Duality relation among periodic-potential problems in the lowest Landau level

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Using a momentum representation of a magnetic von Neumann lattice, we study a two-dimensional electron in a uniform magnetic field and obtain one-particle spectra of various periodic short-range potential problems in the lowest Landau level. We find that the energy spectra satisfy a duality relation between a period of the potential and a magnetic length. The energy spectra consist of the Hofstadter-type bands and flat bands. We also study the connection between a periodic short-range potential problem and a tight-binding model.
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A physical system sometimes shows a peculiar fractal structure of the spectrum if the system has two scales of periods. In the nearest-neighbor (NN) tight-binding model with a magnetic flux, Hofstadter obtained one-particle spectra of butterfly shape and discovered a multifractal structure.[1] Experimentally, it may be a challenging theme to observe the spectrum of the fractal structure. Actually, interesting phenomena have been observed in experiments of lateral superlattices.[2–4] The system with the cosine potential has been well studied theoretically.[5] In the case of the antidot array, however, the potential term includes a large number of cosine functions with different wavelengths.[6] The steeper the potential of the antidot, the harder it is to solve the eigenvalue problem. Silberbauer[7] and Kühn *et al.*[8] studied a steep antidot potential problem. Huang *et al.*[9] solved the periodic short-range potential problem in a finite system numerically. Analytic properties of the problem are unknown.

In the present paper, we study eigenvalue problems of the two-dimensional electron systems defined on an infinite plane with the periodic short-range potentials in the lowest Landau level (LLL) analytically and numerically. We find Hofstadter-type bands and flat bands. The former bands satisfy a duality relation. Namely, the spectrum at one value of t is connected with that of $1/t$, where t is the flux Φ penetrating the unit cell of the periodic potential normalized by the unit flux Φ_0, i.e., $t=\Phi/\Phi_0$. Our duality is concerned with flux and is different from Aubry and Andre's duality,[10] which is concerned with hopping strengths in the NN tight-binding model. We also study the connection between a periodic short-range potential problem and a tight-binding model.

The magnetic von Neumann lattice[11,12] is a representation of the quantum system in a uniform magnetic field and has quite useful properties in studying the system with periodic potentials. In particular, its lattice structure varies and we can select the suitable one in accordance with the potential lattice. Using a momentum representation of the von Neumann lattice,[13] we give a proof of the duality and show energy spectra in the periodic potential problems.

In a two-dimensional system under a uniform magnetic field B, the von Neumann lattice basis is formed by the direct product of harmonic oscillator eigenfunctions $|f_l\rangle$ of relative coordinates (ξ,η) and coherent states $|\alpha_{mn}\rangle$ of guiding center coordinates (X,Y). The coherent states are defined by

$$(X+iY)|\alpha_{mn}\rangle=az_{mn}|\alpha_{mn}\rangle, \tag{1}$$

where $a=\sqrt{2\pi\hbar/eB}$, $z_{mn}=m\omega_x+n\omega_y$ $(m,n\in\mathbf{Z})$, and ω_x,ω_y are complex numbers that satisfy $\mathrm{Im}[\omega_x^*\omega_y]=1$.[14] For our purpose, it is convenient to use the Fourier transformed basis as

$$|l,\mathbf{p}\rangle=\frac{1}{\beta(\mathbf{p})}\sum_{m,n}e^{ip_xm+ip_yn}|f_l\otimes\alpha_{mn}\rangle, \tag{2}$$

where $\beta(\mathbf{p})$ is a normalization factor defined by

$$\beta(\mathbf{p})=(2\ \mathrm{Im}\ \tau)^{1/4}e^{i(\tau/4\pi)p_y^2}\vartheta_1\left(\frac{p_x+\tau p_y}{2\pi}\middle|\tau\right) \tag{3}$$

and $\tau=-\omega_x/\omega_y$. For $\tau=i$, the von Neumann lattice becomes a square lattice. For $\tau=e^{i2\pi/3}$, it becomes a triangular lattice. The Fourier transformed basis is an orthonormal set of extended states and obeys a nontrivial boundary condition

$$|l,\mathbf{p}+2\pi\mathbf{N}\rangle=e^{-i\phi(p,N)}|l,\mathbf{p}\rangle, \tag{4}$$

where $\phi(p,N)=\pi(N_x+N_y)-N_yp_x$.

