^{(0y)})]_z\rangle+O(\rho_M^2). \tag{3.22}$$

Recalling also that

$$\hat{\rho}_e=\frac{4\pi}{i\omega\hat{\varepsilon}_e},$$

we finally obtain the following results for $\hat{\varepsilon}_e$ ($\sigma_M\equiv 1/\rho_M$):

$$\hat{\varepsilon}_e(\varepsilon_I,\hat{\rho}_M)\simeq\frac{\varepsilon_I}{\langle\theta_I(\nabla\times\mathbf{A}^{(0x)})_x\rangle}\hat{I}-\frac{i\omega\varepsilon_I^2}{4\pi\sigma_M}\frac{\langle\theta_M(\nabla\times\mathbf{A}^{(0x)})^2\rangle}{\langle\theta_I(\nabla\times\mathbf{A}^{(0x)})\rangle^2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}-\frac{i\omega\varepsilon_I^2}{4\pi\sigma_M}\frac{\langle\theta_M(\nabla\times\mathbf{A}^{(0z)})^2\rangle}{\langle\theta_I(\nabla\times\mathbf{A}^{(0z)})\rangle^2}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$
$$-\frac{i\omega\varepsilon_I^2H}{4\pi\sigma_M}\frac{\langle\theta_M[(\nabla\times\mathbf{A}^{(0x)})\times(\nabla\times\mathbf{A}^{(0y)})]_z\rangle}{\langle\theta_I(\nabla\times\mathbf{A}^{(0x)})\rangle^2}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}. \tag{3.23}$$

This expression contains averages such as $\langle\theta_{I(M)}(\nabla\times\mathbf{A})^2\rangle$. These quantities are not easy to compute, since they require knowledge of local values of $\mathbf{A}(\mathbf{r})$. However, in dc magnetotransport, one arrives at similar expressions. In Sec. IV, we will use this analogy to construct a scaling theory for the Faraday effect near the percolation threshold.

IV. SCALING THEORY

In this section, we will discuss physical consequences of the results obtained by the different approaches, and we will use a scaling theory in order to discuss the behavior of the permittivity below the percolation threshold by comparing with scaling theories developed earlier for dc magnetotransport in a percolating system.[3,4,15]

As we noted earlier, Eq. (2.1) is the same as the result obtained in Refs. 1 and 13: It is an equation for the bulk effective conductivity of a binary mixture which was studied thoroughly in Ref. 14; in particular, it displays a percolation threshold at $p_c=1-e^{-1/3}\simeq 0.28$. The Faraday effect is contained in Eqs. (2.2) and (2.3) or, equivalently, in Eqs. (2.6) and (2.7). We first note that the percolation threshold for that effect is independent of the magnetic field and, as expected, is the same as the threshold found for the conductivity.

We can easily check those results in two limiting cases. First, for zero magnetic field it is easy to see that one recovers Hori's equation for the effective permittivity of a binary mixture.[13] Then, for low concentration ($p\ll 1$) we find that $\varepsilon_e\simeq\varepsilon_I$ and $\tilde{\varepsilon}_e\simeq p(4\pi\sigma_M/\omega)H/(1+H^2)$, and the rotation angle therefore satisfies $\theta\propto 1/H$ for high fields and $\theta\propto H$ for low fields. This agrees to order $O(p)$ with the low-density expansion results for spherical inclusions as obtained, for example, from the Clausius-Mossotti-type approximation of Ref. 8. We note that here θ is independent of the frequency and is always very small.

We now apply our discussion from Sec. II to the case of a nondilute metal-dielectric mixture: For the metal, $\alpha_1=\sigma_M/(1+H^2)$ and $\beta_1=\alpha_1 H$, and for the dielectric (which is assumed to have a negligible Faraday effect) $\alpha_2=i\omega\varepsilon_I/4\pi$ and $\beta_2\simeq 0$. We will concentrate on the critical region near the percolation threshold p_c, where $\Delta p\equiv p-p_c$ is small (p is the metal volume fraction). We will now consider ε_e and $\tilde{\varepsilon}_e$ separately for the weak-field regime ($H\ll 1$) and the strong-field regime ($H\gg 1$).

