Resistance of Josephson-junction arrays at low temperatures

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We study the motion of vortices in arrays of Josephson junctions at zero temperature where it is controlled by quantum tunneling from one plaquette to another. The tunneling process is characterized by a finite time and can be slow compared to the superconducting gap (so that $\tau\Delta \gg 1$). The dissipation which accompanies this process arises from rare processes when a vortex excites a quasiparticle above the gap while tunneling through a single junction. We find that the dissipation is significant even in the case $\tau\Delta \gg 1$; in particular it is not exponentially small in this parameter. We use the calculated energy dissipation for the single vortex jump to estimate the physical resistance of the whole array. [S0163-1829(98)03938-1]

I. INTRODUCTION

In recent years dynamics of Josephson-junction arrays has attracted much interest.[1–6] The Josephson-junction arrays (which are artificially fabricated networks of superconducting islands weakly coupled by tunnel junctions) became model systems for the study of quantum phase transitions, i.e., transitions occurring at $T \to 0$.

The simplest physical picture of the phase transition in a two-dimensional short-ranged Josephson-junction array is the following. The temperature is lower than the bulk transition temperature of the islands, so that each individual island is superconducting and is characterized by a phase of the superconducting order parameter. The absolute value of the order parameter, the superconducting gap Δ, is the largest energy scale in the problem. The phase variable is conjugate to the Cooper pair charge on the island. When the phase is well defined, the charge fluctuates and the array is superconducting. That happens in the limit where the Josephson energy E_J, associated with the Cooper pair tunneling, is much greater than the Coulomb energy E_C, which determines the electrostatic coupling between the islands that tends to localize the charge carriers. In terms of vortices that means that in the limit $E_J \gg E_C$ the vortices form the Abrikosov lattice. In the opposite limit, $E_C \gg E_J$, the Coulomb blockade pins Cooper pairs to the islands, so at low temperatures the array is insulating. Since in this phase the charge is fixed, the phase variable fluctuates and vortices form a superfluid.

Both phases were observed by preparing samples with different values of E_C and E_J.[1] The insulating phase exhibits high values of resistance at finite temperatures, which grow as $T \to 0$. The opposite behavior indicates the superconducting phase. The transition can also be induced in the same sample by varying magnetic field. The field-induced transition can be experimentally observed in arrays[2] and also in granular superconducting films.[3]

The conventional theoretical picture of the superconductor to insulator (S-I) transition suggested by Fisher[5] is based on duality between vortices and charges. In this picture the transition point between the two phases is characterized by finite resistance, which is predicted to have universal value, proportional to the quantum resistance $R_q = h/4e^2$.

Experimentally reported values[1–3] of the transition point resistance, however, while being of the same order as the predicted universal value, differ by as much as a factor of 5. Moreover, recent experiments[7] show that the superconducting and insulating phases are separated by the wide metallic region, characterized by nonzero dissipation. In particular it was found that at low temperatures ($T < T_0 = 100$ mK) and in a noncommensurate magnetic field array resistance becomes temperature independent and remains finite down to the lowest temperatures accessible (10 mK).

The metallic behavior of the arrays cannot be described by the usual duality picture, since it ignores the presence of dissipation. Two issues have to be addressed. In terms of vortices, a metal corresponds to a *normal liquid*, rather than the superfluid which characterizes an insulator. Vortices, however, are interacting bosons and at low temperatures tend to form the Bose condensate. Therefore the first question is how can the zero-temperature normal liquid exist. The second question is what is the origin of dissipation at zero temperature.

In this paper we will focus on the second question. We consider vortex motion at zero temperature where it is controlled by quantum tunneling of single vortices from one plaquette to another. It turns out that during the tunneling process a vortex can excite a quasiparticle state above the gap with the probability which is not exponentially small in the parameter $\tau\Delta \gg 1$, where Δ is the superconducting gap on the island and τ is the tunneling time. The relaxation of the excited quasiparticle then provides the dissipation in the system.

In order to calculate the matrix element for quasiparticle excitation during vortex tunneling we first solve a simpler quantum-mechanical problem. We consider a particle in a quasiclassical potential barrier which is also coupled to a single harmonic oscillator. The probability to tunnel through the barrier is given (in the simplest approximation) by the WKB approach. The initial state of the whole system (the particle and the oscillator) is that before particle tunneling the oscillator was in its ground state. After the tunneling the oscillator could remain in its ground state or it could be in one of its excited states; the latter case corresponds to dissipation because for any nonzero coupling to environment the oscillator will eventually relax to the ground state. Note that this relaxation cannot affect the tunneling since it has already

happened. For such a problem the dissipation is determined by the *conditional probability* of the oscillator excitation (given the fact of the tunneling) which we calculate below in Sec. IV.

The solution of this quantum-mechanical problem can be applied to the case of vortex tunneling. When the vortex moves the phase on the islands changes. The time derivative of the phase acts as an effective field acting on the quasiparticles and thus may result in quasiparticle excitations. We note that the processes of excitation of different quasiparticle modes are independent, so the result for the total dissipation is given by the sum over all modes.

Besides the calculation of the matrix element, we have to make sure that in the process of quasiparticle excitation the energy is conserved. In real arrays[7] the quasiparticle gap Δ is larger that both E_C and E_J, so at zero temperature a single vortex does not have enough energy to excite a quasiparticle. However, in the vortex liquid dissipation does happen. Experimental evidence[8] suggests that the vortex lattice melts easily due to frustration (since in the incommensurate magnetic field the vortex lattice does not match the underlying array). That means that kinetic energy of vortices in the liquid is rather small, so that the liquid retains the short-range order. This viewpoint is supported by numerical simulations[9] of two-dimensional (2D) melting, which show that below the melting point the vortices are mostly in large, ordered clusters. When vortices move, these clusters move as a whole and the extra momentum due to the interaction with quasiparticles in the island is transferred to the whole cluster. The energy of the cluster is much larger than the gap in the quasiparticle spectrum, therefore the energy conservation is satisfied.