If the magnetic von Neumann lattice has a periodicity commensurate with the periodicity of the external potential $V(\mathbf{x})$, the one-body potential problem becomes easy to treat. This happens when t is equal to q/p with relatively prime integers p,q. Let us consider an arbitrary regular lattice of short-range potentials of $t=q/p$:

$$V(\mathbf{x})=a^2V_0\sum_N\ \delta^{(2)}\left(z+aN_xq\omega_x+a\frac{N_y}{p}\omega_y\right). \tag{5}$$

The matrix element of the potential in the LLL is given by

$$\langle 0,\mathbf{p}|V(\mathbf{x})|0,\mathbf{p}'\rangle = \frac{V_0}{q}\sum_{r_p,r_q,\mathbf{N}} \beta\left(p_x - 2\pi\frac{r_p}{p}, p_y\right) \times\beta^*\left(p_x - 2\pi\left(\frac{r_p}{p}+\frac{r_q}{q}\right), p_y\right)(2\pi)^2 \times\delta\left(p_x' - p_x + 2\pi\left(\frac{r_q}{q}+N_x\right)\right) \times\delta(p_y' - p_y + 2\pi N_y)e^{i\phi(p',N)}, \tag{6}$$

where $r_p=0,1,\ldots,p-1$ and $r_q=0,1,\ldots,q-1$. Here we study the eigenvalue equations in the LLL. The eigenvalue equation becomes

$$D^\dagger D\psi = \epsilon\psi, \tag{7}$$

where a $p\times q$ matrix D is defined by

$$[D(\mathbf{p})]_{r_p r_q} = \beta^*\left(p_x - 2\pi\left(\frac{r_p}{p}+\frac{r_q}{q}\right), p_y\right) \tag{8}$$

and $D^\dagger D$ is a $q\times q$ Hermitian matrix. The magnetic Brillouin zone (MBZ) is the region where $|p_x|<\pi/q$ and $|p_y|<\pi$. Each band has a p-fold degeneracy. Consequently, the fundamental region of the MBZ is the region where $|p_x|<\pi/pq$ and $|p_y|<\pi$.

We should note that the rank of $D^\dagger D$ is generally $\min(p,q)$. If $p<q$, the band splits into p subbands and one flat band, which corresponds to the zero modes of D. The number of zero modes is $q-p$. By a linear transformation $\psi' = D\psi$, the eigenvalue equation (7) becomes

$$DD^\dagger\psi' = \epsilon\psi'. \tag{9}$$

$DD^\dagger$ is a $p\times p$ Hermitian matrix. Equation (7) is equivalent to Eq. (9) except for the zero modes. Apparently, Eq. (9) is obtained by taking the complex conjugate and interchanging p and q of Eq. (7). Therefore, there exists a duality relation between two problems of $t=q/p>1$ and $t=p/q<1$. Actually, the energy spectra E obey the relation

$$E(t) = \frac{1}{t}E\left(\frac{1}{t}\right), \tag{10}$$

except for the flat bands. If $p>q$, Eq. (10) is also obtained. Thus the duality relation is proved in an arbitrary regular lattice of short-range potentials. The self-dual point is $t=1$ and the physical quantity has critical behavior near the point.

Before solving the eigenvalue problems numerically, we clarify the connection between a periodic short-range potential problem and a tight-binding model in the LLL. Let us suppose a square lattice potential $V(\mathbf{x}) = V_0\exp[2\pi i(mx+ny)/b]$, where $t=(b/a)^2=q/p$. The potential energy term in the second quantized form becomes

$$V_0\int_{\mathrm{MBZ}}\frac{d^2p}{(2\pi)^2}\sum_{r_q} c^\dagger_{r_q}(\mathbf{p})c_{r_q+m}(\mathbf{p})\exp\left[-\frac{\pi}{2t}(m^2+n^2) + in\left(p_x - \frac{2\pi}{t}r_q\right) - i\frac{m}{t}(p_y+\pi n)\right], \tag{11}$$

