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Analytical predictions for the magnetoelectric coupling in piezomagnetic materials reinforced by piezoelectric ellipsoidal inclusions

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In this paper, a number of problems of fundamental importance in a piezoelectric-piezomagnetic composite material are solved. The problems covered range from the derivation of the analytical expressions for the magneto-electro-elastic Eshelby tensors to the analysis of the magnetoelectric coupling effect which is a new property exhibited in the piezoelectric-piezomagnetic composite. In particular, when both the matrix and the inclusions of the composite are transversely isotropic with different magneto-electro-elastic moduli, and shapes of inclusions are of elliptic cylinder, circular cylinder, disk, and ribbon, closed-form solutions for the magnetoelectric coupling coefficients are presented compactly. The resulting solutions are a function of the shape of inclusion, phase properties, and volume fraction of the inclusions. These results could provide us with insight into how a piezoelectric-piezomagnetic composite material consisting of inclusions will perform and would be helpful in understanding the magneto-electric-elastic behavior of the composite. [S0163-1829(98)07825-4]

I. Introduction. Over the past few years, piezoelectric and piezomagnetic (magnetorestrictive) materials have received widespread uses in applications to many transducers and sensors with the responsibility of electromechanical and magnetomechanical energy conversion. These kind of materials constitute an important branch of the recently emerging technologies of modern engineering materials. Historically, improvements in piezoelectric or piezomagnetic materials have been focused on the development of improved single-phase materials. Recently, considerable research has been directed toward the development of piezoelectric-piezomagnetic composite materials that are defined as the combination of distinct phases of piezoelectric and piezomagnetic materials. Examples include piezoelectric ceramic particles or rods dispersed in a piezomagnetic matrix and piezoelectric-piezomagnetic laminated composites. Due to a superior product property that is so-called the magnetoelectric coupling effect resulting from the interaction between magnetism, electricity, and mechanism, piezoelectric-piezomagnetic composites have received considerable recent attention for designing intelligent or active structures. Applications of the composite are in the development of materials for electronic packaging, magneto-electro-mechanical transducers, and magnetic sensors. Particularly, applications in magnetic sensors could be very helpful for detecting both ac and dc magnetic fields. This suggests potential usage in magnetic storage and read-out devices, in magnetic imaging technology, and for shielding and protecting database by sensing and shielding from damaging magnetic fields.[1]

The development of piezoelectric-piezomagnetic composites has its roots in the early work of van Schtelen[2] who proposed that the combination of piezoelectric-piezomagnetic phases may exhibit a new material property—the magnetoelectric coupling effect. Since then, the magnetoelectric coupling effect of $BaTiO_3$-$CoFe_2O_4$ composites has been measured by many researchers: van Run *et al.*,[3] van den Boomgaad *et al.*,[4,5] Bracke and van Vliet,[6] among others. They have shown that there indeed exists a remarkable magnetoelectric coupling effect in such piezoelectric-piezomagnetic composites. Much of the theoretical work for the investigation of the magnetoelectric coupling effect has only recently been carried out by Harshe *et al.*,[7] Avellanede and Harshe,[1] Nan,[8] Nan and Jin,[9] Benvensite,[10] and Huang and Kuo.[11] It appears that these approaches have not provided a means to find closed-form solutions of the magnetoelectric coupling effect. Thus, the present work is an attempt to provide the needed information.

In the present paper, a presentation of some notations used, the basic theory, and equations on which the rest of this paper is built is in Sec. II. In Sec. III, the analytical solution for the coupled magneto-electro-elastic behavior of piezoelectric-piezomagnetic composites developed by the author's previous work[11] is utilized to derive a set of nine tensors for an ellipsoidal inclusions in an infinite piezomagnetic matrix. These tensors will be referred to as the magneto-electro-elastic Eshelby tensors analogous to the Eshelby[12] tensor in elasticity. In addition, closed-form expressions for the magneto-electro-elastic Eshelby tensors for some inclusions such as elliptic cylinder, circular cylinder, penny shape, and ribbon embedded in a transversely isotropic ($6mm$ symmetry) piezomagnetic medium are presented. These four inclusions are practically important in applications and are usually discussed together by the micromechanics and composite communities. Section IV takes the results derived in previous sections and applies them to determine the magnetoelectric coupling existing in piezoelectric-piezomagnetic composites. As a result, the magnetoelectric coupling is obtained in closed forms for the composite reinforced by elliptic cylinderical, circular cylinderical, penny-shape, and ribbonlike inclusions, respectively.