The rest of the paper is organized as follows. In Sec. II we describe Josephson-junction arrays in terms of phase variables. In Sec. III we derive the form of the interaction between vortices and quasiparticles. In Sec. IV we solve a quantum-mechanical problem of a particle in a barrierlike potential coupled to a harmonic oscillator, and in Sec. V apply the results to the vortex tunneling. In Sec. VI we discuss the vortex lattice melting in the presence of frustration. In Sec. VII we obtain the array resistance due to the dissipation found in Sec. V. The conclusions follow in Sec. VIII.

II. JOSEPHSON-JUNCTION ARRAYS

The microscopic theory of superconductivity in each individual island will be reviewed in the next section; here we shall describe an array of small superconducting islands assuming that the amplitude of the order parameter in each island is constant and it is entirely controlled by a single phase variable, i.e., we ignore its spatial dependence on the length scale of the size a of the islands. This is true when the magnetic field does not penetrate the bulk of the islands, which is guaranteed by the condition that the flux through one island is less than the flux quantum $Ha^2 < \phi_0$.

The array Hamiltonian consists in general of three parts. Time variations of the phase in each island result in voltage differences between the islands, which are electrostatically coupled to each other and to the ground plane. That defines the first part of the Hamiltonian, the electrostatic energy as $(1/2)\Sigma_{ij}\tilde{C}_{ij}^{-1}\dot{\phi}_i\dot{\phi}_j$, where $\tilde{C}_{ij}$ is proportional to a capacitance matrix $\tilde{C}_{ij}^{-1}=(2e)^2C_{ij}^{-1}$. In the real experiment[7] the main contribution comes from the junction capacitance C. Taking into account also the self-capacitance C_0 (the capacitance to the ground plane) we approximate C_{ij} by a matrix in which only nonzero elements are diagonal $C_{ii}=C_0+4C$ and those corresponding to nearest neighbors in the array $C_{ij}bf =-C$. The junction capacitance defines the energy scale $E_C=e^2/2C$, which is usually referred to as the charging energy.

The second part of the array Hamiltonian is the Josephson coupling between the neighboring islands. The coupling defines the other energy scale in the system E_J. The dissipation arises from the coupling of the phase variable to some other degrees of freedom in the array. We will denote that part of the Hamiltonian as $H_{\rm int}$ and will derive its form in the next section, where we consider coupling of the time-dependent phase to the quasiparticles in the islands. The Hamiltonian therefore is given by

$$H=\frac{1}{2}\sum_{ij}\tilde{C}_{ij}^{-1}\dot{\phi}_i\dot{\phi}_j-E_J\sum_{\langle ij\rangle}\cos(\phi_i-\phi_j)+H_{\rm int}. \qquad (1)$$

At small Coulomb energy the ground state of Eq. (1) corresponds to the constant phase. Excitations around that ground state are small spin-wave-like fluctuations and topological defects, vortices, where the sum of all gauge-invariant phases around a vortex adds up to 2π. In the superconducting state, while the Coulomb energy is small, vortices appear in bound pairs (with antivortices). As the Coulomb energy increases, pairs unbound, resulting in the transition to the insulating state of the array. In the field-tuned transition, E_C is kept constant and vortices are created by the magnetic field. In the superconducting state they form a lattice, which melts at the transition point. Melting occurs at E_C which is smaller than needed to unbound the vortex-antivortex pairs. The density of vortices is controlled by magnetic field and remains small even in the liquid phase. The flow of this vortex liquid results in finite resistivity.

Vortices forming this liquid are distinguished by two important features. First, they do not have a normal core region, which would be the source of dissipation in homogeneous superconductors. Second, since the Coulomb energy (which is a measure for the kinetic energy of vortices) is small, the vortex motion at low temperatures is due to quantum-mechanical tunneling through the cosine potential.

The tunneling rate can be determined by calculating the instanton action corresponding to a vortex moving from one site to the neighboring site. The instanton action was determined by several authors[10–12] each for slightly different models without dissipation, which also differ from Eq. (1) by another form of capacitance matrix; note, that theoretical calculation[10] involves also approximation of the cosine Josephson interaction by a piecewise parabolic potential, i.e., the Villain's approximation. The results are similar, the instanton action is $S_{\rm inst}=\alpha\sqrt{E_J/E_C}$, where the number α is of the order of unity and depends on a particular model. To determine the resistivity in the array we need to take into account also the dissipation [described by the $H_{\rm int}$ term in Eq. (1)], which is the main subject of this paper.

For better comparison with experiments we need to determine the constant α in the realistic model with experimental values of coupling constants. We have repeated the direct numerical instanton action calculation for the Hamiltonian Eq. (1) without the dissipation term $H_{\rm int}$. First we find the phase configuration in the array corresponding to one vortex in a particular (arbitrary, but known) plaquette. To do that we set the magnetic field through the array so that the total flux is exactly equal to one flux quantum and then minimize the energy Eq. (1). Tunneling corresponds to changing the phase configuration to the one with the vortex in the neighboring plaquette. In terms of phases the vortex tunneling can be described as tunneling of each individual phase in the vortex configuration from it's value corresponding to the original position of the vortex to the value corresponding to the final position of the vortex. We set these two vortex configurations as the boundary conditions for time evolution of individual phases in the array and minimize the action, corresponding to the Hamiltonian Eq. (1), taking the phases to be functions of imaginary time.

For array sizes 6×6, 8×8, and 10×10 we determined the value of the coefficient $\alpha=0.7$ (for $C_0=0$). The phases that change the most during tunneling are the ones in the plaquette with the vortex. Therefore even with relatively small array sizes the calculation gives the answer that does not change with increasing the size.

The tunneling rate $\Gamma_0\sim\exp(-S_{\rm inst})$ is then given by

$$\Gamma_0\sim\frac{1}{\hbar}\sqrt{E_J E_C}\exp(-0.7\sqrt{E_J/E_C}) \tag{2}$$

and provides a measure for the vortex mass.

