Applying voltage sources to a Luttinger liquid with arbitrary transmission

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The Landauer approach to transport in mesoscopic conductors has been generalized to allow for strong electronic correlations in a single-channel quantum wire. We describe in detail how to account for external voltage sources in adiabatic contact with a quantum wire containing a backscatterer of arbitrary strength. Assuming that the quantum wire is in the Luttinger liquid state, voltage sources lead to radiative boundary conditions applied to the displacement field employed in the bosonization scheme. We present the exact solution of the transport problem for arbitrary backscattering strength at the special Coulomb interaction parameter $g=1/2$. [S0163-1829(98)01340-X]

I. INTRODUCTION

One-dimensional (1D) materials have received much attention in the past few years. The discovery of novel 1D conductors and the failure to model these according to Fermi-liquid theory have raised many interesting questions. The generic behavior of electrons in a 1D conductor is described by the Luttinger liquid (LL) model,[1,2] when one considers externally screened short-range interactions. Experimental realizations of the phenomena predicted by the LL model could be based on carbon nanotubes,[3,4] quantum wires in semiconductor heterostructures operated in the single-channel limit,[5] long chain molecules,[6] or edge states in a fractional quantum Hall bar.[7]

In this paper, we study electrical transport in 1D conductors and incorporate external voltage sources adiabatically connected to the quantum wire. We focus on the simplest case of a spinless LL described by the standard interaction parameter $g \leqslant 1$, where $g=1$ is the noninteracting limit and a small value of g equals strong correlations. The extension to spin-$\frac{1}{2}$ electrons or to nanotubes is then straightforward. Taking into account backscattering effects due to impurities, the generic behavior at low-energy scales can be studied by considering a pointlike scatterer of arbitrary strength λ. Then $\lambda=0$ corresponds to a clean conductor, while $\lambda\to\infty$ is the limit of perfect reflection. Again the extension to a more complicated situation, e.g., several impurities, is straightforward and not further discussed here.

This underlying geometry is shown in Fig. 1. For clarity, we discuss the case of a gated single-channel quantum wire (QW) extending from $-L/2<x<L/2$. The screening backgate is responsible for short-range interactions within the QW, and in the single-channel limit under consideration here, a LL is formed.[8,9] Possibly with minor modifications, however, the theory applies to all 1D correlated electron systems.

At the ends of the QW, reservoirs are assumed to be adiabatically connected. We consider ideal reservoirs as in the standard Landauer approach for Fermi-liquid conductors.[10,11] The reservoirs are held at chemical potentials $\mu_{1,2}$, and since one has good screening in the (2D or 3D Fermi liquid) reservoirs, the difference $U=(\mu_1-\mu_2)/e$ is the applied two-terminal voltage. For simplicity, we consider only time-independent voltages in this paper. It is shown in Sec. III that the presence of the voltage sources leads to boundary conditions, which in turn allow for the application of powerful theoretical techniques, e.g., bosonization,[1] refermionization,[2] or boundary conformal field theory.[12]

Other work has also dealt with similar questions as the ones addressed here. (1) There have been attempts to describe the effects of a reservoir by a 1D LL with $g=1$, both for the clean case[13] and allowing for impurity backscattering.[14] Albeit such calculations can explain the experimentally observed absence of a conductance renormalization in a perfectly clean system,[5] it remains unclear whether this approach can properly account for ideal reservoirs. Furthermore, calculations become bulky if one includes the impurity backscattering. (2) Other studies have simply assumed a local voltage drop at the impurity.[15] As discussed in Sec. III, this assumption is justified only if the impurity backscattering is effectively very strong. (3) In the clean case, Kubo-formula-based theories have been presented[16] to explain the perfect conductance $G=2e^2/h$. (4) Yet another approach for a clean system models reservoirs by charges conjugate to the chemical potentials of the voltage sources.[17]

We believe that our approach may offer the simplest and most general answer to the question of how to incorporate external voltage sources. It is the natural extension of Landauer's original approach designed for uncorrelated electrons[10] to a strongly correlated situation. The theory applies for arbitrary interaction strength g and impurity back-

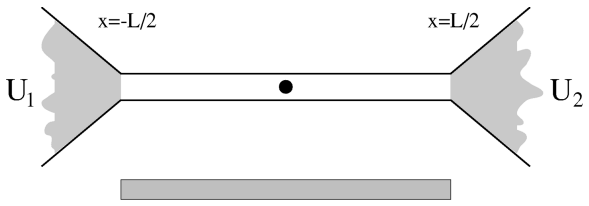

FIG. 1. Gated quantum wire as a model of the Luttinger liquid. Reservoirs held at chemical potential $\mu_{1,2}=eU_{1,2}$ are adiabatically connected to the 1D conductor at $x=\mp L/2$. A backscatterer of strength λ (indicated by the filled circle) is located at $x=0$. The screening gate is responsible for externally screened interactions within the quantum wire.

scattering λ. In particular, as an example of experimental relevance, we determine the full crossover from perfect conductance quantization in a clean wire to the anomalous power-law conductance suppression in a disturbed wire[15] as the impurity strength is varied.

