Mean-field theory for the spin-ladder system

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In the present paper, we propose a mean-field approach for spin ladders based upon the Jordan-Wigner transformation along an elaborately ordered path. We show on the mean-field level that ladders with even-numbered legs open an energy gap in their low-energy excitation with a magnitude close to the corresponding experimental values, whereas the low-energy excitation of the odd-numbered-leg ladders are gapless. It supports the validity of our approach. We then calculate the gap size and the excitation spectra of a two-leg-ladder system. Our result is in good agreement with both the experimental data and the numerical results. [S0163-1829(98)09901-9]

I. INTRODUCTION

The study of the low-dimensional Heisenberg antiferromagnetic model is one of the most active research fields in condensed-matter physics. Haldane[1] conjectured that for integer-spin one-dimensional antiferromagnetic chains an energy gap exists in the low-energy excitation spectrum, but for a half-integer-spin case the excitation spectrum is gapless. The spin-1/2 antiferromagnetic chain can be solved exactly by the Bethe ansatz.[2] The excitation spectrum is found to be gapless. The measurement of realistic ladder material such as $SrCu_2O_3$ (two-leg), $Sr_2Cu_3O_5$ (three-leg), or $(VO)_2P_2O_7$ (two-leg)[3] shows that the spin-excitation gap is opened in the ladders with an even numbers of legs, while for ladders with an odd numbers of legs, no gap is found in spin excitation. This conclusion was predicted by early numerical calculations[4] and was explained qualitatively by Khveshchenko.[5] In Khveshchenko's explanation a topological term appears in the effective Hamiltonian of the long-wavelength dynamics in an odd-leg ladder and is absent in an even-leg ladder. Recently, Sierra[6] has mapped the ladder problem onto an effective one-dimensional (1D) nonlinear sigma model. An extra topological term appears in the odd-leg system and is absent in the even-ladder system. This difference between odd- and even-leg ladders is essentially an extension of the difference between the half integer and integer Heisenberg chains.[7]

We also have the experience from the 1D quantum Heisenberg spin chain, that it is not so trivial to incorporate the subtle physics of the topological term in a mean-field approach. For a two-leg ladder, the existence of an energy gap in the spin excitation for nonzero interchain coupling J' is confirmed by various methods such as Lanczos, quantum Monte Carlo,[4,8] renormalization group,[9] variational method,[10] strong-coupling expansion,[11] spin-liquid mean-field approach,[12] as well as the mean-field approach based on the Jordan-Wigner transformation.[13] For $J=J'$ (J is the exchange coupling along the chains), which is the physical parameter of the real ladder compounds, the energy gap obtained by numerical calculation is $0.5J$ which is very close to the experimental value. The spin-wave excitation spectra have also been obtained by the numerical calculation which shows the minimum of the spectra is located at the wave number π.[4,8] This was also confirmed by recent neutron-scattering experiment.[14]

A mean-field treatment of the two-leg Heisenberg ladder with the application of a Jordan-Wigner (JW) transformation has been proposed by Azzouz *et al.*[13] In their approach the Jordan-Wigner transformation is introduced to map the spin-1/2 system to the spinless Fermion system. The gap is obtained in their mean-field approach which is about $0.7J'$ under the case of $J=J'$ which does not fit well with the experimental value. The excitation spectra are also calculated in their approach, which contains a minimum at the wave number π. But the shape of the spectra is not consistent with the numerical result which predicts the maximum of the spectra locating between 0 and $\pi/2$.[8]

In the present paper, we propose a mean-field approach also based on the JW transformation.[15] Our mean-field approach is quite different from the approach used in Ref. 13. When performing a JW transformation in the ladder system, one must put the sites in a queue. Then the spin operator can be expressed as $S_i^+=c_i^+e^{i\pi\hat{\Phi}(i)}$, where $\hat{\Phi}(i)$ is nothing but the summation of number of the sites from $-\infty$ to the $(i-1)$th site in the particularly ordered queue which are occupied by spinless fermions. The difference between the particularly elaborated queues used in our approach and those of Ref. 13 are shown in Figs. 1(a) and 1(b). In our approach the sites in the odd number rungs are labeled from upward to downward but from downward to upward in the even number rungs [shown as Fig. 1(a)], whereas in Ref. 13 all rungs are labeled from upward to downward [shown as Fig. 1(b)].

