Experimental studies of nonlinear continuous waves and pulses in disordered media showing Anderson localization

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The physics of disordered systems, involving Anderson localization, and the physics of nonlinear systems merge for systems which are both disordered and nonlinear. A fundamental question concerning continuous (single frequency) wave or pulse propagation within such systems is whether or not Anderson localization is weakened by the nonlinearity. Theoretical predictions for the answer are different for the two cases of continuous waves and pulses. For continuous waves, it is expected that localized eigenstates will not be affected by the introduction of nonlinearity. For nonlinear pulses the localization may or may not occur depending on the relative magnitude of the Anderson localization length and a characteristic ''nonlinearity'' length which describes the pulse. We have experimentally studied two different types of disordered and nonlinear acoustic systems involving both pulsed and continuous waves and have obtained results which show that in all cases studied theoretical predictions are verified. [S0163-1829(98)01541-0]

I. INTRODUCTION

Two interesting areas of research involve the physics of disordered systems and the physics of nonlinear systems, both having been the subject of experimental and theoretical studies for some time. Fundamental studies within the area of disordered systems treat a wave propagating through a random array of scatterers,[1] and research on this problem has revealed several interesting phenomena such as coherent backscatter[2] and the exponential localization of the wave, known as Anderson localization.[3] An understanding of wave propagation in disordered media is crucial in medical imaging, acoustic geophysical survey (including oil exploration), testing of composites, alloys and porous materials, and mesoscopic electronic systems. In nonlinear science, important developments have included the introduction of the concept of ''chaos,'' which describes the unpredictable behavior of a system governed, paradoxically, by simple and completely deterministic equations, and the application of a systematic method for solving nonlinear differential equations, which produces an explanation of the ''soliton,'' a stable wave pulse which has particlelike behavior. The overlap between the sciences of disorder and nonlinearity is a relatively new area of research, motivated by the importance of systems which are both disordered and nonlinear.[4] These systems are common both in nature and in technological applications. A few examples are many-electron mesoscopic devices,[5] high intensity optical systems, biological and polymer systems, solitons in Josephson transmission lines,[6] and stress waves in composite materials.[7] To understand many of these systems it is of fundamental importance to study the propagation of nonlinear waves within disordered media. A primary question of interest is: Does nonlinearity weaken Anderson localization? Recently there has been significant theoretical progress in solving this difficult problem,[8,9,11–16] but there have been few direct experimental studies. In this paper we describe two experiments which were undertaken to provide experimental answers to the question of the effect of nonlinearity on Anderson localization. Before stating the results, the question itself should be clarified by means of some introductory material.

II. ANDERSON LOCALIZATION

Wave propagation in a linear one-dimensional system containing a sequence of scatterers may be treated in a general manner, commonly found in quantum mechanics text books.[17] Whether the problem is stated in terms of quantum or acoustic variables, the effect of a wave encountering a single scatterer or a sequence of scatterers may be described by a two-by-two matrix. For static scattering fields it is convenient to work in the temporal frequency domain, rather than the time domain, so we treat a monotonal wave field with time dependence $\exp(-i\omega t)$, or $\exp(-iEt/\hbar)$ for a quantum particle. The wave equation describing the propagation of the wave is second order and therefore must have two linearly independent solutions with coefficients u_{j-1} and v_{j-1} on one side of the scatterer and u_j and v_j on the other side. The u and v refer to the forward propagating solution and backward propagating solutions, respectively. For one scatterer, it can be shown that the two coefficients on one side of the scatterer are related to the two coefficients on the other side by a two-by-two matrix of the form

$$\begin{pmatrix} u_{j-1} \\ v_{j-1} \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j^* & \alpha_j^* \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix}. \tag{1}$$

For general scatterers, the α's and β's can be given in terms of complex reflection and transmission coefficients: $\alpha_j = 1/T_j$ and $\beta_j = (R_j/T_j)^*$. For a sequence of scatterers one wishes to find the overall reflection and transmission coefficients. This is done by assuming the boundary conditions of an incoming wave of amplitude u_0, a reflection wave of amplitude v_0, and a transmitted wave of amplitude 1. Starting with the assumed transmission coefficient vector

$$\begin{pmatrix} u_N \\ v_N \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

and working backward from the position of the final scatterer at x_{N+1}, the matrices for each scatterer are multiplied together until reaching the input end at $x_0 = 0$ with the wave field determined by the coefficients u_0 and v_0:

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \prod_{j=1}^{N} \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j^* & \alpha_j^* \end{pmatrix} \begin{pmatrix} u_N \\ v_N \end{pmatrix}. \tag{3}$$