The structure of this paper is as follows. In Sec. II the LL concept is introduced using the gated QW as an example. In Sec. III, boundary conditions describing the applied voltage sources are presented. Several general consequences of the voltage sources are collected in Sec. IV. In Sec. V, the full transport problem for arbitrary backscattering strength λ is solved for the special LL parameter $g=1/2$. Finally, some conclusions can be found in Sec. VI. A brief account of some results presented in Secs. III and IV has been given before in Ref. 18.

II. LUTTINGER LIQUID

We start by summarizing the Luttinger-liquid concept, which allows for a convenient description of 1D conductors. Due to Coulomb repulsion, the electrons in a 1D quantum wire have a tendency to occupy states on a Wigner lattice. As familiar from lattice excitations, the electronic configurations can be described in terms of a ''displacement field'' $\theta(x)$. Remarkably, as far as the low-energy physics is concerned, one can always find a one-to-one transmutation relating the 1D interacting fermion system to an equivalent bosonic system described by this displacement field (''bosonization''). In the bosonization scheme, the right- and left-moving component ($p=R/L=\pm$) of the electron operator is expressed in terms of the displacement field,[2]

$$\psi_p(x)=(2\pi a)^{-1/2}\exp[-ipk_Fx-ip\sqrt{\pi}\theta(x)-i\sqrt{\pi}\phi(x)], \tag{2.1}$$

where $a\approx 1/k_F$ corresponds to the lattice spacing of the associated Wigner lattice and k_F is the Fermi momentum. The dual field $\phi(x)$ obeys the commutation relation

$$[\phi(x),\theta(x')]_-=-(i/2)\mathrm{sgn}(x-x'). \tag{2.2}$$

At low-energy scales, any particular dispersion relation can be linearized around the Fermi energy, and the kinetic energy is then formally given by a massless Dirac Hamiltonian. Applying Eq. (2.1) then yields

$$H_0=\frac{\hbar v_F}{2}\int dx[(\partial_x\phi)^2+(\partial_x\theta)^2], \tag{2.3}$$

with the Fermi velocity v_F. The low-energy excitations are simply harmonic charge-density wave oscillations.

In the LL model, one considers externally screened short-ranged interactions which, for the purpose of a low-energy theory, can be represented by the interaction potential $U_c(x-x')=u_0\delta(x-x')$. On length scales larger than the screening length imposed by the gate, the interaction potential acting in the QW will always take this form. The standard LL parameter g is then given by

$$g=(1+u_0/\pi\hbar v_F)^{-1/2}, \tag{2.4}$$

and the interaction Hamiltonian reads

$$H_I=\frac{u_0}{2}\int dx\,\rho^2(x). \tag{2.5}$$

Here $\rho(x)$ is the total density relative to the zero-voltage equilibrium value $\rho=k_F/\pi$. Including the mixed components of the density operator, e.g., $\psi_R^\dagger\psi_L$, in Eq. (2.5) corresponds to retaining the electron-electron backscattering. Following standard arguments,[2] this is an irrelevant perturbation which can be taken into account by a simple renormalization of the LL parameters. The density $\rho(x)$ entering H_I is then only due to the densities $\rho_{R/L}$ of right- and left-moving fermions, $\rho=\rho_R+\rho_L$, for which Eq. (2.1) yields $\rho(x)=\partial_x\theta(x)/\sqrt{\pi}$. Thereby the LL Hamiltonian emerges,

$$H_{\mathrm{LL}}=\frac{\hbar v}{2}\int_{-L/2}^{L/2}dx[g(\partial_x\phi)^2+g^{-1}(\partial_x\theta)^2]. \tag{2.6}$$

For simplicity, we assume full translation invariance in the QW such that the sound velocity $v=v_F/g$. Remarkably, the low-energy excitations of the interacting system are still harmonic oscillations, and therefore H_{LL} allows for an exact solution.

Next consider a scatterer sitting at, say, $x=0$. The important backscattering[19] comes from the mixed component of the density operator, $H_{\mathrm{imp}}\sim\psi_R^\dagger(0)\psi_L(0)+\mathrm{H.c.}$, which, expressed in terms of the displacement field, leads to

$$H_{\mathrm{imp}}=\lambda\cos[\sqrt{4\pi}\theta(0)]. \tag{2.7}$$

The energy λ is a measure of the impurity backscattering strength. The Hamiltonian $H=H_{\mathrm{LL}}+H_{\mathrm{imp}}$ has been subject of intense theoretical effort in the past few years. The purpose of our paper is to clarify how this strongly correlated model should incorporate applied voltage sources.

For the gated QW in Fig. 1, the interaction contribution H_I in Eq. (2.5) can be interpreted as the charging energy $(e^2/2c)\int dx\,\rho^2(x)$ of the gate-QW capacitor, where $c=e^2/u_0$ is the capacitance per unit length. The electrostatic potential $\varphi(x)$ in the QW then follows by comparison with

$$H_I=(e/2)\int dx\,\rho(x)\varphi(x) \tag{2.8}$$

as

$$e\varphi(x)=u_0\rho(x). \tag{2.9}$$

Since we have an effectively short-ranged interaction, the Poisson equation is replaced by Eq. (2.9) here. The electrostatic potential directly gives the local potential drop between the QW and the screening backgate. The noninteracting limit $u_0=0$ ($g=1$) then implies that the backgate is located within the wire. The electrostatic potential is thus zero everywhere. In contrast, if no gate is present, i.e., in the limit of unscreened Coulomb interactions, we can put $g\to 0$ in the long-wavelength limit.