At low fields, we use the scaling result for α_e, and thus find that the effective permittivity ($\varepsilon_e=4\pi\alpha_e/i\omega$) is given by

$$\varepsilon_e\simeq\frac{q_c}{3}\left(\frac{\varepsilon_I}{|\Delta p|}-i\frac{Aq_c^2}{12\pi}\frac{\omega\varepsilon_I^2}{\sigma_M|\Delta p|^3}\right) \tag{4.1}$$

for the regime where $\omega\varepsilon_I/\sigma_M\Delta p^2\ll 1$ and $p<p_c$. The real part of ε_e thus diverges like $1/|\Delta p|$ (exponent s equal to its SEMA value 1), and the imaginary part is proportional to $\omega\varepsilon_I^2/\sigma_M|\Delta p|^3$. From Eq. (2.11), we obtain the following result for $\tilde{\varepsilon}_e$ ($\tilde{\varepsilon}_e=4\pi\beta_e/\omega$):

$$\tilde{\varepsilon}_e\propto-\frac{\omega\varepsilon_I^2}{4\pi\sigma_M|\Delta p|^2}H. \tag{4.2}$$

In this regime, the Faraday coefficient will be

$$\mathcal{F}\propto-\frac{\omega\varepsilon_I^{3/2}H}{4\pi\sigma_M|\Delta p|^{3/2}}. \tag{4.3}$$

In the strong-field regime, $\beta_1\simeq\sigma_M/H$ and $\alpha_1\simeq\sigma_M/H^2$. Using the scaling result for α_e in this regime, we find that the effective permittivity is given by

$$\varepsilon_e\simeq\frac{q_c}{3}\left(\frac{\varepsilon_I}{|\Delta p|}-i\frac{Aq_c^2}{12\pi}\frac{\omega\varepsilon_I^2}{\sigma_M|\Delta p|^3}\right). \tag{4.4}$$

This scaling behavior is the same as what was found above in the weak-field regime. The first term should indeed be independent of H, since it corresponds to the universal behavior of the dc permittivity near the percolation threshold. However, the fact that the imaginary parts of Eqs. (4.4) and (4.1) are the same to the order shown here is accidental, and is probably due to the nature of the approximations used (a similar accident also occurs in the SEMA results). Indeed,

we can see from Eq. (3.23) that these parts depend on the current distribution in the metallic inclusions, hence they should be different in the weak- and strong-field regimes. The important physical conclusion is that, in both regimes, ε_e is independent of H.

In the strong-field regime, from Eq. (2.15) we obtain the surprising result

$$\tilde{\varepsilon}_e \propto -\frac{\omega \varepsilon_I^2}{4\pi\sigma_M(\Delta p)^2} H, \tag{4.5}$$

which is again the same scaling behavior as in the weak-field regime. In both regimes, the Faraday coefficient thus reads

$$\mathcal{F} \propto -\frac{\omega \varepsilon_I^{3/2} H}{4\pi\sigma_M |\Delta p|^{3/2}}. \tag{4.6}$$

This means that the Faraday rotation (the angle is proportional to $\mathcal{F}$) is proportional to the applied magnetic field, even for strong fields, i.e., when $H \gg 1$. It is thus clear that we can obtain a large value of the rotation angle using such a composite.

We now consider the results obtained by means of the high-contrast or low-frequency expansion. The scaling behavior of the averages which appear in Eq. (3.23) can be deduced by comparing (3.20) to scaling theories previously developed for dc magnetotransport coefficients,[3,4] and from the property that

$$\mathbf{J}^{(0)} = \nabla \times \mathbf{A}^{(0)} \propto \begin{cases} 1\mathbf{r} \in \text{ percolating cluster}, & p > p_c \\ 0\mathbf{r} \in \text{ elsewhere}, & p > p_c \\ 1\mathbf{r} \in \text{ anywhere}, & p < p_c . \end{cases} \tag{4.7}$$

These considerations lead to

$$\langle \theta_I (\nabla \times \mathbf{A}^{(0)}) \rangle \propto \begin{cases} \Delta p^s, & p < p_c, \quad \text{any } H \\ 0, & p > p_c, \quad \text{any } H, \end{cases} \tag{4.8}$$

$$\langle \theta_M (\nabla \times \mathbf{A}^{(0\alpha)})^2 \rangle \propto \begin{cases} \Delta p^{-t}, & \text{any } pH \ll 1 \\ \Delta p^{-t_H} F\left(\dfrac{\xi_p}{\xi_H}\right), & \text{any } pH \gg 1, \end{cases} \tag{4.9}$$

$$\langle \theta_M [(\nabla \times \mathbf{A}^{(0x)}) \times (\nabla \times \mathbf{A}^{(0y)})]_z \rangle \propto \begin{cases} \Delta p^{-g}, & \text{any } pH \ll 1 \\ \Delta p^{-g_H} G\left(\dfrac{\xi_p}{\xi_H}\right), & \text{any } pH \gg 1. \end{cases} \tag{4.10}$$