The complex reflection and transmission coefficients for the whole system are then given by $R = v_0/u_0$ and $T = 1/u_0$. The system of scatterers may be characterized by exciting a continuous wave $\cos(\omega t)$ at one end of the system and examining the transmission spectrum $T(\omega)$ at the other end. Alternatively, a linear pulse may be incident on the system and the temporal response at the exit point, $\psi(t)$, may be obtained with an inverse Fourier transform from the frequency domain into the time domain. A key point is that the behavior of the system, whether excited with continuous waves or pulses, involves a product of matrices containing information about the whole system. The product of matrices may represent a sequence of periodic scatterers or a sequence of disordered scatterers. For a periodic distribution of scatterers, the eigenfunctions of the system, described by the set of coefficients u_j and v_j, are the Bloch wave functions. For a random distribution of coupled scatterers, we write the product of matrices in a simpler form

$$P_j = \prod_{j'=1}^{j} M_{j'} \tag{4}$$

and make use of Furstenberg's theorem.[18] This reveals that an ensemble average (over different realizations of the random distribution) of a product of random matrices may be written as

$$\overline{P_j} = e^{j\overline{M}}, \tag{5}$$

a solution which grows or decays exponentially as j goes to plus or minus infinity. For discrete eigenfrequencies ω one finds integrable eigenfunctions which decay exponentially in both the positive and negative directions, i.e., the Anderson localized states. When averaged over the different ensembles, fluctuations which may be present in the individual eigenfunctions disappear and the system behaves as though the eigenfunctions were localized with a pure exponentially decaying eigenfunction such as $|\psi(x)| = \exp(-2|x - x_l|/\Lambda)$, where x_l is the site of localization in one dimension and Λ is the localization length. These localized eigenstates are a consequence of the presence of disorder to the array of scatterers, and may be studied using established theoretical tools.

III. EFFECTS OF NONLINEARITY

Including nonlinear effects greatly complicates the problem of wave propagation in disordered media. With respect to the question of nonlinearity weakening Anderson localization, theoretical papers discussing the question are divided between those that predict that Anderson localization will be weakened by nonlinearity[8,9,11,12] and those that predict that it will not be weakened.[12–16] While seeming to be a controversy, the division between the theories reflects a feature of many nonlinear problems: the answer is not unique and depends on precisely how the question is posed. In this case, the answer depends on whether one is discussing a pulse propagating through a disordered media[8,9,11] or an extended single frequency wave.[12,13] As discussed above, the transmission spectrum for a continuous linear wave and the temporal response for a linear pulse are related to each other by a Fourier transform. In a nonlinear system a simple relation between $T(\omega)$ and $\psi(t)$ does not exist, so that careful consideration of the two cases must be made. To fully understand these important differences, both pulses and continuous waves were used in our experimental investigation of the effect of nonlinearity on Anderson localization. For the case of a continuous (extended, single frequency) wave, there is still more than one answer concerning the effect of nonlinearity on Anderson localization. The measurement of the transmission of an extended single frequency wave through a disordered one-dimensional region as the length of that region increases may be done in two ways; the incident wave amplitude may be held constant, or the output amplitude may be held constant. In the first method the transmitted power is not necessarily unique, although it has been shown[13] that the transmission may still decay exponentially with length as for the linear disordered system. When using the second method, one may obtain the unique result that the exponential decay is replaced with a power-law decay.[12] The first method, a more typical way of defining a transmission measurement, was used for the continuous-wave experiments described in this paper.

For the continuous-wave experiment, the boundary condition of holding the input amplitude constant corresponded to the theory of Frölich, Spencer, and Wayne (FSW).[14] Their paper considers the existence of exponentially localized solutions of a Hamiltonian with a nonlinear term, and concludes that under general conditions Anderson localization exists in the presence of nonlinearity. They do not, however, consider the possibility of the nonlinearity weakening or destroying the Anderson localization by enhancing resonant hopping, that is, by the nonlinear parametric excitation of an eigenfunction localized at a distant site but which has a nearby eigenfrequency. In a linear disordered system, resonant hopping from one state to another is unlikely since the two states would have to be near neighbors for there to be a significant overlap between the exponentially decaying, Anderson localized eigenfunctions. For the linear disordered system then, there are large resonance free regions in the spectrum and diffusion is prohibited. A nonlocal nonlinearity, as present in the continuous-wave experiments described in this paper, introduces the possibility of enhanced resonant hopping between localization sites. We found that, for a one-dimensional system under the conditions stated in the FSW paper, the nonlocal nonlinearity does not weaken the Anderson localization, even with conditions which favored nonlinear enhanced resonant hopping.

For the pulse experiments, the presence of nonlinearity introduces a second length scale, in addition to the Anderson localization length, to the problem of pulse propagation in disordered media. As mentioned above, the behavior of a linear pulse, which is equivalent to a superposition of eigenstates, depends on satisfying conditions throughout the system. By contrast, a nonlinear pulse has extra degrees of freedom which may be adjusted so that conditions need be satisfied only locally, within a characteristic distance (e.g., the width of a soliton), referred to as the ''nonlinearity'' length. If the nonlinearity of the pulse is weak so that the nonlinearity length is greater than the Anderson localization length, then the pulse transmission exponentially decays with the size of the system L, as for a linear pulse. If the nonlinearity of the pulse is strong, so that the nonlinearity length is much less than the Anderson localization length, then the effective extent of the disorder is insufficient to yield localization, and the pulse may show no exponential decay for large L. When the nonlinearity length and the Anderson localization length are comparable, theory predicts that the pulse will be transmitted for short L with moderate decay, but for larger L it will exponentially decay. It is this interesting case which is relevant for the nonlinear pulse experiment described in detail below. The experiment was performed by examining the changes in the pulse transmission versus length as the nonlinear length, defined as the characteristic width of the pulse, decreased with increasing nonlinearity. For weak nonlinearity, the Anderson localization length was expected to be less than the characteristic nonlinear length, and the pulse transmission was found to decay exponentially. For the interesting intermediate nonlinearity, the predicted break in the transmission versus length was observed.