Suppose now that the densities ρ_R^0 and ρ_L^0 of right- or left-moving electrons are injected into the QW. This will charge the gate-QW capacitor and imply a voltage drop according to Eq. (2.9). The electrostatic potential (2.9) shifts the band bottom by $-e\varphi(x)$. With the density of states $1/\pi\hbar v_F$ this implies a shift of the total density by

$-e\varphi(x)/\pi\hbar v_F$. Therefore the actual density in the QW has to be self-consistently determined from Eq. (2.9) and the relation

$$\rho=\rho_R+\rho_L=\rho_R^0+\rho_L^0-e\varphi/\pi\hbar v_F. \tag{2.10}$$

Using Eq. (2.4) the solution is

$$\rho(x)=\rho_R(x)+\rho_L(x)=g^2[\rho_R^0(x)+\rho_L^0(x)]. \tag{2.11}$$

Since the electrostatic potential is only linked to the total density via Eq. (2.9), the difference of the R/L moving densities stays invariant,

$$\rho_R-\rho_L=\rho_R^0-\rho_L^0. \tag{2.12}$$

This difference determines the current flowing through the QW,

$$I=ev_F(\rho_R-\rho_L), \tag{2.13}$$

which can be computed at any point x due to the continuity equation. We note in passing that for the ac case, a displacement current has to be added to Eq. (2.13).

III. VOLTAGE SOURCES

Next we wish to include the adiabatically connected external voltage sources indicated in Fig. 1. The left reservoir held at chemical potential $\mu_1=eU_1$ injects the bare density

$$\rho_R^0(-L/2)=eU_1/2\pi\hbar v_F \tag{3.1}$$

of right-movers into the left end of the QW. Similarly, the right reservoir with $\mu_2=eU_2$ injects a bare density

$$\rho_L^0(L/2)=eU_2/2\pi\hbar v_F \tag{3.2}$$

of left-movers into the right end. These bare injected densities cannot depend on the intrinsic properties of the QW. In particular, they must be independent of the LL parameter g and of the backscattering strength λ. With the density of states $1/\pi\hbar v_F$, and noting that a factor 1/2 arises because only the left- or right-moving density is injected, Eqs. (3.1) and (3.2) readily follow. The outgoing particle densities are not fixed by the reservoirs. Outgoing particles are assumed to enter ideal reservoirs without reflection at the interface between QW and reservoir.

According to Eqs. (2.11) and (2.12), we can express the bare injected densities in terms of the true right- and left-moving densities,

$$\rho_R^0(x)=\frac{g^{-2}+1}{2}\rho_R(x)+\frac{g^{-2}-1}{2}\rho_L(x), \tag{3.3}$$

$$\rho_L^0(x)=\frac{g^{-2}-1}{2}\rho_R(x)+\frac{g^{-2}+1}{2}\rho_L(x). \tag{3.4}$$

From these relations and Eqs. (3.1) and (3.2), it is immediately clear that the external reservoirs can be completely described in terms of boundary conditions for the asymptotic true right- and left-moving densities $\rho_{R/L}$ in the QW. These boundary conditions should be imposed for the (ground-state or thermal) expectation value of the densities. As a shorthand notation, however, the appropriate $\langle\cdots\rangle$ brackets are mostly omitted in the sequel.

Employing Eq. (2.1), the densities ρ_R and ρ_L can now be expressed in terms of the displacement field $\theta(x,t)$,

$$\rho_R+\rho_L=\frac{1}{\sqrt{\pi}}\partial_x\theta, \tag{3.5}$$

$$\rho_R-\rho_L=\frac{1}{\sqrt{\pi}v_F}\partial_t\theta. \tag{3.6}$$

Thereby we arrive at *radiative boundary conditions* for the displacement field,

$$\left(\frac{1}{g^2}\partial_x+\frac{1}{v_F}\partial_t\right)\langle\theta(x=-L/2,t)\rangle=\frac{eU_1}{\sqrt{\pi}\hbar v_F}, \tag{3.7}$$

$$\left(\frac{1}{g^2}\partial_x-\frac{1}{v_F}\partial_t\right)\langle\theta(x=L/2,t)\rangle=\frac{eU_2}{\sqrt{\pi}\hbar v_F}, \tag{3.8}$$

which have to be fulfilled at all times t in the stationary nonequilibrium state. They hold provided ideal reservoirs are *adiabatically* connected to the QW and one is in the *low-energy regime*, where both the applied voltage $U=U_1-U_2$ and the temperature are very small compared to the bandwidth. The latter is of the order of the Fermi energy $E_F\approx\hbar v_F k_F$. The consequences of the boundary conditions (3.7) and (3.8) are investigated in the next two sections. In the remainder of this section, we focus on the two limiting cases of perfect transmission and perfect reflection.