III. SUPERCONDUCTIVITY IN A SINGLE ISLAND

In this section we briefly derive the $H_{\rm int}$ part of the phase Hamiltonian Eq. (1) which couples phase fluctuations to quasiparticles. In this derivation we follow the standard microscopic description of superconductivity based on the BCS Hamiltonian.[13,14]

We start with the BCS Hamiltonian with an effective local, attractive interaction

$$H_{\rm BCS}=-\int d^3x\,\psi_\sigma^\dagger(x)\frac{\nabla^2}{2m_e}\psi_\sigma(x) - \frac{g_0}{2}\int d^3x\,\psi_\sigma^\dagger(x)\psi_{-\sigma}^\dagger(x)\psi_{-\sigma}(x)\psi_\sigma(x). \tag{3}$$

A summation over spins is implied. The order parameter $\Delta(x,\tau)$ is introduced by means of the Hubbard-Stratonovich transformation. The grand canonical partition function $Z_G=\mathrm{Tr}_\psi\{\exp[-\beta(H-\mu N)]\}$ becomes

$$Z_G=\mathrm{Tr}_\psi\left\{\int\mathcal{D}[\Delta,\Delta^*]T\,\exp\left(-\int_0^\beta d\tau H_{\rm eff}(\tau)\right)\right\}, \tag{4}$$

where the effective Hamiltonian is given by

$$H_{\rm eff}(\tau)=\int d^3x\left\{\psi_\sigma^\dagger(x)\left(-\frac{\nabla^2}{2m_e}-\mu\right)\psi_\sigma(x) - \Delta^*(x,\tau)\psi_\uparrow(x)\psi_\downarrow(x) - \Delta(x,\tau)\psi_\downarrow^*(x)\psi_\uparrow^*(x)+\frac{1}{g_0}|\Delta(x,\tau)|^2\right\}. \tag{6}$$

We can compactify our notations by introducing Nambu spinor[15]

$$\hat\psi=\begin{pmatrix}\psi_\downarrow\\ \psi_\uparrow^*\end{pmatrix}. \tag{7}$$

The effective Hamiltonian becomes

$$H_{\rm eff}(\tau)=\int d^3x\left[\hat\psi^\dagger(K\hat\tau_3-\hat\Delta)\hat\psi+\frac{1}{g_0}|\Delta(x,\tau)|^2\right], \tag{8}$$

where $K=-\nabla^2/2m_e-\mu$ is the kinetic operator and the order parameter is described by the matrix $\hat\Delta=|\Delta|e^{-i\phi\hat\tau_3}\hat\tau_1$, $|\Delta|$ and ϕ are the absolute value and the phase of the order parameter. In our approximation $|\Delta|$ is constant and $\phi(\tau)$ is a function of time.

The Hamiltonian is now quadratic in fermion fields and we can formally perform the trace over the fermion variables in Eq. (4). The partition function becomes the integral over the order parameter

$$Z_G=\int\mathcal{D}[\Delta,\Delta^*]\exp(-S[\Delta]), \tag{9}$$

where the action is

$$S[\Delta]=-\mathrm{Tr}\,\ln\,\hat G^{-1}+\int_0^\beta d\tau\frac{1}{g_0}|\Delta(\tau)|^2. \tag{10}$$

Here $\hat G$ is a 2×2 matrix Green's function in the particle-hole space[15] typical for superconductivity where the inverse is given by

$$\hat G^{-1}(x,\tau;x',\tau')=\left\{-\frac{\partial}{\partial\tau}-K\hat\tau_3+\hat\Delta\right\}\delta(x-x')\,\delta(\tau-\tau'). \tag{11}$$

The action Eq. (10) is a standard BCS action written in a form convenient for the following. It does not contain any dissipation as yet, therefore there are no nonlocal terms discussed in Ref. 14. The dissipation appears after an additional assumption about time dependence of the phase variable, namely that while on average it changes slowly, this change occurs with rare but large enough jumps, due to the lattice structure of the array. Thus in the following we shall not assume that $\phi(t)$ is a smooth function of time; such an assumption would eliminate all dissipation sources in this problem.

The dependence of the Green's function (and therefore the action) on the phase of the order parameter can be displayed through the gauge transformation

$$\mathcal{G}^{-1}(x,\tau;x',\tau')=e^{-i\phi\hat\tau_3/2}\hat G^{-1}(x,\tau;x',\tau')e^{i\phi\hat\tau_3/2}, \tag{12}$$

where $\mathcal{G}^{-1}$ is obtained from Eq. (11) by the replacement $\partial/\partial\tau\to\partial/\partial\tau-(i/2)(\partial\phi/\partial\tau)\hat\tau_3$. This transformation shows that a constant ϕ contributes nothing to the action Eq. (11).

The fermion contribution to the action can be represented by the path integral over Grassman variables

$$\exp(\mathrm{Tr}\,\ln \mathcal{G}^{-1}) = \int \mathcal{D}[\psi_\sigma]\exp(-S_\psi), \tag{13}$$

where the fermion action

$$S_\psi = \int d\tau d\tau' \int d^3x d^3x' \hat{\psi}^\dagger \mathcal{G}^{-1}(x,\tau;x',\tau')\hat{\psi} \tag{14}$$

is explicitly given by

$$S_\psi = \int d\tau d^3x \hat{\psi}^\dagger \left[-\left(\frac{\partial}{\partial\tau} - \frac{i}{2}\frac{\partial\phi}{\partial\tau}\hat{\tau}_3 \right) - K\hat{\tau}_3 + |\Delta|\hat{\tau}_1 \right]\hat{\psi}. \tag{15}$$

At zero temperature we can write the real time action as

$$S_\psi = i\int dt d^3x \hat{\psi}^\dagger \left[-i\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial\phi}{\partial t}\hat{\tau}_3 - K\hat{\tau}_3 + |\Delta|\hat{\tau}_1 \right]\hat{\psi}. \tag{16}$$