Here $\xi_H = H^{\nu_H}$ is the magnetic-field-dependent correlation length, $\xi_p \propto \Delta p^{-\nu}$ is the percolation correlation length, and $F(z)$ and $G(z)$ are scaling functions, which tend to nonzero constants when $z \ll 1$, and to asymptotic forms that return the Δp exponents to their $H \ll 1$ values when $z \gg 1$ (see Ref. 4):

$$F(z) \propto \begin{cases} \text{const}, & z \ll 1 \\ z^{\frac{t - t_H}{\nu}}, & z \gg 1, \end{cases} \tag{4.11}$$

$$G(z) \propto \begin{cases} \text{const}, & z \ll 1 \\ z^{\frac{g - g_H}{\nu}}, & z \gg 1. \end{cases} \tag{4.12}$$

The values of the critical exponents which appear in the above scaling expressions, as determined by simulations of three-dimensional percolating network models, are[16,7,4]

$$\nu \cong 0.88, \quad t \cong 2.0, \quad s \cong 0.7, \quad g \cong 0.38,$$

$$\nu_H \cong 0.5, \quad t_H \cong 6.0, \quad g_H \cong 5.0, \tag{4.13}$$

where ν_H, t_H, and g_H are exponents describing high field behavior of percolating systems.[4] Finally, we obtain the following results for the Faraday coefficient of a percolating mixture below p_c:

$$-\frac{\omega\varepsilon_I^{3/2}H}{4\pi\sigma_M}\frac{\langle\theta_M[(\nabla\times\mathbf{A}^{(0x)})\times(\nabla\times\mathbf{A}^{(0y)})]_z\rangle}{\langle\theta_I(\nabla\times\mathbf{A}^{(0x)})_x\rangle^{3/2}}\propto\begin{cases}-\dfrac{\omega\varepsilon_I^{3/2}H}{4\pi\sigma_M}\Delta p^{-g-3s/2}, & H\ll 1\\ -\dfrac{\omega\varepsilon_I^{3/2}H^{1+(g_H-g)\nu_H/\nu}}{4\pi\sigma_M}\Delta p^{-g-3s/2}, & H\gg 1 \quad \text{but} \quad \xi_H\ll\xi_p\\ -\dfrac{\omega\varepsilon_I^{3/2}H}{4\pi\sigma_M}\Delta p^{-g_H-3s/2}, & \xi_H\gg\xi_p .\end{cases} \tag{4.14}$$

These results are consistent with Eq. (4.6), which was obtained using the replica method, where we expect to find the SEMA values $s=1$ and $g=g_H=0$. We can also derive a number of physical consequences which follow from both approaches [from Eq. (3.23) in the high contrast expansion, and from Eqs. (4.1) and (4.4) in the replica approach].

(1) The diagonal part of $\hat{\varepsilon}_e$ has an imaginary part that is proportional to $\omega\varepsilon_I^2/\sigma_M$ and is independent of H [up to $O(\rho_M)$ in the high contrast expansion]. This means that there will be some dissipation.

(2) In the same order, $\hat{\varepsilon}_e$ has an antisymmetric part which is imaginary and proportional to $\omega\varepsilon_I^2 H/\sigma_M$. These results are valid both for weak and strong fields, as long as both $\omega\varepsilon_I\ll\sigma_M$ and $\omega\varepsilon_I H\ll\sigma_M$.

We note that the scaling predictions for magnetotransport in a percolating system have been tested experimentally only for weak fields, and only in systems that were above the percolation threshold p_c.[17] Measurements of the induced Faraday effect in a metal-dielectric mixture below p_c could therefore provide an important test of those predictions. Another prediction which follows from Eq. (3.23) is that the induced Faraday effect in a nonconducting metal-dielectric composite, which is not necessarily near any percolation threshold, is linear in $H\equiv\omega_c\tau$ even when $H\gg 1$, in agreement with the replica approach near p_c.