Starting with the *clean case*, $\lambda=0$, we first observe that all densities are x independent along the QW. From Eqs. (2.11) and (2.12), the true right- and left-moving densities are given by

$$\rho_R=\tfrac{1}{2}(\rho_R^0-\rho_L^0)+\frac{g^2}{2}(\rho_R^0+\rho_L^0), \tag{3.9}$$

$$\rho_L=\tfrac{1}{2}(\rho_L^0-\rho_R^0)+\frac{g^2}{2}(\rho_R^0+\rho_L^0). \tag{3.10}$$

Even if no left movers are injected ($U_2=0$), the shift of the band bottom due to the charging of the gate-QW capacitor will induce a change in the density ρ_L of left movers. These relations directly imply from Eq. (2.13) the current

$$I=(e^2/h)U, \tag{3.11}$$

which is the perfect conductance quantization observed experimentally.[5] There is no renormalization of the d.c. conductance of a clean QW by the electron-electron interaction.[13,16–18]

The excess density $\rho=\rho_R+\rho_L$ charging the gate-QW capacitor is given by

$$\rho=\frac{g^2e(U_1+U_2)}{2\pi\hbar v_F}, \tag{3.12}$$

and the electrostatic potential drop between the QW and the backgate is then found from Eq. (2.9),

$$\varphi=(1-g^2)\frac{U_1+U_2}{2}. \tag{3.13}$$

The rather incomplete screening in one dimension[9] implies that only a fraction $(1-g^2)$ of the average potential shift $(U_1+U_2)/2$ is compensated by the backgate, leaving a fraction g^2 of the bare density as true charge density. For a long-ranged $1/r$ interaction, one has $g\to 0$ in the long-wavelength limit, and perfect electroneutrality ($\rho=0$) is recovered. In that case, the electrostatic potential follows the chemical potential, $e\varphi=\mu$. On the other hand, for the non-interacting case $g=1$, the electrostatic potential vanishes, and the density is fully given by the injected density. We note that for any g, there is no electric field acting along the QW since the electrostatic potential is constant. The current flowing through the QW is of purely chemical origin.

Next we turn to the case of *perfect reflection*, $\lambda\to\infty$. Since no current can flow, we have $\rho_R(x)=\rho_L(x)$, and Eqs. (2.11) and (2.12) then yield

$$\rho(x<0)=g^2eU_1/\pi\hbar v_F,$$

$$\rho(x>0)=g^2eU_2/\pi\hbar v_F.$$

The density drop across the insulating barrier is then given by

$$\Delta\rho=\rho(x<0)-\rho(x>0)=g^2eU/\pi\hbar v_F. \tag{3.14}$$

From Eq. (2.9) we find the electrostatic voltage drop across the barrier,

$$\Delta\varphi=(1-g^2)U, \tag{3.15}$$

which is the applied two-terminal voltage reduced by the characteristic underscreening factor $(1-g^2)$. Note that the potential drop between the QW and the gate is $(1-g^2)U_1$ for $x<0$, and $(1-g^2)U_2$ for $x>0$, respectively. This yields again Eq. (3.15). Of course, Eq. (3.14) can be decomposed into a chemical potential part and an electrostatic part,

$$\Delta\rho=\frac{\Delta\mu-e\Delta\varphi}{\pi\hbar v_F}, \tag{3.16}$$

where $\Delta\mu=\mu_1-\mu_2=eU$. Electroneutrality is recovered only for $g=0$, with $\Delta\varphi=U$. Finally, for $g=1$, there is no electrostatic potential drop across the barrier.

IV. GENERAL EFFECTS OF THE VOLTAGE SOURCES

Next we discuss general consequences of the applied voltage $U=U_1-U_2$ for the system depicted in Fig. 1. Extending the reasoning of Ref. 18 to the real-time case, we introduce a new field $q=\sqrt{4\pi}\theta(0)$ by means of a Lagrange multiplier field η. This has the advantage of rendering the $\theta(x)$ degree of freedom in a Gaussian form, and the nonlinearity due to $H_{\rm imp}$ affects only q. We shall employ a path-integral representation in the following.

Since it is convenient to integrate out the θ field, all fields have to be defined on the Keldysh contour $\mathcal{C}$ extending from time $z=-\infty$ to $z=\infty$ (forward path) and back from $z=\infty$ to $z=-\infty$ (backward path). For instance, the field $q(z)$ consists of a forward path $q_f(t)$ and a backward path $q_b(t)$, where the time variable t now runs from $-\infty$ to ∞. The action then reads

$$S=\int_{\mathcal{C}} dz\, L[\theta(z),q(z),\eta(z)], \tag{4.1}$$

with the Lagrange function

$$L=\frac{\hbar v}{2g}\int dx\left[\frac{1}{v^2}(\partial_z\theta)^2-(\partial_x\theta)^2\right]-\lambda\cos q(z)$$
$$-\eta(z)[q(z)-\sqrt{4\pi}\theta(0,z)]. \tag{4.2}$$

The $\theta(x)$ field can now be eliminated by Gaussian integration subject to the radiative boundary conditions (3.7) and (3.8). This is achieved by solving the Euler-Lagrange equation,

$$\left(\frac{1}{v^2}\partial_z^2-\partial_x^2\right)\theta(x,z)=\sqrt{4\pi}\eta(z)g^2\delta(x)/\hbar v_F. \tag{4.3}$$

The solution to this equation can always be decomposed into a particular solution θ_p subject to the boundary conditions plus the homogeneous solution obtained for $U_1=U_2=0$. The latter is in fact well known, see Ref. 15. A particular solution obeying both Eq. (4.3) and the boundary conditions (3.7) and (3.8) is

$$\theta_p(x,z)=\frac{g^2e[(U_1+U_2)x-V|x|]}{\sqrt{4\pi}\hbar v_F}+\frac{e(U-V)z}{\sqrt{4\pi}\hbar}, \tag{4.4}$$

for both the forward and the backward path. The quantity V appears as the zero mode of the Lagrange multiplier field $\eta(z)$. The physical meaning of V is the *four-terminal voltage* as becomes clear from the following discussion.