II. Effective magneto-electro-elastic moduli. Suppose that a sufficiently large two-phase composite consists of randomly oriented piezoelectric ellipsoidal inhomogeneities $(\Omega_1, \Omega_2, \ldots, \Omega_N)$ with magneto-electro-elastic moduli L^1_{iJMn} and volume fraction f. The surrounding matrix is piezomagnetic and has magneto-electro-elastic moduli L^0_{iJMn}. Hereafter the superscripts 0 and 1 denote quantities in the matrix and the inhomogeneity, respectively. The effective magneto-electro-elastic moduli $\overline{L}_{iJMn}$ of the composite have been obtained by Huang and Kuo[11] through the Mori-Tanaka[13] theory incorporated with the equivalent inclusion method[12] as

$$\overline{L}_{iJMn}=L^{0}_{iJMn}+fL^{0}_{iJAb}V^{-1}_{AbqR}(L^{1}_{qRMn}-L^{0}_{qRMn}), \qquad (1)$$

where V^{-1}_{AbiJ} is the inverse of V_{iJAb} defined by

$$V_{iJAb}=(1-f)(L^{1}_{iJMn}-L^{0}_{iJMn})S_{MnAb}+L^{0}_{iJAb}, \qquad (2)$$

with S_{MnAb} being a collection of nine tensors that are referred to as the magneto-electro-elastic Eshelby tensors analogous to the Eshelby[12] tensor for elastic inclusion problems. As will be seen in the subsequent development, the magneto-electro-elastic Eshelby tensors are the key ingredients necessary for determining the magnetoelectric coupling of piezoelectric-piezomagnetic composites. Consequently, it is necessary to express S_{MnAb} explicitly in terms of magneto-electro-elastic moduli L^{0}_{iJMn} of the piezomagnetic matrix, i.e.,

$$S_{mnab}=\frac{1}{8\pi}\{C^{0}_{ijab}(G_{mjin}+G_{njim})+q^{0}_{iab}(G_{m5in}+G_{n5im})\},$$

$$S_{mn4b}=\frac{-1}{8\pi}\kappa^{0}_{ib}(G_{m4in}+G_{n4im}),$$

$$S_{mn5b}=\frac{1}{8\pi}\{q^{0}_{bij}(G_{mjin}+G_{njim})-\Gamma^{0}_{ib}(G_{m5in}+G_{n5im})\},$$

$$S_{4nab}=\frac{1}{4\pi}(C^{0}_{ijab}G_{4jin}+q^{0}_{iab}G_{45in}),$$

$$S_{4n4b}=-\frac{1}{4\pi}\kappa^{0}_{ib}G_{44in}, \quad S_{4n5b}=\frac{1}{4\pi}(q^{0}_{bij}G_{4jin}-\Gamma^{0}_{ib}G_{45in}),$$

$$S_{5nab}=\frac{1}{4\pi}(C^{0}_{ijab}G_{5jin}+q^{0}_{iab}G_{55in}),$$

$$S_{5n4b}=-\frac{1}{4\pi}\kappa^{0}_{ib}G_{54in},$$

$$S_{5n5b}=\frac{1}{4\pi}(q^{0}_{bij}G_{5jin}-\Gamma^{0}_{ib}G_{55in}), \qquad (3)$$

in which G_{MJin} is defined by[11]

$$G_{MJin}=\int_{-1}^{1}\int_{0}^{2\pi}N_{MJ}(\overline{\xi})D^{-1}(\overline{\xi})\overline{\xi_i\xi_n}\,d\theta\,d\overline{\varsigma}_3, \qquad (4)$$

with $N_{MJ}(\overline{\xi})$ and $D(\overline{\xi})$ being the cofactor and the determinant of the 5×5 matrix $[L^{0}_{iJMn}\xi_i\xi_n]$.