where

$$c_{r_q}(\mathbf{p}) = b_0\left(p_x + \pi - 2\pi\frac{p}{q}r_q, p_y\right)e^{-i\pi(p/q)r_q} \tag{12}$$

and $b_l(\mathbf{p})$ is the annihilation operator of the state $|l,\mathbf{p}\rangle$. Equation (11) is equivalent to the tight-binding Hamiltonian with the hopping term

$$V_{\mathrm{hop}} = V_0\exp\left(-\frac{\pi}{2t}[(m_2-m_1)^2+(n_2-n_1)^2] + i\frac{\pi}{t}(m_2+m_1)(n_2-n_1)\right), \tag{13}$$

where $m_2-m_1=m$ and $n_2-n_1=n$. This hopping term has a flux per unit cell of a square lattice $2\pi/t$ and the hopping strength varies with t. If $(m,n)=(\pm1,0),(0,\pm1)$, Eq. (11) becomes the NN tight-binding Hamiltonian. In a square lattice of short-range potentials, the summation is taken over all (m,n) with an equal weight. Therefore, the result becomes a tight-binding Hamiltonian with finite-range hopping terms

$$\frac{1}{t}\sum_{m,n} c^\dagger(m_2,n_2)V_{\mathrm{hop}}(m_2,n_2;m_1,n_1)c(m_1,n_1). \tag{14}$$

The hopping range is about $\sqrt{t}$ and the number of relevant terms increases linearly with t.

In a regular lattice, the one-particle spectrum is given by the solution of Eq. (7). Since the matrix element of $D^\dagger D$ is given in the analytic form, it is easy to solve numerically. Figure 1 shows the spectra for the square and triangular lattices. The points at $E=0$ correspond to the original LLL. There are two marked structures in these figures: Hofstadter-type bands and a large gap above the flat bands in $t>1$. The origin of the large gap is easily understood in a dilute potential limit $t\to\infty$. In this limit, the potential approaches one short-range potential. Its spectrum consists of a bound state trapped at the potential and a flat band. The bound state has the energy $E=V_0$ and the flat band has the energy $E=0$. In a dense potential limit $t\to0$, the potential approaches constant V_0/t. Therefore, the asymptotic form of the spectrum in $t\to0$ and $t\to\infty$ except for the zero modes is given by

$$E(t)\sim V_0\left(1+\frac{1}{t}\right), \tag{15}$$

which satisfies the duality relation (10).

To check the duality, we replace E with Et for $t<1$ and replace t with $2-1/t$ otherwise in Fig. 1. Then the figures become symmetric with respect to $t=1$ owing to the duality. The results for the square lattice are shown in Fig. 2. The duality is clear in this figure. The patterns in Figs. 1 and 2 are very similar to that of the NN and next-nearest-neighbor (NNN) tight-binding model on the square and the triangular lattice. However, in contrast with the NN or NNN tight-binding model, a periodicity with respect to the flux does not exist in our model. Instead, the duality between t and $1/t$ does exist.

As an example of an irregular lattice, we study a honeycomb lattice potential numerically. A honeycomb lattice can be regarded as a sum of two triangular lattices. This property causes the following eigenvalue equation at $2t = (\text{flux per hexagon})/\Phi_0 = q/p$:

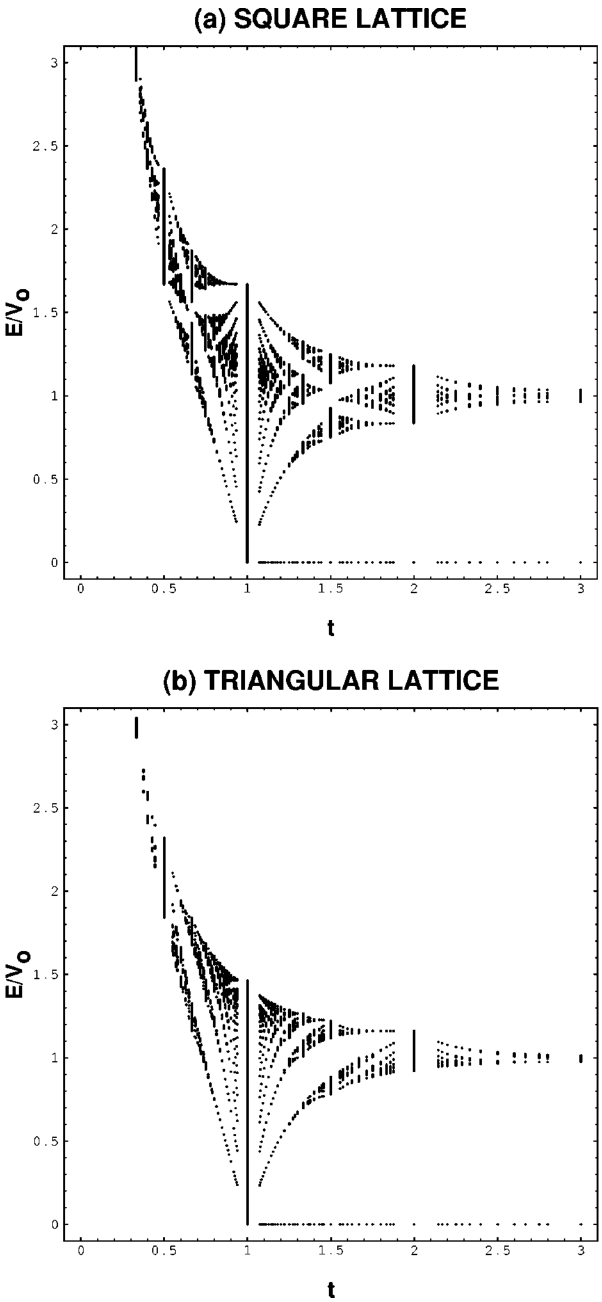

FIG. 1. Energy spectra of the periodic short-range potential for (a) the square lattices and (b) the triangular lattices. The horizontal variable t is the flux per unit cell in the flux unit. V_0 is the strength of the potential.

$$(D_1^\dagger D_1 + D_2^\dagger D_2)\psi = \epsilon\psi, \tag{16}$$

where D_1 and D_2 are defined by

$$D_1 = D(p_x, p_y), \quad D_2 = D\left(p_x - \frac{2\pi}{3p}, p_y + \frac{2\pi q}{3}\right). \tag{17}$$

Since both $D_1^\dagger D_1$ and $D_2^\dagger D_2$ are positive definite, the zero modes must satisfy both $D_1\psi = 0$ and $D_2\psi = 0$. If $q - 2p > 0$, the zero modes exist and the LLL splits into $2p$ subbands and one flat band. Otherwise, the LLL splits into q subbands. This system does not satisfy the duality relation (10). However, the asymptotic behavior of the one-particle energy spectrum is also given by Eq. (15) from arguments similar to that of the regular lattice. The numerical result of the spectrum is shown in Fig. 3. The spectrum also has a Hofstadter-type structure and a large gap above the flat bands.

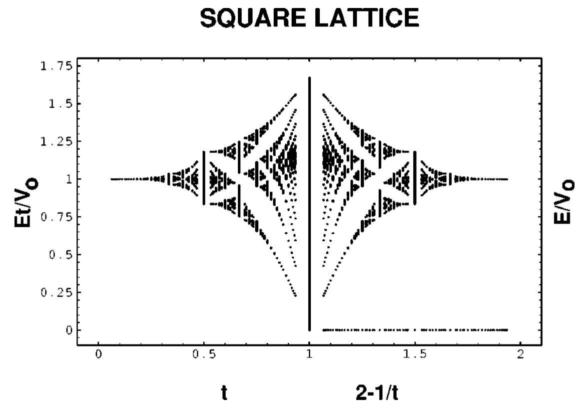

FIG. 2. Energy spectra of Fig. 1(a) are deformed to show the duality clearly. The vertical variable is tE/V_0 for $t<1$ and E/V_0 otherwise. The horizontal variable is $2-1/t$ for $t>1$ and t otherwise. As a result, the spectrum becomes symmetric with respect to $t=1$.