We perform the JW transformation with such a specially ordered queue. The advantages of our approach are the following. First of all if we replace the phase of the hopping term by its average value, we can easily obtain the mean-field Hamiltonian for n-leg ladders in which the effective hopping terms of the spinless fermions have alternative signs along both the chain and rung directions. This mean-field Hamiltonian is similar to the one used in Ref. 13 for the two-leg case. But in their approach a further assumption, i.e., adding a π flux to each plaquette is needed to obtain such a mean-field Hamiltonian, whereas in our approach the mean-field Hamiltonian is obtained directly by replacing the phase factor in the hopping terms with their average values. We

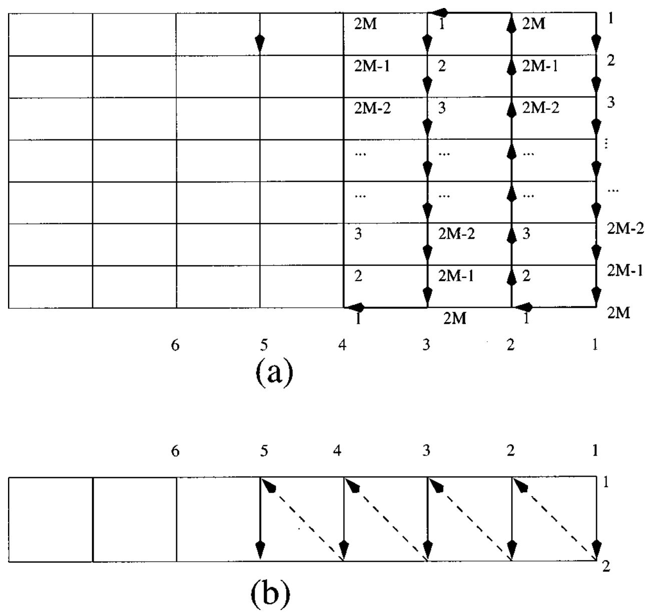

FIG. 1. To perform the Jordan-Wigner transformation the sites are put in a particular queue in our present study (a) and in Ref. 13 (b).

can then easily show that the spin gap is only opened in even-leg ladders. To a certain degree, it supports the validity of our way of performing the Jordan-Wigner transformation construction.

Secondly we can treat the phase factor in the hopping terms more carefully by introducing a self-consistent procedure for the two-leg ladder. We find that both the gap magnitude and the spin-excitation spectra are in good agreement with the numerical and experimental results in the case of $J'=J$ which is of the physical parameter for the real ladder compounds. Actually the energy gap is found to be $0.46J$ in the case $J'=J$ which is very close to the experimental result $0.47\pm 0.2J$. The spin-wave excitation spectra obtained by our mean-field calculation is also consistent with the numerical result with the minimum at the wave number π and the maximum at the wave number 0.356π. Compared with the approach used in Ref. 13, our mean-field approach is much better in the case of $J'>0.5J$. In the weak-coupling regime our approach does not work as well. The energy gap persists even in the case of $J'=0$. In spite of the unsatisfactory aspect in the weak coupling regime, our approach is still valuable because it works very well in the intermediate- and strong-coupling regimes, which have the experimental correspondence.

In Sec. II, we calculate the spin-excitation gap of the spin ladders with various numbers of legs and show that in the mean-field level the energy gap only exists in ladder systems with an even number of legs. In Sec. III, we propose a more careful treatment of the two-leg spin ladder. The spin-excitation gap, as well as the excitation spectra is calculated. Finally, we make concluding remarks in Sec. IV.