The corresponding Hamiltonian in momentum space is given by the 2×2 matrix

$$H_\psi = \begin{pmatrix} -\epsilon_k - \varphi & |\Delta| \\ |\Delta| & \epsilon_k + \varphi \end{pmatrix}, \tag{17}$$

where $\varphi = (1/2)(\partial\phi/\partial t)$. In the BCS theory $\varphi = 0$ and the Hamiltonian can be diagonalized by the Bogolyubov transformation, which is just a rotation of the fermion variables. When $\varphi \neq 0$ we still can perform the rotation, but the resulting action will no longer be diagonal due to the time dependence of φ. The rotation matrix, which diagonalizes the Hamiltonian at each moment of time is

$$\mathcal{R} = \frac{1}{\sqrt{2\lambda_k(\lambda_k+\epsilon_k+\varphi)}} \begin{pmatrix} \lambda_k+\epsilon_k+\varphi & -|\Delta| \\ |\Delta| & \lambda_k+\epsilon_k+\varphi \end{pmatrix}. \tag{18}$$

After the rotation the Hamiltonian becomes diagonal

$$\hat{\mathcal{H}} = \begin{pmatrix} -\lambda_k & 0 \\ 0 & \lambda_k \end{pmatrix}, \tag{19}$$

with the eigenvalue $\lambda_k = \sqrt{(\epsilon_k+\varphi)^2 + |\Delta|^2}$. The action becomes

$$S_\gamma = i\int dt d^3k \hat{\gamma}^\dagger(k,t) \left[-i\frac{\partial}{\partial t} + \hat{\mathcal{H}} - \frac{i}{2\lambda_k(\lambda_k+\epsilon_k+\varphi)} \times \begin{pmatrix} \epsilon_k+\varphi & -|\Delta| \\ |\Delta| & \epsilon_k+\varphi \end{pmatrix} \frac{\partial\varphi}{\partial t} \right] \hat{\gamma}(k,t), \tag{20}$$

where $\hat{\gamma}(k,t)$ are the variables in the rotated basis (which for $\varphi = 0$ correspond to Cooper pairs).

The additional term in Eq. (20) appeared due to the time dependence of φ. The diagonal part is small compared to the eigenvalues λ and can be ignored. The nondiagonal part, however, describes a new process: a quasiparticle excitation [the Cooper pairs correspond to diagonal part of Eq. (20)]. This is the interaction term which is responsible for dissipation. Upon integrating out the fields $\hat{\gamma}(k,t)$ it becomes the interaction part of the action, corresponding to the H_{int} part of the array Hamiltonian Eq. (1).

$$S_{\mathrm{int}} = i\int dt d^3k \hat{\gamma}^\dagger(k,t) \frac{1}{2(\lambda_k+\epsilon_k)} \begin{pmatrix} 0 & -|\Delta| \\ |\Delta| & 0 \end{pmatrix} \frac{\partial\phi}{\partial t}\hat{\gamma}(k,t). \tag{21}$$

Here we have integrated the interaction term by parts, in order to express the result in terms of the phase fluctuations ϕ. This brings the extra factor $2\lambda_k$ from the time dependence of $\hat{\gamma}(k,t)$. Also, we neglected φ in the prefactor which forms the coupling constant $g_k \approx |\Delta|/2(\lambda_k+\epsilon_k)$ because we consider only adiabatically slow motion.

The phase now can be treated as independent variable, describing a ''particle'' in the periodic potential and coupled to the quasiparticles in the island through the action Eq. (21). Note that the action Eq. (21) is diagonal in momentum k, so for each k the quasiparticle action is that of a two-level system. Since the probability to excite a quasiparticle is small, the phase fluctuations excite only one two-level system at a time, so these excitation processes are independent and the total probability can be found as a sum of probabilities to excite each individual two-level system. Therefore we can consider the quantum-mechanical problem of a particle coupled to the two-level system and then integrate the results over k.

Furthermore, dissipation resulting from exciting a two-level system is not different from the one of an oscillator because excitations of the latter to higher levels can be neglected. The latter problem has a slightly broader application. Note, however, the important difference between this problem and the Caldeira-Legget[16] model. Here the motion of the particle is coupled to a single oscillator with large level spacing, which corresponds to the large quasiparticle gap, whereas the Caldeira-Legget[16] model is a system coupled to a large number of small oscillators, so it is not difficult to excite each one individually and interesting physics arise from exciting a large number of them simultaneously.

Thus we reduced our problem of calculating the probability to excite a quasiparticle in an island to a problem of exciting a harmonic oscillator during tunneling. In the next section we consider this simpler problem and then in the following section we apply the obtained results to the case of coupling to a two-level system and then sum the probability over momenta k to obtain the final probability to excite a quasiparticle.

IV. SIMPLE MODEL—PARTICLE COUPLED TO HARMONIC OSCILLATOR

In this section we consider the quantum-mechanical problem of a particle coupled to a harmonic oscillator and tunneling through some barrier. The Hamiltonian is

$$H = \frac{\hat{p}^2}{2m} + V(x) + \frac{\hat{P}^2}{2M} + \frac{1}{2}M\omega_0^2 Q^2 + g\hat{p}Q, \tag{22}$$

where $\hat{p}$ and $\hat{P}$ are momentum operators of the particle and the oscillator, m and M are their respective masses, $V(x)$ is the potential which we assume has the form of a barrier, and

g is the coupling constant. Here we chose the coupling (using the momentum operator $\hat{p}$ rather then the position operator) which has the same form as the one in the action Eq. (21). We need to obtain the probability to find the oscillator in its first excited state after the particle has tunneled through the barrier, if before the tunneling the oscillator was in the ground state. The coupling g is taken to be small enough so that the oscillator states are unchanged.

We look for the wave function of the system in the form

$$\Phi=\sum_n \Psi_n(x)|n\rangle, \tag{23}$$

where $|n\rangle$ denotes oscillator wave functions corresponding to the nth energy level.

