Level statistics in a two-dimensional disk with diffusive boundary scattering

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We calculate the energy level statistics in a two-dimensional disk with diffusive boundary scattering by the means of the recently proposed ballistic nonlinear σ model [B. A. Muzykantskii and D. E. Khmelnitskii, JETP Lett. **62**, 76 (1995)]. [S0163-1829(98)02724-6]

The purpose of this paper is to use the recently proposed nonlinear σ model[1] for ballistics in disordered conductors with long mean free path and find out how it works. In a recent paper[2] Andreev and Altshuler (AA) suggested a general method of calculation of level statistics in a disordered system beyond the limits of the random matrix theory. Their calculations were performed for a diffusive disordered system with the mean free path l shorter than the system size R. For this case and for energy difference ω exceeding the mean level spacing Δ ($\omega \gg \Delta$) their method was based on using the nonlinear σ model[3] and accounting for a perturbative contribution from the vicinity of several stationary points of the action.[3] AA have also conjectured a general form which the level statistics obeys. In this paper we follow the same general strategy, addressing this problem for a quantum particle in a two-dimensional disk with no scattering in the bulk and strong boundary scattering.[4] Such a problem is a natural target for the recently proposed field theory for quantum ballistics.[1,5] Consideration is restricted solely to unitary symmetry. In all relevant parts our results coincide with those independently obtained by Blanter, Mirlin, and Muzykantskii.[6]

In order to apply the general approach of AA, we begin with replacing the nonlinear diffusive supermatrix σ model[3] by its ballistic generalization.[1] The partition function of this field theory is determined as a functional integral over a supermatrix $g(\mathbf{n},\mathbf{r})=U^{-1}\Lambda U$ on the energy shell $E=\mathbf{p}^2/2m$ ($\mathbf{n}=\mathbf{p}/p$) in the phase space:

$$Z=\int_{g^2=1}\mathcal{D}g\ \exp\{-F\}, \tag{1}$$

$$F=\frac{\pi\nu}{4}\mathrm{str}\int d\mathbf{r}\Big[i\omega\Lambda\langle g(\mathbf{r})\rangle-2v_F\langle\Lambda U^{-1}\mathbf{n}\boldsymbol{\nabla}U\rangle + \int d\mathcal{O}d\mathcal{O}'W_{\mathbf{n},\mathbf{n}'}g(\mathbf{r},\mathbf{n})g(\mathbf{r},\mathbf{n}')\Big], \tag{2}$$

where $\langle\cdots\rangle=\int\cdots d\mathcal{O}/2\pi$, $\mathcal{O}=d\mathcal{O}_{\mathbf{n}}$, $d\mathcal{O}'=d\mathcal{O}_{\mathbf{n}'}$ and the scattering probability $W(\mathbf{n},\mathbf{n}')$ in the bulk is connected with the mean free time τ and transport mean free time τ_{tr} as

$$\frac{1}{\tau}=\int d\mathcal{O}W(\mathbf{n},\mathbf{n}'),\quad \frac{1}{\tau_{\mathrm{tr}}}=\int d\mathcal{O}W(\mathbf{n},\mathbf{n}')(1-\mathbf{n}\mathbf{n}'). \tag{3}$$

In this paper we will consider a clean disk with no scattering in the bulk and strong boundary scattering. Therefore both τ and τ_{tr} will be taken as being infinitely large. Integration in Eq. (2) is not defined unless boundary conditions are imposed on supermatrix g at the inner boundary of the sample. Since supermatrix g has a meaning of a distribution function of electrons, the boundary condition it obeys is similar to that which is applied to the distribution function in classical kinetics.[7,8] The general boundary condition for matrix functions (see Ref. 9 for an example) is pretty complicated. Fortunately, for the purposes of this paper (calculation of the spectral correlation function with the precision to the first nonvanishing term beyond random matrix theory) the problem could be significantly simplified, because most of the properties of the energy levels could be determined by the the values of g matrix close to the special point $g(\mathbf{rn})=\Lambda$. If $U=1-w/2+w^2/8+\cdots$, then the free energy F could be expressed through matrices w, which gives in the quadratic approximation

$$F_0\{\hat{w}\}=-\frac{\pi\nu}{4}\int d\mathbf{r}d\mathcal{O}_{\mathbf{n}}\mathrm{str}[w_{21}(\hat{L}-i\omega)w_{12}], \tag{4}$$