V. CONCLUSION

We studied the Faraday effect in a metal-dielectric composite in the quasistatic regime, close to but below the percolation threshold of the metal component. The response of the dielectric component was assumed to be independent of magnetic field; all the field dependence of the macroscopic response is due to the Hall effect in the metal component. We presented two different approaches leading essentially to the same conclusions. The first approach relies on the replica method and allowed us to derive in a nonperturbative way equations for the effective permittivity tensor. The second approach is the result of a high contrast expansion in powers of ρ_M/ρ_I [ρ_M (ρ_I) is the impedance of the metallic (dielectric) component], combined with scaling *Ansätze* near p_c.

First of all, both approaches are consistent with each other, the only difference is that the scaling exponents predicted by the replica approach have their SEMA values. Second, the main result is the following: the scaling of the Faraday coefficient is the same for the weak and strong-field regimes (as long as both $\omega\varepsilon_I\ll\sigma_M$ and $\omega\varepsilon_I H\ll\sigma_M$). In particular, we found that the Faraday angle is proportional to the magnetic field $\mathbf{B}$ even for a strong field, as long as $p<p_c$ and $\varepsilon_I\omega/\sigma_M\Delta p^2\ll 1$. We can thus predict that it should in principle be possible to obtain large values of the rotation angle in such a system. For instance, for Δp of order 0.1 (which is realistic in experiments), ε_I of order unity and σ_M/ω of order 100 for semiconductors, and with ω in the microwave region (the ratio σ_M/ω should not be too large since $\tilde{\varepsilon}_e$ is proportional to its inverse), one obtains for the Faraday coefficient

$$\frac{\tilde{\varepsilon}_e}{\sqrt{\varepsilon_e}}\simeq 10^{-1}H, \tag{5.1}$$

which can be made of order unity using currently available magnetic fields and high-mobility doped semiconductors. One should recall that in homogeneous dielectrics, the Faraday coefficient is usually much less than 1: Typical values for a 1-T magnetic field, and for a wavelength in the visible spectrum ($\lambda\simeq 0.6$ μm), are of order 10^{-6} for dielectrics like quartz, and of order 10^{-2} for thin ferromagnetic metallic iron films. Measurements of the Faraday effect below p_c in a percolating metal-dielectric composite could provide an important test of the scaling predictions in both the strong and weak-field regimes. Such experiments would have to involve either propagation or reflection of microwaves by a metal-dielectric composite with metallic inclusions that are smaller than the relevant skin depth. Both approaches also predict that the transverse diagonal elements of $\hat{\varepsilon}_e$ have an imaginary part that is proportional to $\omega\varepsilon_I^2/\sigma_M$, and are independent of H [up to terms of order $O(\rho_M)$ in the high-contrast expansion]. This means that there will be some dissipation.

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APPENDIX A

We want here to solve the Maxwell equation satisfied by the electric field $\mathbf{E}$ (where $k_0=\omega/c$)

$$[\nabla\times\nabla\times+\hat{\varepsilon}(\mathbf{r})k_0^2]\mathbf{E}(\mathbf{r})=0. \tag{A1}$$

The tensor $\hat{\varepsilon}$ is a random variable equal to $\hat{\varepsilon}_1$ with probability p and to $\hat{\varepsilon}_2$ with probability $q=1-p$ [$\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are tensors of the form given in Eq. (1.1)]. In order to obtain an

integral equation for the electric field, we first write $\hat{\varepsilon}(\mathbf{r}) = \varepsilon_0 \hat{I} + \delta\hat{\varepsilon}(\mathbf{r})$ (where ε_0 is an arbitrary constant which will disappear at the end of the calculation). One can then easily show that $\mathbf{E}(\mathbf{r})$ is also the solution of the equation

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \int d\mathbf{r}' \hat{G}(\mathbf{r}-\mathbf{r}') \delta\hat{\varepsilon}(\mathbf{r}') \mathbf{E}(\mathbf{r}'), \quad \text{(A2)}$$

where $\hat{G}$ is the dipolar tensor for the uniform medium of permittivity ε_0, and where the quantity $\mathbf{E}_0$ depends only on the boundary conditions and is assumed to be uniform.