Neglecting the coupling completely we should have

$$\Phi_0=\Psi_0(x)|0\rangle, \tag{24}$$

which means that the oscillator is in the ground state. The particle wave function under the barrier is given in the WKB approximation by

$$\Psi_0(x)\approx u_0(x)\exp\left(-\int_{x_a}^{x}\sqrt{2m(V-E)}dx\right). \tag{25}$$

In the first order in g we have for the wave function $\Phi_1=\Psi_0(x)|0\rangle+\Psi_1(x)|1\rangle$. The Schrödinger equation for the correction $\Psi_1(x)$ is

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V(x)-E+\omega_0\right]\Psi_1(x)-igQ_{10}\frac{\partial}{\partial x}\Psi_0(x)=0, \tag{26}$$

where $Q_{10}=\langle 1|Q|0\rangle=1/\sqrt{2M\omega_0^2}$ is the oscillator matrix element. The oscillator ground-state energy $\omega_0/2$ is included in the definition of E. It is convenient to express the solution in the form $\Psi_1(x)=u_1(x)\Psi_0(x)$ with the boundary condition $u_1(x_a)=0$, noting that the oscillator was in the ground state prior to tunneling. The equation becomes

$$\frac{\Psi_0}{2\Psi_0'}u_1''+u_1'-m\omega_0\frac{\Psi_0}{\Psi_0'}u_1+imgQ_{10}=0. \tag{27}$$

Compare now first and second derivative terms. The typical particle energy is $\Omega=\sqrt{V/mL^2}$, where $L\sim(x_b-x_a)$. The typical time $\tau\sim 1/\Omega$. The ratio Ψ_0/Ψ_0' can be estimated using Eq. (25) as $1/\sqrt{mV}$. Therefore the second derivate term can be estimated as $(\Psi_0/2\Psi_0')\,u_1''\sim u_1\,(1/L^2\sqrt{mV})$. The first derivative term is simply $u_1'\sim u_1/L$, so that their ratio is $(\Psi_0/2\Psi_0')\,u_1''/u_1'\sim(1/L\sqrt{mV})\ll 1$. Therefore we can drop the second derivative term in Eq. (27). Solving the remaining first-order equation we get for the function u_1 right after the tunneling

$$u_1(x_b)=-imgQ_{10}\int_{x_a}^{x_b}\left[\exp\left(-\int_x^{x_b}\frac{m\omega_0}{\sqrt{2m(V-E)}}ds\right)\right]dx. \tag{28}$$

Consider now two limiting cases. When $\omega_0\ll\Omega$ (fast transition), the integrand in the exponential in Eq. (28) is small, therefore $u_1(x_b)\approx -imgQ_{10}L$ so there is no additional suppression other than the smallness of g. In the opposite limit $\omega_0\gg\Omega$ we can evaluate the integral in the exponential noting that the main contribution comes from the region near the ending point x_b. We can then expand the integrand around x_b to get $m\omega_0/\sqrt{2m(V-E)}\sim(2m\omega_0/\sqrt{2mV'})\sqrt{x_b-x}$, where $V'=dV/dx\,|_{x=x_b}$. Evaluating the integral is now straightforward and we get

$$u_1(x_b)=-imgQ_{10}L\frac{V'}{m\omega_0 L}\sim -imgQ_{10}L\left(\frac{\Omega}{\omega_0}\right)^2, \tag{29}$$

which means that in the limit of slow tunneling $u_1(x_b)$ is indeed small in Ω/ω_0 but only as a power law.

The probability of exciting the oscillator is proportional to the square of $u_1(x_b)$ and is given by

$$\mathcal{P}\approx g^2m^2L^2Q_{10}^2\left(\frac{\Omega}{\omega_0}\right)^4. \tag{30}$$

The average energy $\mathcal{W}$ dissipated in one jump is equal to the energy needed to excite the oscillator (ω_0) times the transition probability Eq. (30)

$$\mathcal{W}\approx(gmLQ_{10})^2\omega_0\left(\frac{\Omega}{\omega_0}\right)^4. \tag{31}$$

V. TUNNELING OF PARTICLE COUPLED TO QUASIPARTICLE SYSTEM

We now return to the full problem, formulated in Sec. II, namely to the action Eq. (21). As we have already mentioned, for each value of k we can treat the quasiparticle system as a two-level system and then sum over all possible k. Again we treat the phase variable on the island as a coordinate of a ''particle,'' which tunnels through some barrier. The result for this case is exactly the same as for the case of the oscillator since we have neglected excitations of higher levels. In the case of a two-level system there are no higher levels at all and the result Eq. (31) is the full answer, in which we have to substitute the corresponding matrix element for Q_{10} and the value of the gap for ω_0.

Therefore in order to apply our results to the case of the phase coupled to the quasiparticle system Eq. (21) we only need to rewrite it in a Hamiltonian form equivalent to Eq. (22). In the Hamiltonian formalism the derivative $\partial\phi/\partial t$ in Eq. (21) is replaced by the momentum operator $\hat{p}/m$ (m is the ''particle'' mass) leading to the effective interaction constant

$$g_k=\frac{1}{m}\frac{|\Delta|}{2(\lambda_k+\epsilon_k)}. \tag{32}$$

The energy Ω is now the typical frequency of the phase variation and is equal to the inverse tunneling time, $\Omega\approx\sqrt{E_JE_C}$.

The probability to excite the quasiparticle at each k then follows from Eq. (30). The oscillator frequency ω_0 is now exchanged for the quasiparticle gap, which is $2\lambda_k$ at the same k. The matrix element corresponding to Q_{10} is just 1. The probability then is

$$\mathcal{P}_k \approx (g_k m 2\pi)^2 \left(\frac{\Omega}{2\lambda_k}\right)^4. \tag{33}$$

Here we used the factor of 2π for the effective length L, which is by how much the phase can be changed. The dissipation contribution for each k follows by multiplying the probability by the energy gap

$$\mathcal{W}_k \approx (g_k m 2\pi)^2 2\lambda_k \left(\frac{\Omega}{2\lambda_k}\right)^4. \tag{34}$$

Integrating this expression over momentum we get

$$\mathcal{W} \approx \pi^5 \Omega \left(\frac{\Omega}{|\Delta|}\right)^3 \mathcal{N}, \tag{35}$$

where $\mathcal{N}$ is the number of particles on the island. This defines the energy transferred to the quasiparticle system during tunneling and therefore the dissipation in a single vortex jump between two neighboring plaquettes.