where indices 1 (2) relate to ''retarded'' (''advanced'') degrees of freedom, and $\hat{L}$ denotes operator of the kinetic equation. Since the free energy in Eq. (4) is quadratic, it results in a classical linear equation. The boundary condition which should be imposed upon $w_{12}(\mathbf{r},\mathbf{n})$ is now a direct analog of the condition imposed upon the distribution function in classical kinetics. Extremely strong boundary scattering is popularly modeled by the diffusive boundary condition,[7,8] which assumes that the distribution function for outgoing particles does not depend on angular variable $\mathbf{n}$ and is coupled to that for incoming particles by flux conservation. If $\mathcal{N}$ is an outward normal to the sample's boundary, then the diffusive boundary condition reads as

$$w_{12}(\mathbf{n}\mathcal{N}<0)=\int_{\mathbf{n}'\mathcal{N}>0}\frac{d\mathcal{O}_{\mathbf{n}'}}{\pi}\mathbf{n}\mathcal{N}w_{12}(\mathbf{n}'). \tag{5}$$

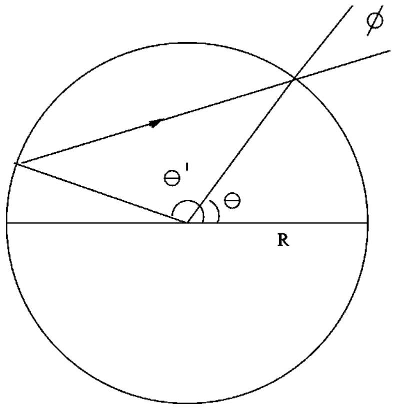

FIG. 1. Typical electron trajectory.

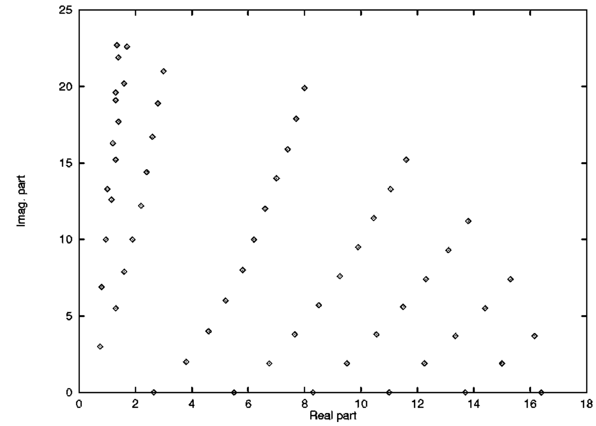

FIG. 2. Eigenvalues of Liouvillean operator.

According to AA, the level statistics is determined by the determinant of a linear operator $\hat{L}$ from Eq. (4). The eigenvalue condition is

$$\mathbf{n} v_F \boldsymbol{\nabla} w = \lambda w, \tag{6}$$

subject to boundary condition

$$2w_< = -w_< \int_{\pi/2}^{3\pi/2} d\phi \cos \phi = \int_{-\pi/2}^{\pi/2} d\phi \cos \phi w_>(\phi), \tag{7}$$

where $w_<$ and $w_>$ are the values of ''distribution function'' $w(\mathbf{n})$ at the disk boundary at $\mathbf{n}\mathcal{N}>0$ and $\mathbf{n}\mathcal{N}<0$, respectively, and $w_<$ does not depend on its argument.

The left hand side of Eq. (6) consists of a derivative $\partial w/\partial l$ along the trajectory of a particle inside the disk (see Fig. 1).

Its solution has the form of a simple exponential

$$w(l) = w(0) \exp\left[\frac{\lambda l}{v_F}\right]. \tag{8}$$

Solution (8) should be substituted into the boundary condition (7). It is also convenient to express the direction of momentum $\cos \phi$ of incident electron at point θ of the disk boundary Eq. (7) through that coordinate on the boundary θ', where this electron was diffusively scattered from $\cos \phi = \sin[(\theta' - \theta)/2]$. This all leads to the eigenvalue equation in the form

$$4w_<(\theta) = \int_\theta^{\theta+2\pi} d\theta' w_<(\theta') \times \exp\left[\frac{2\lambda R}{v_F} \sin\left(\frac{\theta' - \theta}{2}\right)\right] \sin\left(\frac{\theta' - \theta}{2}\right). \tag{9}$$