IV. ACOUSTIC ANALOGS

Advances in understanding waves propagating in disordered media have occurred through studies of the localization of light,[10] and the wave properties of electrons in disordered solids. However electron systems, for example, are complicated by effects of screening, spin interactions, and inelastic scattering at finite temperatures.[5] An advantageous way of studying complicated systems is to study classical analog systems which model the salient features of the complicated system.[19–23] One of the easiest classical wave systems which may be used is an acoustic system. Indeed, by studying an acoustic analog system, many variables of interest that may be difficult to obtain in quantum or even other classical systems, such as the amplitude and phase of the wave function, may be precisely controlled and/or measured. Other important advantages of studying an acoustic analog are that the system may be relatively lossless, ensuring a coherent phase structure throughout the system; that it may be driven parametrically, allowing the simulation of inelastic effects;[19] and that it may be driven with finite amplitude waves, allowing the study of nonlinear effects. Since mathematically the effects are the same whether the waves are quantum mechanical or classical (acoustic) in nature, acoustic analogs are simple fundamental experiments which may be relevant to electron systems. In any case, the classical analog systems are, in many ways, interesting in their own right. As described in the sections below, we have used acoustic analogs to study the effects of nonlinearity, both local and nonlocal, on one-dimensional Anderson localization. The acoustic analogs involved transverse waves of a mass loaded wire[21] and the propagation of third sound on a substrate with a controlled distribution of scatterers.[22]

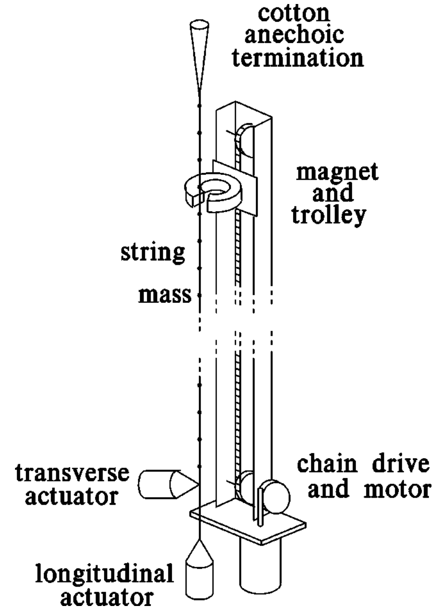

FIG. 1. The one-dimensional mass-loaded wire Anderson localization experiment. The wire under tension with small masses along its length is an accurate realization of the one-dimensional wave equation with a Kronig-Penny potential field.

V. CONTINUOUS WAVES ON A MASS-LOADED WIRE

The acoustic analog which involved a nonlocal nonlinearity was relatively straightforward in construction. The one-dimensional wave medium consisted of a steel wire which had a sequence of small masses attached to it. The wire had mass per unit length $\mu = 2\times 10^{-3}$ g/cm, and was stretched to a tension T_0, giving the speed of low amplitude transverse waves as $c_0 = \sqrt{T_0/\mu} = 400$ m/s. Continuous transverse waves were excited with an electromechanical actuator placed against one end of the wire, as shown in Fig. 1. The vibration field of the wire-mass system was measured with an electrodynamic transducer which could be moved along a track running parallel to the wire, recording the amplitude and phase of the vibration of the wire as a function of position. The small $m = 0.12$ g masses attached to the wire accurately simulated a Kronig-Penny potential field consisting of a series of δ functions with strength $m\omega^2/T_0$ where ω is the temporal frequency of the transverse waves on the wire. When the placement of the small masses along the wire was periodic, extended eigenfunctions and the corresponding band structure were observed, verifying that the system was an appropriate acoustic analog to electrons in solids. When the masses were spaced at random positions along the wire, Anderson localized eigenstates were observed. For this case, the average spacing of the masses was $a = 20$ cm, and the positions deviated randomly from periodic lattice positions within a limit of $0.02a$. For small-amplitude transverse waves, the disordered potential field of the masses was found to produce Anderson localized eigenstates with localization lengths on the order of $6a$.[19] Together, the wire and mass system was relatively lossless; the resonances of the system at low amplitudes had quality factors of ~ 1500.

The frequencies used to make measurements were in the neighborhood of what would have been the second transmission band if the system had been periodic; that is, the frequencies were such that approximately one-half wavelength fit between the masses. Measurements were made by selecting an amplitude for the drive actuator, leaving the receive transducer in one position and slowly sweeping the frequency while the spectral response was recorded. Frequencies which corresponded to Anderson localized states at low amplitude could be selected from the spectral response, and the wire excited at the selected frequency while the receive transducer was translated along the wire, recording the wavefield amplitude and phase. The measurements were repeated for a sequence of increasing drive amplitudes, and the effect of the increasing nonlinearity on the selected Anderson localized states was examined.