We will use the Fourier transform of $G_{\alpha\beta}(\mathbf{r})$ given by

$$G_{\alpha\beta}(k) = -\frac{k_\alpha k_\beta}{\varepsilon_0 k^2} + \frac{k_0^2}{k^2 - \varepsilon_0 k_0^2}\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right]. \quad \text{(A3)}$$

We work in the quasistatic limit, which means that we take the limit k_0 going to zero. In this case, the dipolar tensor is given by

$$G_{\alpha\beta}(k) = -\frac{k_\alpha k_\beta}{\varepsilon_0 k^2} \quad \text{for } k \neq 0. \quad \text{(A4)}$$

At $k=0$, the value of this tensor is $G_0 = -\delta_{\alpha,\beta}/(d\varepsilon_0)$, where d is the space dimension, here equal to 3. For values of k_0 that are too large, we cannot define an effective permittivity tensor, and we have to introduce the notion of spatial dispersion (for a review, see e.g., Ref. 18, and for a study using the replica method see Ref. 19).

Averaging Eq. (A2), after inverting it, and averaging it before inverting it, leads to the exact relation

$$\langle M^{-1}_{\alpha\beta}(k=0) \rangle = (1 - G_0 \delta\hat{\varepsilon}_e)^{-1}_{\alpha\beta}, \quad \text{(A5)}$$

where $\hat{M}^{-1}$ is the inverse of the random operator

$$M_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') - (\hat{G}(\mathbf{r}-\mathbf{r}')\delta\hat{\varepsilon}(\mathbf{r}'))_{\alpha\beta}. \quad \text{(A6)}$$

Calculation of the effective permittivity tensor is thus reduced to a calculation of the average over the disorder of the inverse of a random operator, namely, $\hat{M}$. In order to do this, we will use the replica method which allows us to express the elements of $\hat{M}^{-1}$ (after using a Gaussian inversion formula) in terms of the functional integral

$$M^{-1}_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \int \mathcal{D}(\overline{\psi},\psi)\, \overline{\psi}^a_\alpha(\mathbf{r}) \psi^a_\beta(\mathbf{r}') \times \exp\left(\int d\mathbf{r}\, d\mathbf{r}' \sum_{\alpha\beta,a} \overline{\psi}^a_\alpha(\mathbf{r}) M_{\alpha\beta}(\mathbf{r},\mathbf{r}') \psi^a_\beta(\mathbf{r}') \right), \quad \text{(A7)}$$

where ψ^a_α (with $a = 1, \ldots, n$) (and its conjugate $\overline{\psi}$) are replicated Grassman fields satisfying the usual anticommutation relations

$$\{\psi(\mathbf{r}), \psi(\mathbf{r}')\} = \{\psi(\mathbf{r}), \overline{\psi}(\mathbf{r}')\} = \{\overline{\psi}(\mathbf{r}), \overline{\psi}(\mathbf{r}')\} = 0. \quad \text{(A8)}$$

The limit $n=0$ is implicitly taken in Eq. (A7) and, as usual in the replica method, we first consider n as an integer and then take the limit n going to zero at the end of the calculation (without adressing the problem of analytic continuation).

It is now easy to average $\hat{M}^{-1}$ over the disorder, and we obtain

$$\langle M^{-1} \rangle_{\alpha\beta}(\mathbf{r}-\mathbf{r}') = \int \mathcal{D}(\overline{\psi},\psi)\, \overline{\psi}^a_\alpha(\mathbf{r}) \psi^a_\beta(\mathbf{r}') e^{\mathcal{H}_e}, \quad \text{(A9)}$$

where the effective Hamiltonian is given by

$$\mathcal{H}_e = \int d\mathbf{r}\, d\mathbf{r}' \sum_{\alpha,\beta,a} \overline{\psi}^a_\alpha(\mathbf{r}) \{\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') - (\hat{G}(\mathbf{r}-\mathbf{r}')\delta\hat{\varepsilon}_1)_{\alpha\beta}\} \psi^a_\beta(\mathbf{r}') + \int d\mathbf{r}_0 \ln\left[1 + \eta \exp\left(\int d\mathbf{r}' \sum_{\alpha,\beta,\mu,a} \overline{\psi}^a_\alpha(\mathbf{r}_0) G_{\alpha\mu}(\mathbf{r}_0 - \mathbf{r}') \Delta_{\mu\beta} \psi^a_\beta(\mathbf{r}') \right) \right]. \quad \text{(A10)}$$