VI. VORTEX LIQUID

At low magnetic fields vortices form a lattice that melts at higher fields. Because melting is due to the competition between kinetic and interaction energies, it happens when the two are parametrically equal. However in 2D the liquid retains short-range order and the interaction energy loss in melting is numerically much smaller as described by a small Lindemann number. In arrays the vortex lattice is frustrated by the incommensurability with the underlying array structure. This effect reduces the ratio of the kinetic energy over the interaction energy at melting even further.

In the absence of a microscopic theory of melting we use the phenomenological Lindemann criterion, which describes the melting in terms of elastic constants of the vortex lattice. In the array system these constants are renormalized by frustration. To estimate this effect we analyze the experiment on thermal melting,[8] in which the effect of incommensurability on the transition temperature was studied in detail. We emphasize that these measurements were performed on array systems, which are different from the ones discussed throughout this paper. Here we use these experiments to obtain estimates of the renormalization of the elastic constants of the vortex lattice and then use this renormalization to describe quantum melting. Our observations however are general and therefore are applicable to the quantum systems of interest.[7]

We need to estimate how frustration renormalizes the elastic constants. Therefore we estimate the interaction energy in the experimental system,[8] which translates into unrenormalized values of elastic constants, leading to an estimate of unrenormalized melting temperature T_{m0}. Comparing T_{m0} with the experimentally observed T_m we find the frustration factor.

To estimate the interaction energy we relate the superfluid density ρ_s to the observed magnetization M. Energy and current in the Josephson junction in magnetic field can be written as

$$E = \frac{\hbar}{2e} J_1 \cos\varphi, \quad J = J_1 \sin\left(\varphi + \frac{2eAa}{\hbar c}\right), \tag{36}$$

where A is the vector potential and a is the distance between islands. We relate the Josephson energy $E_J = (\hbar/2e)\, J_1$ to the magnetization M at low fields, where the response is linear. We use the relation $j = \rho_s A$ between the supercurrent and the superfluid density ρ_s, and express M via the current. Assuming for simplicity circular geometry the magnetization is given by

$$M = \frac{1}{2c} \int_0^L 2\pi r dr (\vec{j} \times \vec{r}). \tag{37}$$

Taking the integral we relate the magnetization to the magnetic field H and the superfluid density ρ_s

$$M = \frac{1}{2c} \rho_s \frac{\pi}{4} L^4 H. \tag{38}$$

We use Eq. (38) to deduce the value of ρ_s from the data;[8] for its zero-temperature value we get $\rho_s(0) = 6.49 \times 10^{15}$. The superfluid density is a function of reduced temperature, $\rho_s(T) = \rho_s(0)\tau$. Since we need the superfluid density at the true melting temperature T_m, τ is defined from the shift in melting temperature due to frustration and was measured to be $\tau = 0.01$.

Comparing the supercurrent equation with Eq. (36) we obtain the Josephson energy (and the magnitude of current)

$$E = \frac{\hbar}{2e} J_1 = \frac{\hbar^2 c}{4e^2} \rho_s(T_m). \tag{39}$$

We can now estimate the interaction energy to be $E \approx 1.7 \times 10^4$ K. Because the melting temperature is of the order of 1 K, its ratio to the interaction energy $T_c/E = \zeta$ is estimated as $\zeta = 10^{-4}$.

We now compare Lindemann criterion for thermal and quantum melting. For the thermal melting considered above we have

$$\langle \rho\rho \rangle \sim T \int \frac{d^2q}{c_{66} q^2} = a_L^2, \tag{40}$$

where the integral is over the Brillouin zone. Here we have also substituted $c_{66}q^2$ for the actual dispersion law. This rough estimate will be sufficient for our purposes. Assuming that frustration renormalizes the elastic constant by $c_{66} \sim \kappa \rho_s$, we get the renormalization factor $\kappa \sim T/\rho_s a_L^2 \approx \zeta/a_L^2$. Taking for the Lindemann parameter the usual value $a_L \approx 0.1$ we get $\kappa = 0.01$.

For quantum melting we conjecture that because the renormalization κ is due to frustration (induced by the array) the elastic constant will be renormalized by the same factor.

$$\langle \rho\rho \rangle \sim \int \frac{d^2q\, d\omega}{(\omega^2/E_C) + c_{66} q^2} \approx \sqrt{\frac{E_C}{c_{66}}} = \tilde{a}_L^2. \tag{41}$$

Using the renormalization factor κ we get $E_C/\rho_s \sim \kappa \tilde{a}_L^4$. Thus we expect that the quantum melting happens at kinetic energies which are at least three orders of magnitude smaller than the interaction energy.

VII. DISSIPATION EFFECTS ON VORTEX MOTION

Before we turn to the estimate of array resistance, we have to address the question of energy conservation. The quasiclassical calculation of the probability to excite a quasiparticle considered above assumed implicitly that the energy of the vortex is large enough. Quantitatively that means that the vortex energy should at least be larger than the quasiparticle gap, otherwise the vortex would lack the energy to excite the quasiparticle.

The energy acquired by a vortex driven by the Lorentz force is proportional to the applied current so at very low currents it would not be sufficient to excite quasiparticles above the finite gap. For larger currents the probability to excite quasiparticles is constant and is given by Eq. (33) so at these currents vortex dissipation is linear in its velocity leading to Ohmic conductivity. Here we estimate smallest currents j_0, at which the conductivity remains Ohmic.

There are two effects that make j_0 very small. First, a vortex makes many jumps between consecutive emission processes and accumulates energy. Second, vortex liquid is incompressible and retains short range order in a broad range of fields above the melting point, so each emission process slows down not an individual vortex but a large number of them. This effect is similar to the Mössbauer effect in crystals, but here the momentum cannot be transferred to the whole number of vortices since there is no long-range order. The effect is difficult to describe quantitatively, due to the absence of a theory of a strongly correlated vortex liquid at $T=0$. We shall attempt therefore only to show that the effect is indeed large.