The expansion of $w_<(\theta)$ in the Fourier series $w_<(\theta) = \Sigma w_m e^{im\theta}$ transforms the condition (9) into

$$f_m(\mu_{m,k}) = 0, \qquad f_m(\mu) = 1 - \frac{1}{2}\int_0^\pi \exp[2imu + \mu \sin u] \sin u\, du = 0, \tag{10}$$

where $\mu_{m,k} = 2R\lambda_{m,k}/v_F$. One can see from Eq. (10) that one of the eigenvalues with $m=0$ vanishes (say $\mu_{0,0}=0$). This corresponds to w independent of both $\mathbf{n}$ and $\mathbf{r}$ and it is not surprising that the relaxation rate of this eigenmode vanishes. Substitution $m \to -m$ into Eq. (10) makes it clear that $\mu_m = \mu_{-m}$. The equation, complex conjugate to Eq. (10), shows that if μ is an eigenvalue, then μ^* is an eigenvalue as well. None of the eigenvalues has a negative real part. A natural labeling[6] is $k = 0, \pm 1, \pm 2, \ldots$ for even m and $k = \pm 1/2, \pm 3/2, \ldots$ for odd m. For $k=0$ and even m the eigenvalues are real. The asymptote of the eigenvalues is

$$\mu_{m,k} \approx \frac{\ln k}{4} + \pi i\left(k + \frac{1}{8}\right), \quad 0 \leqslant m \ll k. \tag{11}$$

So, for $0 \leqslant m \ll k$ $\mathrm{Im}\mu \gg \mathrm{Re}\mu$ and neither depends on m. (See Fig. 2.)

The purpose of this paper is to calculate the spectral correlation function

$$R_2(\omega) = (\pi \Delta R^2)^2 \langle \nu(\epsilon) \nu(\epsilon + \omega) \rangle - 1, \tag{12}$$

where $\nu(\epsilon)$ is the density of states and $\Delta = 1/\pi R^2 \nu$ is the mean level spacing. The time of ballistic flight along diameter of the disk $t_f = 2R/v_F$ introduces a natural scale for the frequencies.

As has been shown by AA, the deviation of $R_2(\omega)$ from the Wigner-Dyson expression

$$R_2(s) = \delta(s) - \frac{\sin^2 \pi s}{s^2}, \quad s = \frac{\omega}{\Delta} \tag{13}$$

at frequencies $\omega \gg \Delta$ is well described by introducing the spectral determinant $D(s)$,

$$D(s) = \prod_{m,k \neq (0,0)} \frac{\lambda_{k,m}^2}{(\lambda_{m,k} - is\Delta)(\lambda_{m,k} + is\Delta)}, \tag{14}$$

which is closely connected with the spectral function $S(\omega)$, introduced by Altshuler and Shklovskii[10] for diffusive systems:

$$S(\omega) = \sum_m \sum_k (\lambda_{m,k} - i\omega)^{-2}, \tag{15}$$

$$\frac{\partial^2 \ln D(s)}{\partial s^2} = -2\left(\Delta^2 \mathrm{Re} S(s\Delta) + \frac{1}{s^2}\right). \tag{16}$$

The spectral correlation function can be decoupled at $\omega \gg \Delta$ into the sum[2] of a smooth part R_{sm} (Ref. 10) and an oscillating part $R_{\rm osc}$:

$$R_{\rm sm}(s)=\frac{\Delta^2}{2\pi^2}{\rm Re}S(s\Delta), \qquad (17)$$

$$R_{\rm osc}(s)=\frac{1}{2\pi^2 s^2}D(s)\cos 2\pi s. \qquad (18)$$

So, the calculation of the spectral determinant $D(s)$ is a key point of the whole AA program, which we approach now.

It is possible to write down an expression for the spectral determinant without an explicit computation of the eigenvalues. In order to do that, note that the function $f_m(\mu)$, defined by Eq. (10), is an entire function of its argument, which has only simple zeros at $\mu=\mu_{m,k}$ and $f'(\mu)/[\mu f(\mu)]$ vanishes as $\mu\to\infty$. Therefore $f_m(\mu)$ can be represented as an infinite product $(m\neq 0)$.

$$f_m(\mu)=f_m(0)e^{f'_m(0)\mu/f_m(0)}\prod_k\left[1-\frac{\mu}{\mu_{m,k}}\right]e^{\mu/\mu_{m,k}}$$
$$=\frac{4m^2}{4m^2-1}e^{f'_m(0)\mu/f_m(0)}\prod_k\left[1-\frac{\mu}{\mu_{m,k}}\right]e^{\mu/\mu_{m,k}}. \qquad (19)$$