The matrix $\hat{\Delta}$ is equal to $\hat{\varepsilon}_2 - \hat{\varepsilon}_1$ and $\eta = (1-p)/p$ (note that η is the inverse of $\lambda = p/q$, which appears in the main text). As usual in the replica method, the average over disorder introduces coupling between different replicas (if there is no coupling, then the averaging is trivial) and in order to study this complicated effective Hamiltonian, we will use a variational principle.[20,1] This principle consists of finding the best Gaussian approximation $\mathcal{H}_0$ to the effective Hamiltonian $\mathcal{H}_e$. Denoting by $\hat{K}^{-1}$ the kernel of $\mathcal{H}_0$ (the variational approximation thus reads $\langle \hat{M}^{-1} \rangle \simeq \hat{K}$), we have to minimize the following variational free energy Φ with respect to $\hat{K}$:

$$\Phi(\hat{K}) = \mathcal{F}_0 + \langle \mathcal{H}_e - \mathcal{H}_0 \rangle_0, \quad \text{(A11)}$$

where $\mathcal{F}_0$ is the free energy associated with $\mathcal{H}_0$, and where $\langle\ \rangle_0$ denotes an average using $\mathcal{H}_0$. We thus obtain the equation

$$(\hat{K}^{-1})_{\alpha\beta} = \delta_{\alpha\beta} - (\hat{G}(k)\delta\hat{\varepsilon}_e)_{\alpha\beta}, \quad \text{(A12)}$$

with

$$\hat{\varepsilon}_e = \hat{\varepsilon}_1 + \sum_{m \geqslant 1} (-)^{m+1} \eta^m \frac{\hat{\Delta}}{1 - m\hat{\Delta} \int \frac{d^d k}{(2\pi)^d} \hat{K}\hat{G}}. \quad \text{(A13)}$$

The tensor $\hat{K}$ can be inverted, and we obtain

$$\hat{K}=\frac{1}{1+\dfrac{\mathbf{k}\cdot\mathbf{q}}{D}}\left[1+\frac{\mathbf{k}\cdot\mathbf{q}}{D}-\frac{\mathbf{k}\otimes\mathbf{q}}{D}\right], \tag{A14}$$

where $D=\varepsilon_0 k^2$, $\mathbf{q}=\delta\hat{\varepsilon}_e^t\mathbf{k}$ ($\delta\hat{\varepsilon}_e^t$ is the transpose of $\delta\hat{\varepsilon}_e$), and where $\otimes$ denotes the usual dyadic product. We can then compute $\hat{K}\hat{G}$ and we find that it is the dipolar tensor for the effective medium

$$(\hat{K}\hat{G})_{\alpha\beta}=-\frac{k_\alpha k_\beta}{\mathbf{k}\cdot(\varepsilon_e^t\mathbf{k})}=-\frac{k_\alpha k_\beta}{\sum_{\alpha=1}^{3}(\varepsilon_e)_{\alpha\alpha}k_\alpha^2}. \tag{A15}$$

It is then easy to integrate $\hat{K}\hat{G}$, and we obtain

$$\int\frac{d\mathbf{k}}{(2\pi)^3}(\hat{K}\hat{G})_{\alpha\beta}=-\frac{1}{3}\delta_{\alpha\beta}\mu_\alpha=-\frac{1}{3}\mu_{\alpha\beta}, \tag{A16}$$

where $\hat{\mu}$ is a diagonal matrix with diagonal elements $\mu_1=\mu_2=1/\varepsilon_e$ and $\mu_3=1/\varepsilon_{ze}$. The self-consistent equation (A13) can thus be recast as the matrix equation

$$\hat{\varepsilon}_e=\int_0^\infty du\,e^{-u}\frac{\langle\hat{\varepsilon}e^{-u\hat{\varepsilon}\hat{\mu}/3}\rangle}{\langle e^{-u\hat{\varepsilon}\hat{\mu}/3}\rangle}, \tag{A17}$$

where the brackets still denote an average over the disorder. It is then easy to show that

$$\begin{pmatrix}\varepsilon_e & i\tilde{\varepsilon}_e\\ -i\tilde{\varepsilon}_e & \varepsilon_e\end{pmatrix}=\int_0^\infty du\,e^{-u}\frac{\langle\hat{\varepsilon}_2 e^{-u\hat{\varepsilon}_2/3\varepsilon_e}\rangle}{\langle e^{-u\hat{\varepsilon}_2/3\varepsilon_e}\rangle}, \tag{A18}$$

where $\hat{\varepsilon}_2$ denotes the restriction of the tensor $\hat{\varepsilon}$ to the (x,y) subspace:

$$\hat{\varepsilon}_2=\begin{pmatrix}\varepsilon & i\tilde{\varepsilon}\\ -i\tilde{\varepsilon} & \varepsilon\end{pmatrix}. \tag{A19}$$