Next we study the possibility of observing the band structure obtained above. The gap above the flat bands may be observed in the magnetoresistance experiment.[4] We assume that the antidot potential has a height $\tilde{V}_0$ and an area of the base r_0^2. V_0 in Eq. (5) is related to $\tilde{V}_0$ and r_0 as $a^2 V_0 = r_0^2 \tilde{V}_0$. The magnitude of the large gap above the flat bands is of the order of V_0 and the correction of the finite size effect of the antidot to the energy is estimated as $V_0 O(r_0/b)$, where b is a lattice constant of the antidot array. Therefore, the gap above the flat bands should be observed in $r_0 \ll b$. In the current experiments,[3] $r_0/b \approx 0.25$, which is enough to

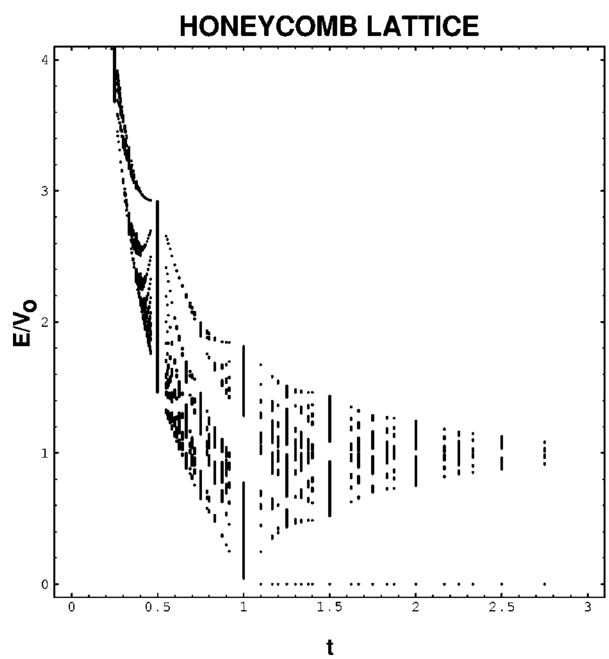

FIG. 3. Energy spectra for the honeycomb lattices. The horizontal variable t is the flux per half of the hexagon in the flux unit. V_0 is the strength of the potential.

observe the gap. Taking the Landau level mixing into consideration, the eigenvalue equation for the potential of Eq. (5) has the form

$$\sum_{l'}\left[E_l\delta_{ll'}+\frac{V_0}{q}D^{(l)\dagger}D^{(l')}\right]\psi^{(l')}=\epsilon\psi^{(l)}. \qquad (18)$$

Because of the $q-p$ zero modes of $D^{(l)}$ for $p<q$, the lth Landau band has a flat band with the energy E_l. Therefore, the flat bands remain flat in the presence of the Landau level mixing effect in the short-range periodic potential problem. The correction of the Landau level mixing to the energy above the large gap is estimated by the second-order perturbative calculation as $E_{\mathrm{LL}}\approx(V_0)^2/\hbar\omega_c$. From $V_0\gg E_{\mathrm{LL}}$, we obtain the condition $\hbar\omega_c\gg\tilde{V}_0(r_0/a)^2$ for the gap to survive. Using the realistic values[2,4] $m=0.07m_e$ and $\tilde{V}_0=0.3$ meV, this condition becomes $2\pi\hbar^2/mr_0^2\gg\tilde{V}_0$. This is satisfied in the current experiments. So far, conditions for observation are satisfied. However, the parameter t in current experiments, e.g., $B=5$ T, $b=200$ nm, and $t\approx 50$, is too large to observe the large gap, the Hofstadter-type bands, and the critical behavior near $t=1$. Thus we hope that the experiment will be made in a finer lattice of the antidot array.

In summary, we solved various periodic potential problems in the LLL using a momentum representation of the von Neumann lattice. In a periodic array of short-range potentials, the energy spectrum has three remarkable structures. (i) The spectrum has a Hofstadter-type structure, which is commonly seen in the periodic potential problems in a magnetic field. (ii) The duality relation exists for a regular lattice potential of Eq. (5). A honeycomb lattice is an irregular lattice and obeys a duality relation asymptotically. (iii) There is a large gap above flat bands[15] that comes from zero modes of the matrix D of Eq. (8). These structures are universal in the periodic short-range potential problem in the LLL. The conditions for the observation of these structures were obtained. In addition, the equivalence between two problems of a square lattice of short-range potentials and a tight-binding model with a inverse flux was shown.

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