The nonlinearity affecting wave propagation at large drive amplitudes was nonlocal, involving interactions between different portions of the steel wire, as we shall now show. Let L_0 be the length of a thin unstretched wire with mass per unit length μ. When the wire is statically stretched to a length $L=L_0+\Delta L$ the tension in the wire will be T_0. This is the equilibrium configuration for the system. The acoustic wave equation for infinitesimal transverse oscillations Ψ in the wire is

$$\mu\frac{\partial^2\Psi}{\partial t^2}-T_0\frac{\partial^2\Psi}{\partial x^2}=0, \tag{6}$$

for which the sound speed c is proportional to the square root of the tension, assuming that transverse stiffness has no effect and that for small displacements the tension in the wire is a constant. A finite amplitude wave traveling along the wire will distort the wire, increasing its arc length and increasing the tension at the site of the finite oscillation. The change in the tension will be propagated along the wire at the speed of longitudinal sound (involving Young's modulus) in the wire material, which is much greater than the speed of the transverse displacement waves. In effect, the change in the tension due to a localized finite amplitude transverse displacement distributed throughout the wire virtually instantaneously. Following the derivation by Morse and Ingard,[24] we obtain an expression for the tension that involves the change in arc length of the entire wire:

$$T=T_0+\frac{T_0}{(\Delta L/L)}\frac{1}{L}\left[\int_0^L\sqrt{1+\left(\frac{\partial\Psi}{\partial x}\right)^2}\,dx-L\right], \tag{7}$$

where $T_0/(\Delta L/L)$ is related to the Young's modulus in the wire and may be experimentally determined. By expanding the square root and keeping only the first-order term, we obtain

$$T=T_0\left[1+\frac{1}{2}\left(\frac{L}{\Delta L}\right)\frac{1}{L}\int_0^L\left(\frac{\partial\Psi}{\partial x}\right)^2dx\right]. \tag{8}$$

The right-hand side of this equation may be integrated by parts to yield

$$T=T_0\left\{1+\frac{1}{2}\left(\frac{L}{\Delta L}\right)\frac{1}{L}\left[\left(\frac{\partial\Psi}{\partial x}\Psi\right)_0^L-\int_0^L\frac{\partial^2\Psi}{\partial x^2}\Psi\,dx\right]\right\} \tag{9}$$

$$=T_0\left[1+\frac{1}{2}\left(\frac{L}{\Delta L}\right)\frac{1}{L}\left(q^2\int_0^L|\Psi|^2dx\right)\right]. \tag{10}$$

The last equation follows with either Neuman or Dirichlet boundary conditions, and with an assumed spatial dependence $\exp(iqx)$. This form is useful for analytic calculations. Replacing T_0 by T in Eq. (6) gives the nonlinear differential equation governing the continuous string:

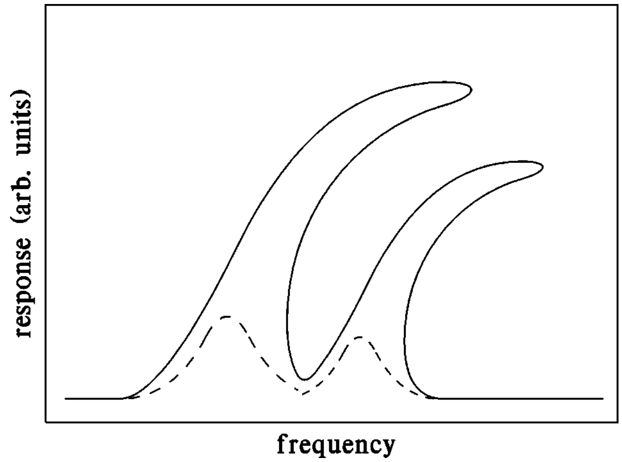

FIG. 2. Schematic of the spectral response for a finite amplitude wave on a wire. The increased tension causes the resonance frequencies for the string to increase, producing the ''bent tuning curves'' shown by the solid line. In the case of nearby resonances, the bent tuning curves may overlap, so that at a single frequency, both modes may be excited, but at different amplitudes. In the experiment, an attempt was made to find a mode, Anderson localized at one site with a large amplitude, which would excite a mode, with a nearby frequency, but at a different site.

$$\mu\frac{\partial^2\Psi}{\partial t^2}-T_0\left[1+\frac{1}{2}\left(\frac{L}{\Delta L}\right)\frac{1}{L}q^2\int_0^L|\Psi|^2dx\right]\frac{\partial^2\Psi}{\partial x^2}=0, \tag{11}$$

which shows that the nonlinear effect arising from the change in the tension will be nonlocal, involving the wave amplitude along the entire length of the wire.[24]

The nonlocal nonlinearity enhances the possibility of hopping between two localization sites, where a large-amplitude transverse displacement at one localization site modulates the tension in the entire wire at twice the eigenstate frequency. This modulated tension may then parametrically excite a response at a distant localization site, even though the eigenfrequency (at low amplitude) of the distant site may be slightly different from that of the original site. This is possible since the finite amplitude displacement also increases the effective static tension of the wire, which causes lines in the spectral response to distort, bending over toward higher frequencies[25] as illustrated in Fig. 2. In this case, a given frequency may correspond to several different eigenstates, and states which have different frequencies for low amplitude displacements may be excited concurrently at the same frequency by finite amplitude displacements. The initial conditions of the experiment [as relevant to the theory of FSW (Ref. 14)] are arbitrary, depending on the state of the system prior to adjusting the frequency of the drive.