III. Evaluation of the Eshelby tensors. As mentioned earlier, the magneto-electro-elastic Eshelby tensors in Eq. (3) are of fundamental importance in the later evaluation of V^{-1}_{AbqR} and $\overline{L}_{iJMn}$; it is useful to evaluate them explicitly in this section. For an ellipsoidal inclusion embedded in a transversely isotropic ($6mm$ symmetry) piezomagnetic medium, the tensors can be divided into three categories. The first category consists of those tensors related to the elastic response under eigenstrains, i.e., S_{1111}, S_{1122}, S_{1133}, S_{2211}, S_{2222}, S_{2233}, S_{1212}, S_{1313}, S_{2323}, S_{3311}, S_{3322}, and S_{3333}. The second involves those related to the piezomagnetic response due to the initial piezomagnetic fields of the same kind, S_{4141}, S_{4242}, S_{4343}, S_{5151}, S_{5252}, and S_{5353}. The third category includes elastic and magnetic interactive terms S_{1153}, S_{2253}, S_{3353}, S_{1351}, S_{2352}, S_{5113}, S_{5223}, S_{5311}, S_{5322}, and S_{5333}. Noted that the tensors in the first and second categories are dimensionless, while interactive terms in the third category relating dissimilar physical quantities are dimensional.

Next, we attempt to explore the magneto-electro-elastic Eshelby tensors analytically for some practical inclusions in the micromechanics and composite communities such as elliptic cylinder, circular cylinder, penny shape, and ribbon embedded in transversely isotropic piezomagnetic materials. To this end, complete explicit expressions of the determinant $D(\xi)$ and the cofactor $N_{MJ}(\xi)$ of the matrix $[\underline{L}^{0}_{iJMn}\xi_i\xi_n]$ are carried out first, followed by substituting $D(\xi)$ and $N_{MJ}(\xi)$ into G_{MJin} given by Eq. (4). Consequently, complete explicit expressions for the corresponding G_{MJin} are then obtained. Having the explicit expressions of G_{MJin} at hand, closed-form expressions of the magneto-electro-elastic Eshelby tensors for piezomagnetic materials can be obtained as follows.

(a) *Elliptic cylinder* ($a_1/a_2=a$, $a_3\to\infty$, where a_1, a_2, and a_3 are the lengths of the semiaxes of the ellipsoid):

$$S_{1111}=\frac{(2+3a)C^{0}_{11}+aC^{0}_{12}}{2(1+a)^2C^{0}_{11}}, \quad S_{2222}=\frac{(3a+2a^2)C^{0}_{11}+aC^{0}_{12}}{2(1+a)^2C^{0}_{11}},$$

$$S_{1122}=\frac{-aC^{0}_{11}+(2+a)C^{0}_{12}}{2(1+a)^2C^{0}_{11}}, \quad S_{2211}=\frac{-aC^{0}_{11}+(a+2a^2)C^{0}_{12}}{2(1+a)^2C^{0}_{11}},$$

$$S_{1133}=\frac{C^{0}_{13}}{(1+a)C^{0}_{11}}, \quad S_{2233}=\frac{aC^{0}_{13}}{(1+a)C^{0}_{11}},$$

$$S_{1153}=\frac{q^{0}_{31}}{(1+a)C^{0}_{11}}, \quad S_{2253}=\frac{aq^{0}_{31}}{(1+a)C^{0}_{11}},$$

$$S_{1212}=S_{1221}=S_{2112}=S_{2121}=\frac{(1+a+a^2)C^{0}_{11}-aC^{0}_{12}}{2(1+a)^2C^{0}_{11}},$$

$$S_{1313}=S_{1331}=S_{3113}=S_{3131}=\frac{1}{2(1+a)},$$

$$S_{2323}=S_{2332}=S_{3223}=S_{3232}=\frac{a}{2(1+a)},$$

$$S_{4141}=S_{5151}=\frac{1}{1+a}, \quad S_{4242}=S_{5252}=\frac{a}{1+a}. \qquad (5)$$