For $m=0$ the function $f_0(\mu)$ vanishes at $\mu=0$. So the same theorem could be applied to the function $-4f_0(\mu)/\pi\mu$. Multiplying $f_m(\mu)$ and $f_m(-\mu)$, taking the product over all m, and analytically continuing to $\mu=\pm i\xi=\pm i\omega t_f$, we arrive, finally, at the expression for the spectral determinant

$$D(\xi)=\frac{\xi^2}{4}\left(\frac{\pi}{2}\right)^6\prod_{m=-\infty}^{+\infty}[f_m(i\xi)\ f_m(-i\xi)]^{-1}, \qquad (20)$$

where it is taken into account that

$$\prod_{m=1}^{\infty}\frac{4m^2-1}{4m^2}=\left(\frac{2}{\pi z}\sin\frac{\pi z}{2}\right)_{z=1}=\frac{2}{\pi}. \qquad (21)$$

Since $\lambda_{m,k}=\mu_{m,k}/t_f$, the spectral determinant $D(s)$ consists of two dimensionless parameters ωt_f and Δt_f. One of these parameters is always small $(\Delta t_f\ll 1)$, while the second one ωt_f could be either larger or smaller than unity. These two limiting cases constitute the limits of high and small frequencies, respectively.

At low frequencies $\omega t_f \ll 1$ the spectral determinant $D(\xi)$ can be simplified and the asymptotes of both the smooth and the oscillatory parts of the spectral correlation functions coincide, as was discovered by Kravtsov and Mirlin.[11] This gives the following expression for the spectral correlation function:

$$R_2(s)=\delta(s)-\frac{\sin^2\pi s}{\pi^2 s^2}+B\frac{\Delta^2 t_f^2}{\pi^2}\sin^2\pi s, \qquad (22)$$

$$B=\sum_{m,k\neq 0,0}\mu_{k,m}^{-2}. \qquad (23)$$

Using the low frequency asymptote of Eq. (20), we can present the spectral function

$$S(\omega)=\sum_m S_m(\omega)$$

in the form of a contour integral

$$S_m(\omega)=\frac{t_f^2}{2\pi i}\oint_C\frac{dz}{(-i\omega t_f+z)^2}\frac{d\ \ln f_m(z)}{dz}$$
$$=-t_f^2\left[\frac{d^2}{dz^2}\ \ln f_m(z)\right]_{i\omega t_f}, \qquad (24)$$

where contour C encloses all zeros of of the function $f_m(z)$. As $\omega\to 0$ we obtain from Eq. (24) the following expression for the coefficient B in Eq. (22):

$$B=-\frac{19}{27}-\frac{175\pi^2}{1152}+\frac{64}{9\pi^2}\approx -1.48. \qquad (25)$$

In order to find the asymptote of the spectral determinant D in the high frequency limit, consider a product $P(i\xi)$,

$$P(i\xi)=\prod_{m=-\infty}^{+\infty}f_m(i\xi). \qquad (26)$$

Its logarithm is presented by the sum

$$\ln P(i\xi)=\sum_{m=-\infty}^{+\infty}\ \ln\left[1-\int_0^{\pi}\frac{du\ \sin u}{2}e^{2imu+i\xi\sin u}\right]. \qquad (27)$$

At this stage, it is convenient to use the identity

$$\sum_{m=-\infty}^{+\infty}F(e^{2imu})=\int_{-\infty}^{+\infty}dx\sum_{n=-\infty}^{+\infty}e^{2i\pi nx}F(e^{2ixu}), \qquad (28)$$

which replaces the sum over m by the sum over n and integral over x. For large values of ξ the integral over u in Eq. (27) is small and the logarithm should be expanded up to the second order in this integral (linear term vanishes). After calculating the sum over n the expression could be simplified to the following form:

$$\ln P(i\xi)\approx -\frac{\pi}{8}\int_0^{\pi}du\ \sin^2 u\ \exp(2i\xi\ \sin u). \qquad (29)$$

To evaluate the spectral determinant, we need to find the product $P(i\xi)P(-i\xi)$. Using the steepest descent method, we arrive at $\xi\gg 1$ at the following asymptote for the spectral determinant:

$$D(\xi)\approx\frac{\xi^2}{8}\left[1+\frac{\pi}{4}\sqrt{\frac{\pi}{\xi}}\cos\left(2\xi-\frac{\pi}{4}\right)\right]. \qquad (30)$$

This gives the smooth part of the spectral correlation function in the form

$$R_{\rm sm}(\omega)=\frac{\Delta^2 t_f^{3/2}}{4\sqrt{\pi\omega}}\cos\left(2\omega t_f-\frac{\pi}{4}\right). \qquad (31)$$

Equation (18) gives the oscillatory part of the spectral function equal to

$$R_{\mathrm{osc}}(\omega)=\frac{\pi^4}{516}\ (\Delta t_f)^2\cos\left(\frac{2\pi\omega}{\Delta}\right). \tag{32}$$

In conclusion, we found that the application of the ballistic nonlinear σ model[1] to the study of level statistics for electrons in a clean disk with strong boundary scattering enables us to solve this problem beyond the limits of the random matrix theory.