With the possible nonlinear effects having been discussed, we can now present the actual experimental results. An informative way to view the results is to examine the spectral response (amplitude at a fixed site as a function of fre-

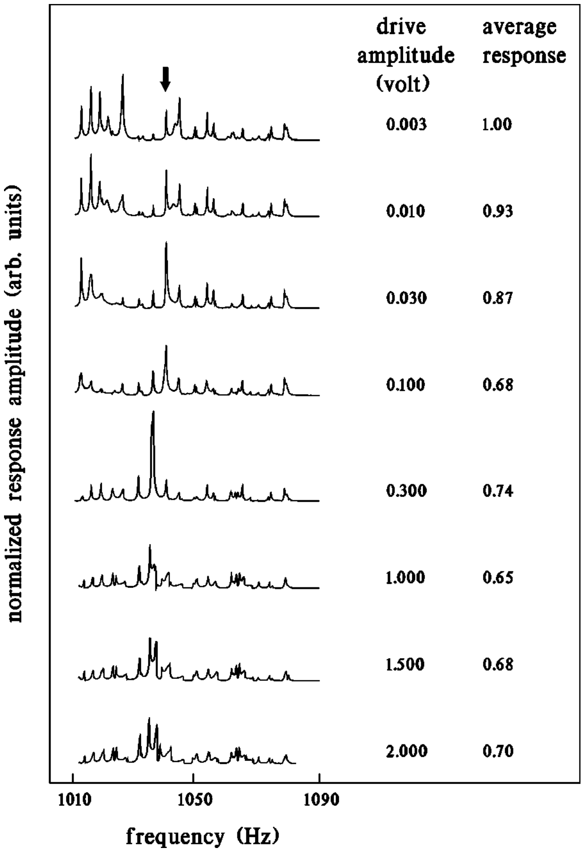

FIG. 3. Normalized spectral response for a sequence of drive amplitudes. The left column of numbers displays the drive amplitude, expressed as the amplitude of the electrical signal applied to the drive actuator in volts. The right column of numbers presents the ''average response,'' defined as the integral of the normalized spectral response over the entire frequency band and normalized to the value at the lowest drive level (0.003 V). The arrow indicates a state whose normalized amplitude increases with drive amplitude, but the effect does not persist.

quency), measured at a distance of about four localization lengths from the drive actuator and normalized by dividing by the drive amplitude, for different drive amplitudes. If the system were strictly linear, then the normalized response would not change. If Anderson localization is weakened by the nonlinearity, then, as the drive amplitude is increased, the normalized response at the distant site should increase.

Our experimental results are presented in Fig. 3, which shows a sequence of normalized spectral response plots for a sequence of increasing drive amplitudes. The drive amplitude, expressed as the amplitude of the electrical signal applied to the drive actuator in volts, is shown in the left column of numbers in Fig. 3. For drive amplitude below 0.003 V, the lowest shown in the figure, there was almost no variation in the spectral response. Changes can be seen in the spectral response for the sequence of increasing drive amplitudes shown. For some of the peaks in the spectrum, for example the one indicated by the arrow in Fig. 3, the normalized response increases with increasing drive amplitude, suggesting that there might be some weakening of the Anderson localization. However, this effect does not seem to persist to the highest drive levels, and examination of the wave fields at the frequency of the increased peaks indicates

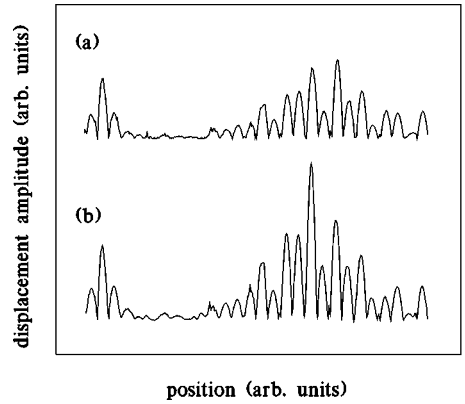

FIG. 4. The wave amplitude, normalized with the drive amplitude, versus position, with the drive actuator to the left of the figure. (a) The wave field for a drive amplitude of 0.01 V. (b) The wave field for a drive amplitude of 0.50 V. While the amplitude increases at some individual sites, the overall localization is hardly changed by finite amplitude effects.

that the effect is due to the growth in amplitude of sections of wire between a few masses only. Figure 4 shows an example of one such wave field whose peak increased with increasing drive amplitude. Figure 4(a) is the wave field (wave amplitude, normalized with the drive amplitude, versus position, with the drive to the left in the figure) for a drive amplitude of 0.01 V, and Fig. 4(b) is the wave field for a drive amplitude of 0.50 V. It can be seen that while the normalized amplitudes of a few sections have increased, the location and size of the Anderson localization site has not changed significantly. It should be noted that for the wave field in Fig. 4, and in all of the measured wave fields, there was no significant harmonic generation observable.