(b) *Circular cylinder* ($a_1=a_2$, $a_3\to\infty$):

$$S_{1111}=S_{2222}=\frac{5C^{0}_{11}+C^{0}_{12}}{8C^{0}_{11}}, \quad S_{1122}=S_{2211}=\frac{-C^{0}_{11}+3C^{0}_{12}}{8C^{0}_{11}},$$

$$S_{1133}=S_{2233}=\frac{C_{13}^0}{2C_{11}^0},\quad S_{1153}=S_{2253}=\frac{q_{31}^0}{2C_{11}^0},$$

$$S_{1212}=S_{1221}=S_{2112}=S_{2121}=\frac{3C_{11}^0-C_{12}^0}{8C_{11}^0},$$

$$S_{1313}=S_{1331}=S_{3113}=S_{3131}=S_{2323}=S_{2332}=S_{3223}=S_{3232}=\tfrac{1}{4},$$

$$S_{4141}=S_{4242}=S_{5151}=S_{5252}=\tfrac{1}{2}. \qquad (6)$$

(c) Disk ($a_1=a_2,\ a_3\rightarrow 0$):

$$S_{1313}=S_{1331}=S_{3131}=S_{3113}=S_{2323}=S_{2332}=S_{3232}=S_{3223}=\tfrac{1}{2},$$

$$S_{1351}=S_{3151}=S_{2352}=S_{3252}=\frac{q_{15}^0}{2C_{44}^0},$$

$$S_{3311}=S_{3322}=\frac{q_{31}^0 q_{33}^0+C_{13}^0\Gamma_{33}^0}{q_{33}^{0\,2}+C_{33}^0\Gamma_{33}^0},$$

$$S_{5311}=S_{5322}=\frac{C_{13}^0 q_{33}^0-C_{33}^0 q_{31}^0}{2(q_{33}^{0\,2}+C_{33}^0\Gamma_{33}^0)},$$

$$S_{3333}=S_{4343}=S_{5353}=1. \qquad (7)$$

(d) Ribbon ($a_1\ll a_2,\ a_1/a_2=a,\ a_3\rightarrow\infty$):

$$S_{1111}=1-\frac{a(C_{11}^0-C_{12}^0)}{2C_{11}^0},\quad S_{1122}=-\frac{a}{2}+\frac{(2-3a)C_{12}^0}{2C_{11}^0},$$

$$S_{2211}=\frac{a(C_{12}^0-C_{11}^0)}{2C_{11}^0},\quad S_{2222}=\frac{a(3C_{11}^0+C_{12}^0)}{2C_{11}^0},$$

$$S_{1133}=\frac{(1-a)C_{13}^0}{C_{11}^0},\quad S_{2233}=\frac{aC_{13}^0}{C_{11}^0},$$

$$S_{1153}=\frac{(1-a)q_{31}^0}{C_{11}^0},\quad S_{2253}=\frac{aq_{31}^0}{C_{11}^0},$$

$$S_{1212}=S_{1221}=S_{2112}=S_{2121}=\frac{1}{2}-\frac{a(C_{11}^0+C_{12}^0)}{2C_{11}^0},$$

$$S_{1313}=S_{1331}=S_{3113}=S_{3131}=\frac{1-a}{2},$$

$$S_{2323}=S_{2332}=S_{3223}=S_{3232}=\frac{a}{2},$$

$$S_{4141}=S_{5151}=1-a,\quad S_{4242}=S_{5252}=a. \qquad (8)$$

IV. Magnetoelectric coupling effects. Due to the coupling interaction between magnetostrition of piezomagnetic phase and piezoelectricity of piezoelectric phase, the magnetoelectric coupling effect λ_{ij} ($=L_{i45n}=L_{i54n}$) which is absent in each constituent exists in piezoelectric-piezomagnetic composites. The goal of this section is to find analytical expressions for the magnetoelectric coupling effect λ_{ij} in the composite. From Eq. (1) with the aid of the analytical expressions for the magneto-electro-elastic Eshelby tensors in Eq. (3), it is readily shown that