Since the expectation value of the density operator $\rho(x)$ at $|x|\gg a$ is determined by the particular solution alone,[18] we obtain from $\rho(x)=\partial_x\theta/\sqrt{\pi}$ the result

$$\langle\rho(x)\rangle=\frac{g^2e(U_1+U_2)}{2\pi\hbar v_F}-\frac{g^2eV}{2\pi\hbar v_F}\mathrm{sgn}\,x. \tag{4.5}$$

The first term is just Eq. (3.12) describing the change in the overall charge density. It can be trivially gauged to zero by choosing $U_1=-U_2=U/2$. The second term is more interesting. It gives the asymmetric charge density in the presence of an applied voltage. The density drop across the barrier is thus

$$\Delta\rho=g^2eV/\pi\hbar v_F, \tag{4.6}$$

such that there is an associated drop in the effective chemical potential of size $\Delta\mu=g^2eV$. Equation (2.9) then yields the electrostatic potential drop at $x=0$,

$$\Delta\varphi=(1-g^2)V. \tag{4.7}$$

In a measurement of the four-terminal voltage,[20] the observed voltage drop is $\Delta\mu/e+\Delta\varphi$, which is just V. Therefore V is indeed the four-terminal voltage. Since V is introduced via the Lagrange multiplier field η, it is in general a fluctuating quantity.

The ensuing steps are rather straightforward. Since the technical details[18,21] are of no interest here, we will only

sketch the analysis. Solving Eq. (4.3) for the homogeneous solution θ_h and inserting $\theta=\theta_p+\theta_h$ back into S, one is left with a Gaussian average over the Lagrange multiplier field [except of the zero mode V, over which we average separately]. Carrying out this Gaussian integration, we obtain the effective action for averaging the local degree of freedom $q(z)$ and the four-terminal voltage V,

$$S_{\rm eff}=i\Phi[q(z)]-\lambda\int_{\mathcal{C}}dz\,\cos[q(z)+e(U-V)z/\hbar] -(eV/2\pi)\int_{\mathcal{C}}dz\,q(z). \tag{4.8}$$

The effects of the external voltage sources are contained in the second and the third term. The first term can be written as

$$\Phi=\int_{\mathcal{C}}dz\int_{z>z'}dz'\,q(z)L(z-z')q(z')+\frac{iA}{2}\int_{\mathcal{C}}dz\,q^2(z), \tag{4.9}$$

where $L(z)$ has the same form as the heat bath kernel in dissipative quantum mechanics,[22]

$$L(z)=\frac{\hbar}{\pi}\int_0^\infty d\omega\,J(\omega)\frac{\cosh[\omega(-iz+\hbar\beta/2)]}{\sinh[\omega\hbar\beta/2]} \tag{4.10}$$

with $\beta=1/k_BT$. The spectral density $J(\omega)$ is of Ohmic form,

$$J(\omega)=\frac{\omega}{2\pi g}\exp[-\hbar\omega/E_F], \tag{4.11}$$

where an exponential bandwidth cutoff has been chosen. Finally, the quantity A in Eq. (4.9) is given by

$$A=\frac{2\hbar}{\pi}\int_0^\infty d\omega\,J(\omega)/\omega. \tag{4.12}$$

The dissipation acting on $q(z)$ effectively comes from the eliminated degrees of freedom away from the scatterer.[15]

The effects of the applied voltage can now be read off from Eq. (4.8). The last term in $S_{\rm eff}$ is a *voltage drop* contribution obtained by making the assumption that there is a local voltage drop V at the impurity. Under this assumption, one can include the coupling to the voltage sources by adding the term

$$\tilde{H}=eV\theta(0)/\sqrt{\pi} \tag{4.13}$$

to the Hamiltonian. Notably, it is in general not the externally applied two-terminal voltage but the fluctuating four-terminal voltage that determines this part. The second effect is a Josephson-like time dependence in the argument of the second term in Eq. (4.8). Most importantly, because of this term one cannot describe all effects of the applied voltage by simply adding terms like Eq. (4.13) to the Hamiltonian. In general, one has to solve the problem under the radiative boundary conditions (3.7) and (3.8).

Let us now briefly discuss the four-terminal voltage V. In the clean case, $\lambda=0$, the field q describes a massless particle such that $V=0$ results from the associated infrared divergence. This is of course in accordance with Eq. (4.6), since there is no density drop if there is no barrier. In the limit of perfect reflection, $\lambda\to\infty$, the four-terminal voltage is $V=U$, as enforced by the rapidly oscillating impurity contribution in Eq. (4.8). This value can also be obtained by comparing Eqs. (3.14) and (4.6). As a function of λ, the four-terminal voltage thus exhibits a crossover from $V=0$ at $\lambda=0$ to $V=U$ for $\lambda\to\infty$. Contrary to the Fermi-liquid case, this crossover now sensitively depends on the energy scales k_BT and eU under consideration, see Sec. V.

The effective action (4.8) may serve as starting point for further calculations, e.g., of the current-voltage characteristics. We shall not pursue this approach here but instead present an exact solution for the special interaction strength $g=1/2$.