The equation along the z axis is decoupled from the preceeding one, and is Eq. (2.1) of the main text. Using the relation

$$e^{\alpha\hat{\varepsilon}_2}=e^{\alpha\varepsilon}\begin{pmatrix}\cosh\alpha\tilde{\varepsilon} & -i\sinh\alpha\tilde{\varepsilon}\\ i\sinh\alpha\tilde{\varepsilon} & \cosh\alpha\tilde{\varepsilon}\end{pmatrix}, \tag{A20}$$

after simple manipulations we obtain Eqs. (2.2) and (2.3) of the main text. It should be noted that all the calculation presented here can be used without any changes for either real or complex values of ε and $\tilde{\varepsilon}$. This justifies the analytical continuation of the formulas obtained in the framework of the Hall effect in order to describe the Faraday effect.

APPENDIX B

The equations for the Green tensor $\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')$ can be solved in almost closed form if the system occupies all space, and if the resistivity tensor $\hat{\rho}$ is constant everywhere and its symmetric part is a scalar tensor, i.e., if

$$\hat{\rho}(\mathbf{r})\cdot\mathbf{v}\equiv\hat{\rho}_0\cdot\mathbf{v}=\rho_0\mathbf{v}+\mathbf{b}\times\mathbf{v} \tag{B1}$$

for any vector $\mathbf{v}$. In that case, the Green tensor depends only on $\mathbf{r}-\mathbf{r}'$, and we can define its Fourier transform by

$$\tilde{G}_{\alpha\beta}^{(\rho_0)}(\mathbf{q})\equiv\int d(\mathbf{r}-\mathbf{r}')G_{\alpha\beta}^{(\rho_0)}(\mathbf{r}-\mathbf{r}')e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{B2}$$

Using Eq. (3.3), it is easily found that this Fourier transform satisfies the linear algebraic equation

$$(\rho_0 q^2-k^2)\tilde{G}_{\alpha\beta}^{(\rho_0)}-\rho_0 q_\alpha(\mathbf{q}\cdot\tilde{G}_{\cdot\beta}^{(\rho_0)})+(\mathbf{b}\cdot\mathbf{q})(\mathbf{q}\times\tilde{G}_{\cdot\beta}^{(\rho_0)})_\alpha=\delta_{\alpha\beta}, \tag{B3}$$

which can be solved to yield ($\varepsilon_{\alpha\beta\gamma}$ is the basic antisymmetric tensor)

$$\tilde{G}_{\alpha\beta}^{(\rho_0)}(\mathbf{q})=\frac{(\rho_0 q^2-k^2)\delta_{\alpha\beta}-[\rho_0(\rho_0 q^2-k^2)+(\mathbf{b}\cdot\mathbf{q})^2]q_\alpha q_\beta/k^2+(\mathbf{b}\cdot\mathbf{q})\varepsilon_{\alpha\beta\gamma}q_\gamma}{(\rho_0 q^2-k^2)^2+q^2(\mathbf{b}\cdot\mathbf{q})^2}. \tag{B4}$$

Clearly, $\tilde{G}_{\alpha\beta}^{(\rho_0)}(\mathbf{q})$ diverges in the limit $k\to 0$. However, if one calculates the Fourier transform of $(\nabla\times G_{\cdot\beta}^{(\rho_0)})_\alpha$, namely,

$$(\mathbf{q}\times\tilde{G}_{\cdot\beta}^{(\rho_0)})_\alpha=\frac{(\mathbf{b}\cdot\mathbf{q})(q^2\delta_{\alpha\beta}-q_\alpha q_\beta)-(\rho_0 q^2-k^2)\varepsilon_{\alpha\beta\gamma}q_\gamma}{(\rho_0 q^2-k^2)^2+q^2(\mathbf{b}\cdot\mathbf{q})^2}, \tag{B5}$$

then the limit $k\to 0$ can be taken without any problems. That is why we had to include the term $k^2G_{\alpha\beta}^{(\rho)}$ in the equation for the Green tensor [see Eq. (3.3)], deferring the limit $k\to 0$ until after the calculation of $(\nabla\times G_{\cdot\beta}^{(\rho)})_\alpha$.