Similar examinations of the wave fields at different frequencies did not show any significant reduction of Anderson localization, in accord with the FSW (Ref. 14) theory. This result did not change when the wave fields of states with nearly the same frequency, but localized at different sites, were examined. These states might be expected to show enhanced resonant hopping from the presence of the nonlinearity, but there was no evidence of this occurring. The highest drive amplitude in our measurements corresponded to a nonlinear shift in the eigenfrequencies by as much as 15% of the band width (quite large by acoustic standards). Analysis at higher drive amplitude was prevented by the onset of chaos in the system; because the Anderson localization concentrates the wave energy in a limited region, the state may act like a simple oscillator which might easily show chaotic behavior.

Evidence for stronger localization may be seen in Fig. 3, where it can be seen that most of the normalized response decreases slightly with increasing drive level. A quantitative measure of this effect may be found with an ''average response,'' defined as the integral of the normalized spectral response over the entire frequency band. The results for each drive level, normalized to the value at the lowest drive level in Fig. 3, are presented in the right column of numbers in Fig. 3. The decrease in the average response of about 30% with increasing drive amplitude suggests that the Anderson localization is slightly enhanced by the nonlinearity. It appears that the presence of the nonlocal nonlinearity causes an

Anderson localized state to become increasingly localized rather than to parametrically excite a distant localization site.

VI. PULSED THIRD SOUND IN SUPERFLUID HELIUM

The propagation of third-sound pulses in superfluid helium, on ordered and disordered substrates, is an acoustic analog with a local nonlinearity. Third sound is a surface wave analogous to shallow water waves, with a velocity of propagation c dependent on the depth of the fluid.[26] In this system the normal fluid component of the helium is locked to the substrate by its viscosity, and only the zero-viscosity superfluid component moves, minimizing the damping and ensuring that the long-range phase coherence is maintained, admitting the possibility of Anderson localization. Experiments by Smith *et al.*[27] and Kono and Nakada[28] have shown that third sound is a viable means of studying Anderson localization as well as other wave phenomena in the linear regime. There are also experimental[29–31] and theoretical[32] studies of third sound in the nonlinear regime, but without a scattering field. The theoretical work on nonlinear third sound has been focused on finding a soliton nature of third-sound pulses, but little work has been done to explain experimental observations. Although it would be satisfying to have a nonlinear theory which accounts for the complexities of finite amplitude third sound, the experimental observations show the nonlinear nature of third sound clearly[29–31] and it is not essential to have a theoretical model in order to measure how the pulses transmit through a disordered sequence of scatterers. The nonlinear nature of third sound may be summarized as follows: When third-sound pulses with sufficient amplitude are generated, the initial part of the pulse saturates, possibly due to the relative motion of the superfluid exceeding a critical velocity. As the energy delivered to the drive transducer increases, a second pulse appears and propagates independently of the initial saturated pulse. It is the second pulse which is observed to be nonlinear in nature, not only because it exists due to finite amplitudes and is unexplained by the linear theory, but also because its velocity of propagation depends on its amplitude.[29] The nonlinearity, in this case, involves local interactions. When a third-sound pulse of finite amplitude ψ is generated, the depth of helium on the surface is changed and c is locally modified. If the modified c is used in the wave equation and expanded to small order in ψ, nonlinear terms involving the local value of ψ are obtained.[33–35]

The effect of the local nonlinearity on Anderson localization was studied by exciting finite amplitude third sound on a disordered substrate. The superfluid film substrate was a $25\times75\times1$ mm glass plate, which was fixed within an enclosed container with its temperature regulated at 1 K. ^{4}He was admitted into the container until a film of several atomic layers formed on the substrate, creating agreeable conditions of temperature and film thickness for the excitation of nonlinear third sound and the subsequent observation of Anderson localization. The third-sound transducers were constructed by depositing 0.2 mm wide strips of aluminum film across the width of the substrate, with electrical connection pads at each end. By adjusting the magnetic field from a superconducting solenoid surrounding the substrate, the aluminum film could be held near its superconducting transition. Each strip could be used as a third-sound generator or as a third-sound receiver. When used as a generator, a pulse of current would drive the aluminum film into its normal state, causing Joule heating and the launch of a third-sound pulse; since each strip acted as a line source, a plane wave-front pulse would then propagate in each direction down the length of the glass substrate. By monitoring the current, voltage, and time duration of the electrical pulse sent to the transducer, the energy used to generate the third-sound pulse could be calculated. When used as receivers, the aluminum film strips acted as conventional superconducting transition-edge bolometers. The receivers were calibrated relative to each other by monitoring the change in effective resistance in each transducer as the temperature of the container was changed by small amounts. A one-dimensional sequence of scatterers was formed by cutting grooves across the width of the glass substrate (parallel to the transducers[22]) with a diamond wire saw. The reflection coefficient for a single groove was measured separately to be about 30%, and more importantly, the reflection was the same regardless of the amplitude of the third-sound pulse. Measurements were made when the sequence of scatterers were arranged periodically and when the array was disordered.