A clean disk with diffusive scattering on its boundaries, unlike other chaotic systems, has an upper limit for the time of flight at $t=t_f\equiv 2R/v_F$. Therefore, if a Fourier transform of a time dependent form factor is calculated, it oscillates as a frequency ω with period $2\pi/t_f$. As was previously shown, the smooth part of the spectral correlation function $R_2(\omega)$ at high frequencies $\omega t_f \gg 1$ is proportional to the square of the relevant form factor. This leads to oscillations of $R_{\mathrm{sm}}(\omega)$ with two times as short a period π/t_f. In our understanding, such oscillations would not appear in a general case.

Another striking result is exhibited in Eq. (32): the amplitude of the oscillatory part of spectral correlation function is small as $(\Delta t_f)^2$, but does not decay with ω, unlike one obtained by AA for a diffusive system. This could be understood if we recall that our disk is clean inside and, therefore, at short times $t \ll t_f$ a certain correlation between electron wave functions remains. This correlation is small and proportional to $(p_F R)^{-2} \sim (\Delta t_f)^2$, but decays much more slowly with the energy difference ω. If our disk has a bulk disorder with the mean free time $\tau \gg t_f$,[12] $R_{\mathrm{osc}} \propto \exp[-\omega\tau]$. A similar result leads to variation of the Fermi velocity with energy: $R_{\mathrm{osc}} \propto \exp[-\omega/E_F]$. The smooth part of spectral correlation function [see Eq. (31)] also exhibits weak dependence on energy difference ω. In our understanding, these results are of a general nature.

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[1] B. A. Muzykantskii and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **62**, 68 (1995) [JETP Lett. **62**, 76 (1995)].

[2] A. V. Andreev and B. L. Altshuler, Phys. Rev. Lett. **75**, 902 (1995).

[3] K. B. Efetov, Adv. Phys. **32**, 53 (1984); *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1997).

[4] This limiting case contrasted with the relatively slight disorder considered by F. Borgonovi, G. Cassati, and B. Li, Phys. Rev. Lett. **77**, 4744 (1996); K. M. Frahm and D. L. Shepelyansky, *ibid.* **78**, 1440 (1997); **79**, 1833 (1997); E. Louis *et al.*, Phys. Rev. B **56**, 2120 (1997).

[5] After the ballistic σ model for disordered conductors (Ref. 1) had been proposed, Andreev and co-workers [A. V. Andreev and B. L. Altshuler, Phys. Rev. Lett. **75**, 902 (1996); A. V. Andreev, B. D. Simons, O. Agam, and B. L. Altshuler, Nucl. Phys. B **482**, 536 (1996)] made an attempt to derive the field theory by the means of averaging over the energy interval E, exceeding the mean level spacing Δ $(E \gg \Delta)$. Although serious difficulties remain [see B. D. Simons (unpublished)], they are irrelevant to the discussion in this paper, because we study a level statistics averaged over disorder, which leads to the boundary scattering.

[6] Ya. M. Blanter, A. D. Mirlin, and B. A. Muzykantskii, Phys. Rev. Lett. **80**, 4161 (1998).

[7] E. M. Lifshits and L. P. Pitaevskii, *Physical Kinetics* (Pergamon, New York, 1981), Sec. 86.

[8] K. Fuchs, Proc. Cambridge Philos. Soc. **34**, 100 (1938).

[9] Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **29**, 853 (1969) [Sov. Phys. JETP **56**, 1590 (1969)].

[10] B. L. Altshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].

[11] V. E. Kravtsov and A. D. Mirlin, Pis'ma Zh. Eksp. Teor. Fiz. **60**, 645 (1994) [JETP Lett. **60**, 656 (1994)].

[12] The smooth part of spectral correlation function in ballistic billiards with bulk was studied by A. Altland and Y. Gefen, Phys. Rev. Lett. **71**, 3339 (1992).