To see this effect and to estimate the correlation length in the liquid state we perform the following calculation, similar to the calculation of the Debye-Waller factor. The idea is that when the momentum is being transferred to the liquid as a whole the quantum state of the liquid does not change, so it is described by the same wave function after the transfer as before. The amplitude of such process is

$$A=\int \Pi dx_i \Psi^*(x_1 \ldots x_N) e^{i(p/N)\Sigma_i x_i} \Psi(x_1 \ldots x_N)\delta(x_1), \tag{42}$$

where N is the number of vortices, p is the transferred momentum, and $\Psi(x_1 \ldots x_N)$ is the macroscopic wave function of the vortex liquid. The factor $\delta(x_1)$ singles out the coordinate of the island, where the quasiparticle was excited. When the vortex interacts with the particular island its coordinate becomes fixed and therefore we do not need to integrate over it.

The amplitude A is a function of momentum p, vortex number, and interaction strength. In a true liquid, where no order is present, the wave function $\Psi(x_1 \ldots x_N)$ depends only on the relative coordinates of vortices and therefore $A=0$. When vortices are organized in clusters at short distances, the correlations decay exponentially like $\exp(-x/\xi)$, where the correlation length ξ defines cluster size. Therefore for a system of the finite size L the amplitude A is of the order of unity when $L\sim\xi$, but when the system becomes large, so that $L\gg\xi$, then the amplitude is exponentially small $A\sim\exp(-L/\xi)$.

Because the vortex-vortex interaction is proportional to the logarithm of distance between vortices the vortex liquid can be approximated by the two-dimensional Coulomb gas. The exact wave functions of the Coulomb gas are unknown. However there exists a three-body Hamiltonian with interaction that is Coulomb for long distances, while different (and three-body) for short distances. The ground-state wave function for this Hamiltonian is known,[17] and it was argued that one can use this known wave function to estimate the properties of the Coulomb gas. The amplitude A then can be estimated numerically. Due to the limited computer availability we performed the calculation for relatively small arrays of up to 20 particles in the circular geometry and small interaction parameters $\alpha=0.5-2.5$ [for comparison, the melting point is at $\alpha=30$ (Ref. 9)]. But even being that far from the melting point we could see the momentum dependence of the amplitude A as described above. Our calculation allows to estimate the correlation length $\xi>5$ in the units of lattice spacing, which would correspond to the cluster size of up to 20–30 vortices and increasing as we increase α.

We now estimate the average energy E_{cl} of a moving cluster due to the external current. When the current is small the force acting on a vortex is given by $F=J\Phi_0/ac$, where a in the lattice spacing, J is the current per junction, and Φ_0 is the flux quantum. The energy acquired by the vortex after one tunneling jump to the neighboring plaquette is equal to the force times the lattice spacing $E=Fa=J\Phi_0/c$. Since the probability to excite a quasiparticle is small the excitation is a rare process and the average number of jumps the vortex makes before it excites a quasiparticle is inverse probability. Therefore at the moment of quasiparticle excitation the vortex would have the energy $E=J\Phi_0/c\mathcal{P}$. Multiplying this energy by the number of particles in the cluster we obtain an estimate of the average energy of the cluster at the moment of quasiparticle excitation $E_{cl}=N_\xi J\Phi_0/c\mathcal{P}$.

The probability can be estimated from Eq. (35) using the experimental[7] values for E_C and $|\Delta|$. We get $\mathcal{P}\approx 0.001$. Therefore the cluster energy can be estimated as $E_{\rm cl}=N_\xi J\cdot 1\times 10^2$ K, where the current is measured in nanoampers.

The system has a linear response when the average cluster energy is larger than the gap $|\Delta|\approx 2$ K. For very small currents, when $E_C<|\Delta|$ the system would exhibit nonlinear current-voltage curves. For a cluster size $N_\xi=100$ the current value where this nonlinearity would be observable can be estimated by setting $E_{\rm cl}$ equal to the gap and using the above estimate for the cluster energy. We get $J_0=0.1\times 10^{-3}$ nA. The currents used in the experiment are of the order of $J\approx 0.1$ nA. Therefore it is likely that in the experimentally observable case the array is in the (pseudo) linear regime.

Thus the coupling to quasiparticles results in dissipation described by the linear response in contrast to the dissipation due to coupling to the acoustic phase modes (spin waves).[18,19] The effective action obtained in Ref. 18 has a dissipation term which is proportional to $\omega^2\ln\omega$. For the slow vortex motion we consider (due to very small currents discussed above) this term is small compared to the linear term Eq. (35) which is determined by the energy scale E_J

rather than the frequency. For larger currents the situation is different and the two effects become comparable.

For the (pseudo) linear-response regime we now consider how the vortex motion is affected by the dissipation. Our goal is to obtain an expression for the total array resistance which arises due to dissipation Eq. (35), therefore we now consider a macroscopic equation of vortex motion, averaged over the whole array following Ref. 20. Note that after averaging this equation [Eq. (43)] does not describe the microscopic coupling between vortices and quasiparticles, which in general is nonlinear.

Under the influence of the driving current J a vortex is moving in a direction perpendicular to the current flow:

$$\Gamma_0^{-1}\ddot{x}+\mathcal{W}\Gamma_0^{-1}\dot{x}=\frac{\Phi_0 Ja}{c}, \tag{43}$$

where x is the vortex position, Φ_0 is the flux quantum, Γ_0 is the tunneling rate, defined in Eq. (2), which provides the measure for the vortex mass, and a is the lattice spacing. The lattice potential was taken into account when we calculated the dissipation and the tunneling rate, therefore it does not appear in Eq. (43).