$$\bar{\lambda}_{11}=-f\{2\kappa_{11}^0 q_{15}^0 V_{4113}^{-1}+(\Gamma_{11}^1-\Gamma_{11}^0)V_{4115}^{-1}\},$$

$$\bar{\lambda}_{22}=-f\{2\kappa_{11}^0 q_{15}^0 V_{4223}^{-1}+(\Gamma_{11}^1-\Gamma_{11}^0)V_{4225}^{-1}\}, \qquad (9)$$

$$\bar{\lambda}_{33}=-f\{\kappa_{33}^0 q_{31}^0(V_{4311}^{-1}+V_{4322}^{-1})+\kappa_{33}^0 q_{33}^0 V_{4333}^{-1}+(\Gamma_{33}^1-\Gamma_{33}^0)V_{4335}^{-1}\},$$

and $\bar{\lambda}_{ij}=0$ otherwise. Once the inverse of the tensor V_{iJAb} is obtained, it can be used with Eq. (9) to obtain closed-form solutions of the magnetoelectric coupling coefficients for elliptic cylinderical, circular cylinderical, penny-shaped, and ribbonlike inclusions. After some straightforward but tedious algebraic manipulations, the closed-form solutions are written out compactly as follows.

(a) *Elliptic cylinder* ($a_1/a_2=a,\ a_3\rightarrow\infty$):

$$\bar{\lambda}_{11}=\frac{-(1+a)^2 f(1-f)(1-a)e_{15}^1\kappa_{11}^0 q_{15}^0\Gamma_{11}^1}{(a+f)^2 q_{15}^{0\,2}Y_{11}+(1-f)^2 e_{15}^{1\,2}Y_{12}+Y_{11}Y_{12}Y_{13}},$$

$$\bar{\lambda}_{22}=\frac{-a(1+a)^2 f(1-f)(1-a)e_{15}^1\kappa_{11}^0 q_{15}^0\Gamma_{11}^1}{a^2 e_{15}^{1\,2}(1-f)^2 Y_{21}+(1+af)^2 q_{15}^{0\,2}Y_{22}+Y_{21}Y_{22}Y_{23}}, \qquad (10)$$

$$\bar{\lambda}_{33}=\frac{-2f(1-f)q_{31}^0 e_{31}^1 Y_{30}}{2(1+a^2)Y_{31}+a(1-f)^2 Y_{32}+aY_{33}}.$$

(b) *Circular cylinder* ($a_1=a_2,\ a_3\rightarrow\infty$):

$$\bar{\lambda}_{11}=\bar{\lambda}_{22}=\frac{-4f(1-f)e_{15}^1 q_{15}^0\kappa_{11}^0\Gamma}{(1+f)\kappa_{11}^0 Y_{11}+(1-f)(\kappa_{11}^1 Y_{11}+Y_{12})},$$

$$\bar{\lambda}_{33}=\frac{2f(1-f)e_{31}^1 q_{31}^0}{(1-f)(C_{12}^0-C_{12}^1-C_{11}^1)-(1+f)C_{11}^0}. \qquad (11)$$

(c) *Disk* ($a_1=a_2,\ a_3\rightarrow 0$):

$$\bar{\lambda}_{11}=\bar{\lambda}_{22}=\frac{-f(1-f)e_{15}^1 q_{15}^0}{fC_{44}^0+(1-f)C_{44}^1},$$

$$\bar{\lambda}_{33}=\frac{-f(1-f)e_{33}^1\kappa_{33}^0 q_{33}^0\Gamma_{33}^1 Y_{30}}{Y_{30}Y_{31}+(1-f)Y_{32}}. \qquad (12)$$