For each measurement sequence in the experiment, one transducer was selected as a third-sound generator, and a second selected as receiver. Pulses were launched with generator energies ranging from 25 nJ or less (small enough to be considered well within the linear regime) to 1200 nJ (large enough to excite nonlinear effects), in 25 nJ steps. For each energy level, 40–100 received wave forms (transducer signal as a function of time) were recorded and sample averaged. In order to magnify the nonlinear part of the received signal (the secondary trailing pulse generated after the initial primary pulse saturates), recorded wave forms for successive 25 nJ separated energy levels were normalized, using the initial linear saturated pulse and subtracted. If the received pulses were strictly linear, then this process would result in a null difference; otherwise, a nonlinear signal is observed. Examples may be seen in Ref. 22, which shows the subtracted wave forms for a transducer separation of 6 mm, and with generator levels ranging from 150 to 400 nJ. The shift in the time-of-flight of the nonlinear signal with increasing generator energy, indicating an amplitude-dependent velocity, was clearly evident. Since the subtraction procedure magnified noise in the data, every wave form shown in Ref. 22 was an average of several wave forms within a narrow range of energies (± 75 nJ) about a nominal energy. This procedure produced one nonlinear signal representing one transducer pair and one nominal value of generator energy. The propagation distance was the separation of the transducers in the pair, and the transmission was taken as the height of the nonlinear pulse. Further information on the subtraction procedure and results for other transducer separations may be found in Ref. 36.

Measurements were first made for linear and nonlinear pulse propagation in a periodic array of 30 scatterers, at a temperature of 1.1 K and with a ^{4}He film thickness of 7.5 atomic layers.[36] For low generator energies (0.013 μJ in Fig. 5) the third-sound pulses on the periodic substrate were linear. The output pulse, for the linear case, was similar in shape to the input pulse, with the addition of a trail of struc-

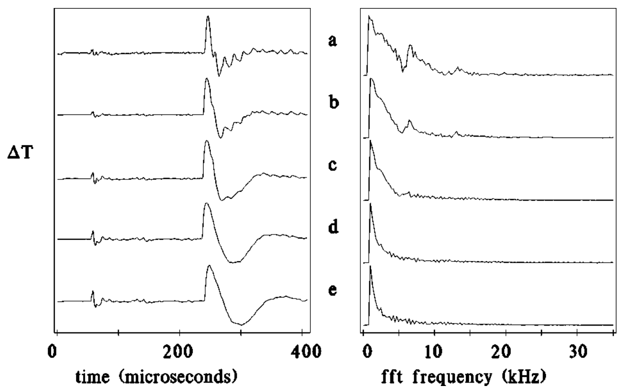

FIG. 5. Transmitted pulses (left) and their Fourier transforms (right) for third sound on a periodic substrate, at a temperature of 1.1 K and with a ^{4}He film thickness of 7.5 atomic layers. Pulse amplitudes and generator energies were: (a) 73 μK, 0.013 μJ, (b) 386 μK, 0.080 μJ, (c) 699 μK, 0.206 μJ, (d) 896 μK, 0.630 μJ, (e) 1220 μK, 1.04 μJ. For low amplitude, (a), the transmitted pulse is trailed by distinct interference structure and the Fourier transform shows band structure. For higher amplitudes, the trailing structure of the pulse disappears as does the band structure in the Fourier transform. These effects may be the result of a decreasing nonlinearity length.

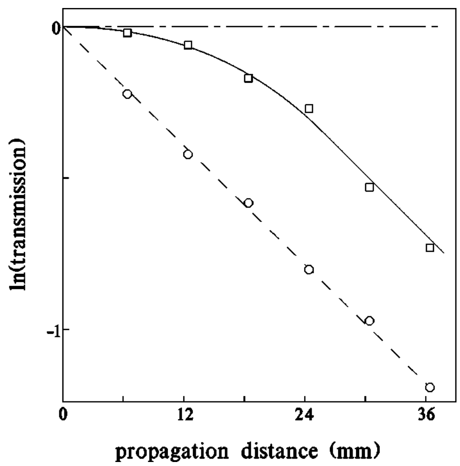

FIG. 6. Logarithm of the pulse transmission versus the propagation distance. The dashed line is for weak nonlinearity, where the nonlinearity length is greater that the Anderson localization length. The dot-dashed line is for strong nonlinearity, where the nonlinearity length is less than the Anderson localization length. The interesting case, illustrated by the solid line, is when the two lengths are comparable. The symbols are data from this experiment.

ture produced by coherent interference between multiple reflections. A Fourier transform of the output pulse showed distinct band structure, as expected for a periodic system. For high generator energies (0.080 μJ and greater in Fig. 5), the interference structure and the band structure disappeared as the amplitude of the third-sound pulse increased, possibly as a result of the decreasing nonlinearity length. For high enough amplitudes (sufficiently short nonlinearity length) the pulse does not sample enough of the periodic array for interference to have effect, and the output pulse loses the trailing interference structure while the band structure in the Fourier transform disappears.