If the driving current is constant then the vortices move with constant velocity. If in the Cartesian coordinate system the current flows along y axis, then vortices move along x (the magnetic field is along the z axis, perpendicular to the xy plane of the array) and their velocity is

$$v_x=\frac{\pi\hbar Ja}{e\mathcal{W}}\Gamma_0. \tag{44}$$

The potential difference caused by the time-dependent phase is

$$U=\frac{\hbar}{2e}\frac{\partial\phi}{\partial t}. \tag{45}$$

When one vortex is moving across the array (in time $t=d/v_x$, where d is the size of the array) the phase changes by 2π. To obtain the total voltage the effect of one vortex should be multiplied by their number $n_v=Bd^2/\Phi_0$

$$U=\pi^2\frac{\hbar}{e^2}\frac{(\hbar\Gamma_0)}{\mathcal{W}}\frac{Bda}{\Phi_0}J. \tag{46}$$

The current per junction J can be obtained from the total current I as $J=Ia/d$ (assuming a square array). The coefficient of proportionality between the voltage U and the current I is the array resistance

$$R=\pi^2\frac{\hbar}{e^2}\frac{(\hbar\Gamma_0)}{\mathcal{W}}\frac{Ba^2}{\Phi_0}. \tag{47}$$

VIII. CONCLUSIONS

We have considered the vortex tunneling in Josephson-junction arrays at zero temperature. Using the simple quantum-mechanical analogy we have showed that such tunneling is accompanied by small dissipation due to quasiparticle excitations in the superconducting islands. Even in the presence of the large quasiparticle gap the probability of such excitations is found to contain only power-law smallness in the (small) ratio of the characteristic tunneling frequency to the gap.

Our main result is the resistance of the vortex liquid Eq. (47), which is due to this dissipation. The result is valid for not too small driving currents (for experimental setup[7] we estimate $J>J_0=0.1\times10^{-3}$ nA), for which the system is in the (pseudo) linear-response regime. Our results provide the quantum-mechanical mechanism of dissipation in Josephson-junction arrays. The resistance Eq. (47) is independent of temperature (due to its quantum-mechanical origin) in agreement with the experiment.[7]

Our argument is applicable to the vortex liquid just above the melting point, where the vortex liquid retains short-range crystalline order. Then the vortices are strongly interacting and the external momentum can be transferred to a large number of vortices. This is similar to the Mössbauer effect in crystals where the external momentum is transferred to the whole crystal. In the vortex liquid such a transfer is impossible due to the absence of long-range order, but the transfer to a finite size cluster remains possible. Such a cluster involves a large number of vortices, therefore its energy is much more than the gap in the quasiparticle spectrum, allowing excitations above the gap. Estimating the minimal cluster size from our numerical data we got that in experimental conditions the energy contained in such a cluster is always larger than the gap; it would get smaller than the gap only for very small currents $J<J_0$.

However these estimates depend crucially on the structure of the strongly correlated vortex liquid formed when the vortex lattice melts. We have argued that the problem is exacerbated by the frustration imposed on the vortex lattice by the underlying array structure. The frustration reduces even further the kinetic energy needed for the melting. The self-consistent theoretical description of the normal liquid of vortices, however, remains to be an unresolved question. In particular it is not clear whether the existence of the normal liquid is due to the dissipation effects or it is in fact possible to form a normal liquid in the absence of dissipation.

One of the possible descriptions of the strongly correlated vortex liquid is a dilute gas of dislocations in the vortex crystal similar to the hexatic phase appearing in 2D thermal melting.[21] In this phase the vortex flow is due to the motion of dislocations. The motion of each single dislocation transfers the whole row of vortices across the system. Here the number of moving vortices scales with the system size so in the thermodynamic limit the combined energy of these vortices becomes infinite and the linear response persists to zero currents. A detailed description of the strongly correlated vortex liquid is the subject of future work.

[1] J. E. Mooij and G. Schön, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992), Chap. 8, and references therein.
[2] H. S. J. van der Zant, F. C. Fritschy, W. J. Elion, L. J. Geerligs, and J. E. Mooij, Phys. Rev. Lett. **69**, 2971 (1992).
[3] A. F. Hebard and M. A. Paalanen, Phys. Rev. Lett. **65**, 927 (1990).
[4] U. Eckern and A. Schmid, Phys. Rev. B **39**, 6441 (1989).
[5] M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).
[6] A. van Otterlo, K-H. Wagenblast, R. Fazio, and G. Schön, Phys. Rev. B **48**, 3316 (1993).
[7] H. S. J. van der Zant, W. J. Elion, L. J. Geerligs, and J. E. Mooij, Phys. Rev. B **54**, 10 081 (1996).
[8] M. A. Itzler, G. M. Danner, R. Bojko, and P. M. Chaikin, Phys. Rev. B **49**, 6815 (1994).
[9] V. M. Bedanov, G. V. Gadiyak, and Yu. E. Losovik, Sov. Phys. JETP **61**, 967 (1985).
[10] S. E. Korshunov, JETP Lett. **46**, 484 (1987).
[11] U. Geigenmüller, in *Macroscopic Quantum Phenomena*, edited by T. D. Clarke *et al.* (World Scientific, Singapore, 1991), p. 131.
[12] R. Fazio, U. Geigenmüller, and G. Schön, in *Quantum Fluctuations in Mesoscopic and Macroscopic Systems*, edited by H. A. Cerdeira *et al.* (World Scientific, Singapore, 1991), p. 214.
[13] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[14] U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B **30**, 6419 (1984).
[15] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[16] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[17] C. L. Kane *et al.*, Phys. Rev. B **43**, 3255 (1991).
[18] A. I. Larkin, Yu. N. Ovchinnikov, and A. Schmid, Physica B **152**, 266 (1988).
[19] U. Eckern and E. Sonin, Phys. Rev. B **47**, 505 (1993); U. Geigenmüller, C. J. Lobb, and C. B. Whan, *ibid.* **47**, 348 (1993).
[20] T. P. Orlando, J. E. Mooij, and H. S. J. van der Zant, Phys. Rev. B **43**, 10 218 (1991).
[21] D. R. Nelson and B. I. Halperin, Phys. Rev. B **19**, 2457 (1979).