V. EXACT SOLUTION

In this section we present the exact solution of the transport problem depicted in Fig. 1 for the special LL parameter $g=1/2$. This value has been discussed previously,[12,15,23–25] essentially by assuming a local voltage drop term, i.e., by using the effective action (4.8) under the assumption $V=U$. However, this assumption is only justified for a strong scatterer or at extremely low-energy scales, and one cannot recover the perfect conductance $G=e^2/h$ of a clean QW using that approach. Our exact solution for arbitrary transmission reported below does not make the voltage drop assumption but instead uses the boundary conditions (3.1) and (3.2) to describe the coupling to the reservoirs. Thereby the full crossover between the perfect conductance quantization and the asymptotic low-energy localization due to the impurity is obtained.

To start, we introduce the chiral boson fields

$$\varphi_R(x)=\sqrt{\pi}\left[\frac{1}{\sqrt{g}}\theta(x)+\sqrt{g}\,\phi(x)\right], \tag{5.1}$$

$$\varphi_L(x)=\sqrt{\pi}\left[-\frac{1}{\sqrt{g}}\theta(x)+\sqrt{g}\,\phi(x)\right]. \tag{5.2}$$

According to Eq. (2.2), they obey the algebra ($p=R,L=\pm$)

$$[\varphi_p(x),\varphi_{p'}(x')]_-=-i\pi p\,\delta_{pp'}\,\mathrm{sgn}(x-x'). \tag{5.3}$$

The right- and left-moving densities in the QW are

$$\rho_{R,L}(x)=\pm\frac{\sqrt{g}}{2\pi}\partial_x\varphi_{R,L}(x), \tag{5.4}$$

and the Hamiltonian $H=H_{\rm LL}+H_{\rm imp}$ reads

$$H=\frac{\hbar v}{8\pi}\int dx\{(\partial_x\varphi_R)^2+(\partial_x\varphi_L)^2\} +\lambda\cos\{\sqrt{g}[\varphi_R(0)-\varphi_L(0)]\}. \tag{5.5}$$

Next we incorporate the applied voltage sources according to the boundary conditions (3.1) and (3.2). Using the relations (3.3) and (3.4), they lead to the conditions

$$(g^{-2}+1)\rho_R(-L/2)+(g^{-2}-1)\rho_L(-L/2)=\frac{eU_1}{\pi\hbar v_F},$$
$$(g^{-2}-1)\rho_R(L/2)+(g^{-2}+1)\rho_L(L/2)=\frac{eU_2}{\pi\hbar v_F}. \tag{5.6}$$

It is then of advantage to switch to new chiral right-moving fields defined by

$$\phi_{p=\pm}(x)=\frac{1}{\sqrt{2}}[\varphi_R(x)\mp\varphi_L(-x)], \tag{5.7}$$

subject to the algebra

$$[\phi_p(x),\phi_{p'}(x')]_-=-i\pi\delta_{pp'}\,\mathrm{sgn}(x-x'). \tag{5.8}$$

They define the densities

$$\tilde{\rho}_\pm(x)=\frac{1}{2\pi}\partial_x\phi_\pm(x)=\frac{1}{\sqrt{2g}}\{\rho_R(x)\mp\rho_L(-x)\}. \tag{5.9}$$

Thereby the boundary conditions (5.6) become conditions for the new chiral densities (5.9). Specializing on $g=1/2$, and taking the sum and difference of the emerging equations, we obtain

$$5\tilde{\rho}_-(-L/2)+3\tilde{\rho}_-(L/2)=e(U_1+U_2)/\pi\hbar v_F, \tag{5.10}$$

$$5\tilde{\rho}_+(-L/2)-3\tilde{\rho}_+(L/2)=eU/\pi\hbar v_F. \tag{5.11}$$

The Hamiltonian (5.5) expressed in terms of the new chiral fields for $g=1/2$ is

$$H=\frac{\hbar v}{8\pi}\int dx\{(\partial_x\phi_+)^2+(\partial_x\phi_-)^2\}+\lambda\cos[\phi_+(0)]. \tag{5.12}$$

It is now apparent that the $\phi_\pm$ fields are completely decoupled. The impurity term in the Hamiltonian (5.12) couples only to ϕ_+, and the applied voltage U also leads to a boundary condition only in the $(+)$ sector, see Eq. (5.11). The ϕ_- field is associated with the shift in the total density arising for $U_1\neq -U_2$. Since there is no backscattering in the $(-)$ sector, the density $\tilde{\rho}_-(x)$ stays constant along the QW, and, according to Eq. (5.10), we again obtain the excess density (3.12) injected by the reservoirs. This shift in the overall density does not lead to interesting physical effects. Putting $U_1=-U_2=U/2$, we only keep the ϕ_+ field in what follows.