(d) *Ribbon* ($a_1\ll a_2,\ a_1/a_2=a,\ a_3\rightarrow\infty$):

$$\bar{\lambda}_{11}=\frac{(1-a)f(1-f)e_{15}^1\kappa_{11}^0 q_{15}^0\Gamma_{11}^1}{(a+f-af)^2 q_{15}^{0\,2}Y_{11}+(1-a)^2(1-f)^2 e_{15}^{1\,2}Y_{12}-Y_{11}Y_{12}Y_{13}},$$

$$\bar{\lambda}_{22}=\frac{af(1-f)(1-a)e_{15}^1\kappa_{11}^0 q_{15}^0\Gamma_{11}^1}{(1-a+af)(Y_{22}-Y_{21}\Gamma_{11}^0)-a(1-f)\Gamma_{11}^1 Y_{21}}, \qquad (13)$$

$$\bar{\lambda}_{33}=\frac{-f(1-f)fq_{31}^0 e_{31}^1 Y_{30}}{2a^2(1-f)^2 Y_{31}+Y_{32}+a(1-f)Y_{33}}.$$

The coefficients $Y_{11}\rightarrow Y_{33}$ in Eqs. (10)–(13) are listed in the Appendix.

It is seen from the above equations that the magnetoelectric coupling coefficients are a function of phase properties, volume fraction, and inclusion shape. It is also of interest to examine the behavior of the present model for the two-phase piezoelectric-piezomagnetic composite in the low (dilute) and high concentration limits. As $f\rightarrow 0$ and $f\rightarrow 1$, the magnetoelectric coupling coefficients vanish. This verifies that the magnetoelectric coupling coefficients are absent in each constituent.

V. Concluding remarks. The most significant work to follow the investigation presented in this work is the development of an analytical prediction for the magnetoelectric coupling coefficients of a piezoelectric-piezomagnetic composite as shown in Eq. (9). The ingredients for such a task are completely contained in the present article. Another valuable result is the closed-form expressions for a set of nine tensors for four practical inclusions in the micromechanics and composite communities: elliptic cylinder, circular cylinder, penny shape, and ribbon. These tensors are referred to as the magneto-electro-elastic Eshelby tensors analogous to Eshelby tensor in elasticity. With these tensors, the magnetoelectric coupling coefficients are then obtained in closed forms as given by Eqs. (10)–(13) for the mentioned four inclusions. These results could provide us with insight into how a piezoelectric-piezomagnetic composite material consisting of inclusions will perform and would be helpful in understanding the magneto-electric-elastic behavior of the composite. Also, the method presented here can be equally applied to piezoelectric materials containing a finite concentration of piezomagnetic inclusions.

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APPENDIX

The coefficients $Y_{11}\rightarrow Y_{33}$ in Eq. (10) are given by

$$Y_{11}=(a+f)\kappa_{11}^{0}+\kappa_{11}^{1}-f\kappa_{11}^{1},\quad Y_{12}=(a+f)\Gamma_{11}^{0}+\Gamma_{11}^{1}-f\Gamma_{11}^{1},$$