The interesting situation is when the nonlinearity length and the Anderson localization length are comparable, for which theory predicts that the pulse will be transmitted for short system length *L* with moderate decay, but for larger *L* it will exponentially decay.[9] It is this case, illustrated by the solid line in Fig. 6, which the experiments with the disordered substrate addressed. In that case, the scatterers had an average spacing of 1 mm, with a random displacement within ± 0.5 mm of the average spacing. Since Anderson localization is a statistical phenomena, different realizations of the disordered array of scatterers might give widely fluctuating measurements of the pulse transmission, even though the disorder in each array may possess the same statistical properties. To obtain measurements which resemble the statement of the theory of Anderson localization, it was necessary to ensemble average over different realizations of the disordered substrate. This was achieved by using different pairs of transducers as generators and receivers, and then average the results from pairs which had the same nominal separation. The nominal separations for the various pairs used in the experiment ranged from 6 to 36 mm, in steps of 6 mm. The calibration for each pair of generator and receiver transducers included the ordinary attenuation of third sound between the transducers (determined separately with a bare substrate to be 0.6 ± 0.1 cm^{-1}), so that the observed decay of the measured pulse amplitude was due to Anderson localization only. Using the disorder as imparted to the substrate and the measured magnitude of the reflection coefficient, a computer simulation indicated that the Anderson localization length was on the order of 18 ± 2 mm. The third sound in the linear regime behaved as expected, with a measured Anderson localization length of 22 ± 2 mm, in reasonable agreement with the computer simulation.

The final, ensemble-averaged results for the nonlinear measurements are presented as the symbols in Fig. 6, which shows the logarithm of the pulse transmission versus the propagation distance *L*. The dashed line in Fig. 6 is the exponential decay associated with a weak nonlinearity. The circles are for a nominal generator energy of 275 nJ; at this energy a nonlinear signal is readily observed, but its transmission exponentially decays, so that the theory would indicate that at this energy the nonlinearity length is greater than the Anderson localization length. The dot-dashed line in Fig. 6 indicates the pulse transmission when the nonlinearity of the pulse is strong with a nonlinearity length that is much less than the Anderson localization length. For this case the effective extent of the disorder is insufficient to yield localization and exponential decay may not be observed, even for large *L*. The solid line in Fig. 6 shows a theoretical prediction for the intermediate case, when the nonlinearity length and the Anderson localization length are comparable. The squares in Fig. 6 are for a nominal generator energy of 975 nJ, for which the transmission is high for short distances and shows exponential decay after about one Anderson localization length, in excellent agreement with theory. The results for intermediate generator energies lie between the curves for the weak nonlinearity case and the intermediate nonlinearity case in Fig. 6. Because there are so few points at the smaller distances, the current data are insufficient to provide a precise study of the transition between the types of behavior. With the third-sound system, we were unable to achieve sufficient amplitude to observe the case where the nonlinearity length was shorter than the Anderson localization length.

VII. CONCLUSIONS

We have studied the effect of nonlinear interactions on continuous wave and pulse propagation in one-dimensional random systems, in particular the effect of a nonlinearity on Anderson localization. The results represent a significant experimental contribution to the study of systems which are both disordered and nonlinear. Whether one considers quantum or classical (acoustic) waves propagating within disordered systems, the problem of Anderson localization in one dimension may be treated theoretically in the same way. A derivation, entailing the multiplication of a sequence of matrices and making use of Furstenberg's theorem,[18] illustrates the existence of the exponentially localized eigenstates which characterize Anderson localization. The two experimental systems studied were acoustic in nature, involving continuous waves and pulses affected by a nonlocal and local nonlinearity, respectively.

For the case of nonlinear continuous waves in a disordered media, a one-dimensional mass loaded wire was studied. We show that the nonlinearity which arises in such a system is a nonlocal one, providing the possibility of resonant hopping between distant localization sites. For these experiments, the boundary condition of holding the input amplitude constant and measuring the output amplitude, as outlined in a paper by Frölich, Spencer, and Wayne,[14] was appropriate, and it was found that the nonlocal nonlinearity does not weaken the Anderson localization, as predicted. The Anderson localization persists, even under conditions which favor nonlinear enhanced resonant hopping between distant localization sites. Our studies involve a range of amplitude of nearly three orders of magnitude, up to the point where the system becomes strongly chaotic. The Anderson localization appears to be even stronger at the largest amplitudes.

Similarly, experiments with nonlinear third-sound pulses propagating on a disordered substrate, involving local nonlinear interactions only, showed different regimes depending on the strength of the pulse nonlinearity. The concept of a nonlinearity length, a measure of the strength of the nonlinearity, is an important feature. The nonlinearity length is long for weak nonlinearity, and short for strong nonlinearity. This idea was tested using a periodic sequence of a number of scatterers before experimenting with disordered arrays of scatterers. For the periodic array, pulses were generated with both low and high generator energies and Fourier transforms of the output pulses examined. For low generator energies, the output pulses showed structure due to interference and the Fourier transform showed distinct band structure, as expected for a periodic system. For high generator energies the interference structure and the band structure disappeared as the amplitude of the third-sound pulse increased, possibly as a result of the decreasing nonlinearity length. For high enough amplitudes (sufficiently short nonlinearity length) the pulse does not sample enough of the periodic array for interference to have effect, and the output pulse loses the trailing interference structure while the band structure in the Fourier transform disappears. In the system with a sequence of disordered scatterers, weakly nonlinear pulses, with a nonlinearity length greater than the Anderson localization length, still showed effects of localization. But more strongly nonlinear pulses, with a shortened nonlinearity length, showed a transition from slight attenuation to exponentially localized behavior as the propagation distance increased. This is in agreement with the theoretical prediction that states that when the nonlinearity length and the Anderson localization length are comparable the pulse will be transmitted for short distances with moderate decay, but for larger distances it will exponentially decay.

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