By means of refermionization,[15,23–25] we can then obtain an exact solution. For that purpose, we first introduce new fermion operators

$$\tilde{\psi}(x)=(2\pi a)^{-1/2}\exp[i\phi_+(x)]. \tag{5.13}$$

Following Matveev,[25] it is convenient to switch in a second step to the fermion operators ψ defined by

$$\tilde{\psi}(x)=(c+c^\dagger)\psi(x), \tag{5.14}$$

where c is an auxiliary fermion. Expressed in terms of these fermion operators, the $(+)$ sector of the Hamiltonian (5.12) reads

$$H=-i\hbar v\int dx\,\psi^\dagger(x)\partial_x\psi(x)$$
$$+(\hbar v\lambda_B/2)^{1/2}(c+c^\dagger)[\psi(0)-\psi^\dagger(0)], \tag{5.15}$$

with the effective impurity strength

$$\lambda_B=\pi a\lambda^2/\hbar v. \tag{5.16}$$

Remarkably, in the refermionized version (5.15) the Hamiltonian attains a very simple form, which can be diagonalized by, e.g., the equation-of-motion method.[24] Switching to Fourier space,

$$\psi(x,t)=\frac{1}{L}\sum_k\exp[ik(vt-x)]\times\begin{cases}a_k & (x<0)\\ b_k & (x>0),\end{cases} \tag{5.17}$$

where k runs over integer multiples of $2\pi/L$ and a_k, b_k denote fermion operators, the equations of motion dictate[24,26]

$$b_k=\tfrac{1}{2}(1+e^{i\alpha_k})a_k+\tfrac{1}{2}(1-e^{-i\alpha_k})a^\dagger_{-k}, \tag{5.18}$$

where the scattering phase shift α_k is defined by

$$e^{i\alpha_k}=e^{-i\alpha_{-k}}=\frac{i\hbar vk-\lambda_B}{i\hbar vk+\lambda_B}. \tag{5.19}$$

So far the analysis has closely followed previous work, see, e.g., Ref. 24. Now we have to take into account the boundary condition (5.11) in order to incorporate the applied voltage U. First we note that the density operator $\tilde{\rho}_+(x)$ defined in Eq. (5.9) can equivalently be expressed in terms of the new fermion operator $\psi(x)$,

$$\tilde{\rho}_+(x)=\psi^\dagger(x)\psi(x). \tag{5.20}$$

Employing Eq. (5.17), the boundary condition (5.11) then leads to

$$\frac{1}{L}\sum_k\{5\langle a_k^\dagger a_k\rangle'-3\langle b_k^\dagger b_k\rangle'\}=eU/\pi\hbar v_F. \tag{5.21}$$

The brackets indicate a stationary nonequilibrium average, and the prime stands for normal ordering with respect to the $U=0$ equilibrium state. Since the a_k correspond to free fermions, they must obey the Fermi distribution function,

$$\langle a_k^\dagger a_k\rangle\equiv n_k(k^*)=[1+\exp\{\hbar\beta v(k-k^*)\}]^{-1}, \tag{5.22}$$

where k^* has to be determined self-consistently. Using Eq. (5.18), we obtain

$$\langle b_k^\dagger b_k\rangle=\tfrac{1}{2}(1+\cos\alpha_k)n_k(k^*)+\tfrac{1}{2}(1-\cos\alpha_k)n_k(-k^*), \tag{5.23}$$

whence Eq. (5.21) with the scattering phase shift (5.19) yields

$$k^*+\frac{6\pi}{L}\sum_k[1+(\hbar vk/\lambda_B)^2]^{-1}\{n_k(k^*)-n_k(0)\}=\frac{eU}{\hbar v_F}. \tag{5.24}$$

In the remainder, we focus on the case of a very long QW, $L\to\infty$, such that sums can be converted into integrals, $(2\pi/L)\Sigma_k\to\int dk$. Carrying out the resulting integration, the condition (5.24) reads

$$k^*+\frac{3\lambda_B}{\hbar v}\mathrm{Im}\,\psi\left(\frac{1}{2}+\frac{\lambda_B+i\hbar v k^*}{2\pi k_B T}\right)=eU/\hbar v_F, \quad (5.25)$$

where $\psi(z)$ is the digamma function. For $\lambda_B=0$, this gives $k^*=eU/\hbar v_F$, while in the opposite limit of a strong scatterer, $\lambda_B\to\infty$, we obtain $k^*=eU/4\hbar v_F$. These two extreme values hold in fact for any value of the temperature T or the length L. The crossover as a function of λ_B between these two limits strongly depends on the energy scales k_BT and eU. Clearly, for $\lambda_B\gg eU$, we could effectively use the strong-coupling value $k^*=eU/4\hbar v_F$. This amounts to making the above-mentioned voltage drop assumption. In the general case, one first has to solve for k^* according to Eq. (5.25) before further calculations.

Let us now study the connection to the *four-terminal voltage* V discussed in the previous section. It can be obtained from the density drop $\Delta\rho=\rho(x<0)-\rho(x>0)$ at $x=0$. Using Eq. (5.9), we have

$$\Delta\rho=\tilde{\rho}_+(x<0)-\tilde{\rho}_+(x>0)=\Delta\tilde{\rho}_+, \quad (5.26)$$

which yields

$$\begin{aligned}\Delta\rho&=\int\frac{dk}{2\pi}\langle[a_k^\dagger a_k-b_k^\dagger b_k]\rangle'\\&=\int\frac{dk}{2\pi}(1-\cos\alpha_k)\{n_k(k^*)-n_k(0)\}\\&=\frac{\lambda_B}{\pi\hbar v}\mathrm{Im}\,\psi\left(\frac{1}{2}+\frac{\lambda_B+i\hbar v k^*}{2\pi k_B T}\right). \quad (5.27)\end{aligned}$$