$$Y_{13}=aC_{44}^{0}+C_{44}^{1}+fC_{44}^{0}-fC_{44}^{1},$$

$$Y_{21}=\Gamma_{11}^{0}+af\Gamma_{11}^{0}-a(-1+f)\Gamma_{11}^{1},$$

$$Y_{22}=\kappa_{11}^{0}+af\kappa_{11}^{0}-a(-1+f)\kappa_{11}^{1},$$

$$Y_{23}=C_{44}^{0}+aC_{44}^{1}+af(C_{44}^{0}-C_{44}^{1}),$$

$$Y_{30}=-aC_{12}^{0}(C_{12}^{0}+C_{11}^{1}-C_{12}^{1})(1-f),\tag{A1}$$

$$Y_{31}=C_{12}^{0}(C_{12}^{0}-C_{11}^{1}-C_{12}^{1})(C_{12}^{0}-C_{11}^{1}-C_{12}^{1}),$$

$$Y_{32}=aC_{44}^{0}+C_{44}^{1}+fC_{44}^{0}-fC_{44}^{1},$$

$$\begin{aligned}Y_{33}=&C_{11}^{0^3}(1+3f^2)+C_{11}^{0}(1-f)[4C_{12}^{0}C_{12}^{1}+3(1-f)\\&\times(C_{11}^{1}-C_{12}^{1})(C_{11}^{1}+C_{12}^{1})-2fC_{12}^{0}(C_{11}^{1}+3C_{12}^{1})\\&-(1-3f)C_{12}^{0^2}]\\&-C_{11}^{0^2}[(1+f)^2C_{12}^{0}+2(1+f)(C_{12}^{1}-3fC_{11}^{1})].\end{aligned}$$

The coefficients Y_{11} and Y_{12} in Eq. (11) are given by

$$\begin{aligned}Y_{11}=&(1+f)^2q_{15}^{0^2}+\{(1+f)C_{44}^{0}+(1-f)C_{44}^{1}\}\\&\times\{(1+f)\Gamma_{11}^{0}+(1-f)\Gamma_{11}^{1}\},\end{aligned}$$

$$Y_{12}=e_{15}^{1^2}(1-f)\{(1+f)\Gamma_{11}^{0}+(1-f)\Gamma_{11}^{1}\}.\tag{A2}$$

The coefficients Y_{30}, Y_{31}, and Y_{32} in Eq. (12) are given by

$$\begin{aligned}Y_{30}=&2C_{33}^{0}q_{31}^{0^2}-4C_{13}^{0}q_{31}^{0}q_{33}^{0}+(C_{11}^{0}+C_{12}^{0})q_{33}^{0^2}\\&-2C_{13}^{0^2}\Gamma_{33}^{0}(C_{11}^{0}+C_{12}^{0})C_{33}^{0}\Gamma_{33}^{0},\end{aligned}$$

$$Y_{31}=\kappa_{33}^{0}\{C_{33}^{1}\Gamma_{33}^{0}q_{33}^{0}+f[q_{33}^{0^2}+(C_{33}^{0}-C_{33}^{1})\Gamma_{33}^{0}]\},\tag{A3}$$

$$\begin{aligned}Y_{32}=&2\kappa_{33}^{0}(C_{33}^{0}q_{31}^{0}-C_{13}^{0}q_{33}^{0})\{fC_{33}^{0}q_{31}^{0}+(1-f)C_{33}^{1}q_{31}^{0}\\&-[fC_{13}^{0}+(1-f)C_{13}^{1}]q_{33}^{0}\}\Gamma_{33}^{1}.\end{aligned}$$

The coefficients Y_{31}, Y_{32}, and Y_{33} in Eq. (13) are given by

$$\begin{aligned}Y_{31}=&(C_{11}^{0}-C_{12}^{0})(C_{11}^{0}+C_{12}^{0}-C_{11}^{1}-C_{12}^{1})\\&\times(C_{11}^{0}-C_{12}^{0}-C_{11}^{1}+C_{12}^{1}),\end{aligned}$$

$$Y_{32}=-2C_{11}^{0}(C_{11}^{0}-C_{12}^{0})(C_{11}^{1}+fC_{11}^{1}-fC_{11}^{1}),$$

$$\begin{aligned}Y_{33}=&(C_{11}^{0}-C_{12}^{0}-C_{11}^{1}+C_{12}^{1})\{(1-f)C_{12}^{0}(C_{12}^{0}-C_{11}^{1}-C_{12}^{1})\\&-(1-3f)C_{11}^{0^2}+C_{11}^{0}[3C_{11}^{1}+3C_{12}^{1}+2fC_{12}^{0}\\&-3f(C_{11}^{1}+C_{12}^{1})]\}.\end{aligned}\tag{A4}$$

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