In order to investigate the symmetry properties of $\hat{G}^{(\rho)}(\mathbf{r},\mathbf{r}')$, we use integration by parts or Green's theorem to get, for any vector fields $\mathbf{A}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$, and second-rank tensor field $\hat{\rho}(\mathbf{r})$,

$$\int_V d\mathbf{r}\{\mathbf{A}\cdot\nabla\times[\hat{\rho}\cdot(\nabla\times\mathbf{B})]-\mathbf{B}\cdot\nabla\times[\hat{\rho}\cdot(\nabla\times\mathbf{A})]\}=-\oint_{\partial V}[(d\mathbf{S}\times\mathbf{A})\cdot\hat{\rho}\cdot(\nabla\times\mathbf{B})-(d\mathbf{S}\times\mathbf{B})\cdot\hat{\rho}\cdot(\nabla\times\mathbf{A})]$$
$$+\int_V d\mathbf{r}[(\nabla\times\mathbf{A})\cdot\hat{\rho}\cdot(\nabla\times\mathbf{B})-(\nabla\times\mathbf{B})\cdot\hat{\rho}\cdot(\nabla\times\mathbf{A})]. \tag{B6}$$

If $\hat{\rho}$ is *symmetric*, then the integrand in the last volume integral vanishes everywhere. Substituting

$$A_\omega(\mathbf{r}) \equiv G^{(\rho)}_{\omega\alpha}(\mathbf{r},\mathbf{r}_1), \quad B_\omega(\mathbf{r}) \equiv G^{(\rho)}_{\omega\beta}(\mathbf{r},\mathbf{r}_2) \tag{B7}$$

into this result, and assuming that $\hat{\rho}$ is a symmetric tensor, we thus obtain, using Eqs. (3.3) and (3.4),

$$G^{(\rho)}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2) = G^{(\rho)}_{\beta\alpha}(\mathbf{r}_2,\mathbf{r}_1) \quad \text{if} \quad \hat{\rho}^t = \hat{\rho}. \tag{B8}$$

Note that if $\hat{\rho}$ is nonsymmetric, then in general $G^{(\rho)}_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is not a symmetric kernel.

[1] M. Barthélémy and H. Orland, J. Phys. I **3**, 2171 (1993).
[2] M. Barthélémy, D. J. Bergman, and H. Orland, Europhys. Lett. **27**, 305 (1994).
[3] D. J. Bergman and D. Stroud, Phys. Rev. B **32**, 6097 (1985).
[4] A. K. Sarychev, D. J. Bergman, and Y. M. Strelniker, Phys. Rev. B **48**, 3145 (1993).
[5] J. P. Straley, J. Phys. C **13**, 4335 (1980).
[6] J. P. Straley, J. Phys. C **13**, L773 (1980).
[7] D. J. Bergman, E. Duering, and M. Murat, J. Stat. Phys. **58**, 1 (1990).
[8] P. M. Hui and D. Stroud, Appl. Phys. Lett. **50**, 950 (1987).
[9] T. K. Xia, P. M. Hui, and D. Stroud, J. Appl. Phys. **67**, 2736 (1990).
[10] L. Landau and E. Lifshitz, *Electrodynamics of Continous Media*, Course of Theoretical Physics Vol. 8 (Addison-Wesley, Reading, MA, 1960).
[11] M. Hori, J. Math. Phys. **18**, 487 (1977).
[12] M. Hori and F. Yonezawa, J. Phys. C **10**, 229 (1978).
[13] J. Shobaki, S. Musameh, F. Rawwagah, and N.A. Yusuf, Phys. Rev. B **54**, 13 063 (1996).
[14] N. A. Yusuf, J. Shobaki, H. Abu-Safia, and I. Abu-Aljarayesh, J. Magn. Magn. Mater. **149**, 373 (1995).
[15] D. J. Bergman and D. Stroud, Solid State Phys. **46**, 147 (1992).
[16] A. Aharony and D. Stauffer, *Introduction to Percolation Theory*, 2nd ed. (Taylor and Francis, London, 1992).
[17] U. Dai, A. Palevski, and G. Deutscher, Phys. Rev. B **36**, 790 (1987).
[18] Yu. E. Lozovik and A. V. Klyuchnik, in *The Dielectric Function of Condensed Systems*, edited by L. V. Keldysh, D. A. Krizhnitz, and A. A. Maradudin (Elsevier, New York, 1987), Chap. 5.
[19] M. Barthélémy, H. Orland, and G. Zerah, Phys. Rev. E **52**, 1123 (1995).
[20] S. F. Edwards and M. Muthukumar, J. Chem. Phys. **89**, 2435 (1988).