Comparing with the general result (4.6), the four-terminal voltage V follows,

$$eV=2\lambda_B\,\mathrm{Im}\,\psi\left(\frac{1}{2}+\frac{\lambda_B+i\hbar v k^*}{2\pi k_B T}\right). \quad (5.28)$$

The generalization to finite length L is straightforward. From our exact solution, one can in principle also compute the *fluctuations* of the four-terminal voltage. When comparing with experiments, however, one may have to include the strong Friedel oscillation contribution.[20]

In the limit of a clean wire, from Eq. (5.28) we find $V=0$, in accordance with the general result for arbitrary g. In the opposite case, $\lambda_B\to\infty$, we obtain $V=U$ from Eqs. (5.25) and (5.28), again in accordance with the general result. The connection between k^* and V can now be read off,

$$k^*=e[U-3V/4]/\hbar v_F. \quad (5.29)$$

We stress that this relation holds for any T and L. Inserting Eq. (5.29) into Eq. (5.28), we can eliminate k^* and obtain a self-consistent equation for the four-terminal voltage,

$$eV/2\lambda_B=\mathrm{Im}\,\psi\left(\frac{1}{2}+\frac{\lambda_B+2ieU-3ieV/2}{2\pi k_B T}\right). \quad (5.30)$$

At zero temperature, this becomes

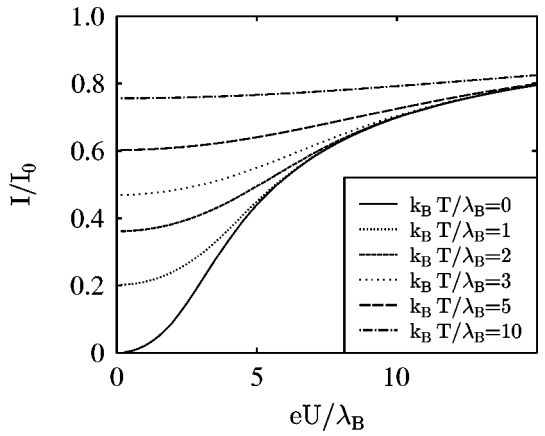

FIG. 2. Current-voltage characteristics for several temperatures T. The current has been normalized to $I_0=(e^2/h)U$. Note the very slow approach towards $I=I_0$ as temperature is raised.

$$eV/2\lambda_B=\tan^{-1}\{[2eU-3eV/2]/\lambda_B\}. \quad (5.31)$$

The relation (5.30) explicitly exhibits *scaling* with the effective impurity strength (5.16) acting as the energy scale, i.e., the energies k_BT, eU, and eV can be turned into dimensionless quantities by measuring them in units of λ_B. Therefore the boundary conditions preserve the important scaling property. For small λ_B, the four-terminal voltage V vanishes, and by increasing λ_B, a crossover to the strong-coupling value $V=U$ is observed.

Finally, we come to the *current-voltage characteristics*. The current flowing through the QW is computed from Eq. (2.13),

$$I=ev_F\langle\tilde{\rho}_+(0)\rangle=\frac{ev_F}{4}\int\frac{dk}{2\pi}\langle(a_k^\dagger+b_k^\dagger)(a_k+b_k)\rangle'.$$

Straightforward algebra yields the general result

$$I(U)=\frac{e^2}{h}(U-V), \quad (5.32)$$

with the four-terminal voltage $V=V(U,T,\lambda_B)$ self-consistently given in Eq. (5.30). Therefore the knowledge of the four-terminal voltage is sufficient to obtain the full nonlinear current-voltage characteristics. In the limit of a clean QW, $V=0$, and we indeed obtain the conductance quantum $G=e^2/h$. In the limit of very small applied voltage, $eU\ll\lambda_B$, and at zero temperature, the voltage drop assumption is correct, and the previous results[12,15,23,24] are recovered.

The exact current-voltage characteristics is plotted in Fig. 2 for various temperatures. Clearly, one has a perfect zero-bias anomaly at $T=0$, with the conductance vanishing $\sim U^2$ as predicted by Kane and Fisher.[15] Notably, Eq. (5.32) gives the full crossover behavior up to the perfect conductance $G=e^2/h$ of a clean QW.

VI. CONCLUSIONS

In this paper, the inclusion of external voltage sources to a one-dimensional single-channel quantum wire with arbitrary transmission has been discussed. This system is a prototypical example for a Luttinger liquid. By deriving radiative boundary conditions, we have demonstrated that the

Landauer approach to mesoscopic transport can be extended to the case of strongly correlated systems. The exact solution of the transport problem at the special value $g=1/2$ reveals that both the previous ''voltage drop'' results (which hold at sufficiently low voltage and temperature) and the perfect conductance quantization in a clean system can be recovered within a unified approach.

An obvious and interesting generalization concerns the ac case. Considering a situation where $U_1=U\cos(\omega t)$ and $U_2=0$, the boundary condition at the left end of the wire would read

$$\rho_R^0=\frac{eU_1\cos(\omega t-\omega x/v)}{2\pi\hbar v_F}. \tag{6.1}$$

The consequences of time-dependent boundary conditions have not been studied so far except in the clean case.[27] Our boundary condition approach also allows for a consideration of more complicated geometries. For instance, the problem of crossed Luttinger liquids allows for an elegant solution by employing this approach.[28]

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