II. MEAN-FIELD TREATMENT FOR THE SPIN GAP IN *n*-LEG LADDERS

We begin with the $2M$-leg antiferromagnetic Heisenberg ladder Hamiltonian:

$$
\begin{aligned}
H &= J'\sum_i \sum_{p=1}^{2M-1} \vec{S}_{i,p}\cdot\vec{S}_{i,p+1} + J\sum_i \sum_{p=1}^{2M} \vec{S}_{i,p}\cdot\vec{S}_{i+1,2M+1-p} \\
&= J'\sum_i \sum_{p=1}^{2M-1} S^z_{i,p}\cdot S^z_{i,p+1} + J\sum_i \sum_{p=1}^{2M} S^z_{i,p}\cdot S^z_{i+1,2M+1-p} \\
&\quad + \frac{J'}{2}\sum_i \sum_{p=1}^{2M-1} S^+_{i,p}\cdot S^-_{i,p+1} + \frac{J}{2}\sum_i \sum_{p=1}^{2M} S^+_{i,p}\cdot S^-_{i+1,2M+1-p} \\
&\quad + \mathrm{H.c.}
\end{aligned}
\tag{1}
$$

In the above Hamiltonian, i represents the site position along the chains and p represents the $2M$ sites of different chains coupled by the interchain coupling constant J'. As shown in Fig. 1, p is labeled from upward to downward at the even sites and downward to upward at the odd sites. This is different from that used in the paper of Azzouz *et al.*[13] Then we introduce the generalized JW transformation:

$$
S^+_{p,i} = c^+_{p,i} e^{i\pi \sum_{n=-\infty}^{i-1} \sum_{l=1}^{2M} c^+_{l,n} c_{l,n} + i\pi \sum_{l=1}^{p-1} c^+_{l,i} c_{l,i}}, \tag{2}
$$

in which c is the spinless Fermion operator. The summation in the phase factor is the number of occupied sites before the ith site along the particular queue shown in Fig. 1(a). Then the quantum spin-1/2 Hamiltonian can be mapped onto a spinless fermion Hamiltonian as

$$
\begin{aligned}
H &= J' \sum_{i,p=1}^{2M-1} \left(\frac{1}{2} - c^+_{i,p} c_{i,p}\right)\cdot\left(\frac{1}{2} - c^+_{i,p+1} c_{i,p+1}\right) \\
&\quad + J \sum_{i,p=1}^{2M} \left(\frac{1}{2} - c^+_{i,p} c_{i,p}\right)\cdot\left(\frac{1}{2} - c^+_{i+1,2M+1-p} c_{i+1,2M+1-p}\right) \\
&\quad + \frac{J'}{2} \sum_{i,p=1}^{2M+1} (c^+_{i,p} c_{i,p+1} + \mathrm{H.c.}) \\
&\quad + \frac{J}{2} \sum_{i,p}^{2M} (c^+_{i,p} c_{i+1,2M+1-p} e^{-i\hat{\Phi}(p)} + \mathrm{H.c.}),
\end{aligned}
\tag{3}
$$

where

$$
\hat{\Phi}(p) = \pi \sum_{l=p+1}^{2m} (n_{i,l} + n_{i+1,2M+1-l}).
$$

In our mean-field approach, we replace $n_{i,l}$ by $\langle n_{i,l}\rangle$. For the present study, we further assume that the finite magnetization in this system is not possible because of the strong quantum fluctuation. This is reasonable for systems with a leg number much less than the site number of each chain. Then we have $\langle S^z_{i,l}\rangle = \langle 1/2 - c^+_{i,l} c_{i,l}\rangle = 0$ which implies $\langle n_{i,l}\rangle = 0.5$. Consequently, the phase factor in Eq. (3) can be approximated by: $\Phi(p) = \pi(2M-p)$. Moreover we decouple the four-fermion interaction term in the above Hamiltonian by the Hartree-Fock approximation. Finally the mean-field Hamiltonian of the spinless fermions has the expression:

$$H=\frac{J'}{2}\sum_{i,p=1}^{2M+1}(c_{i,p}^{+}c_{i,p+1}+\text{H.c.})+\sum_{i,p}^{2M}[c_{i,p}^{+}c_{i+1,2M+1-p}(-1)^{p+1}+\text{H.c.}]. \tag{4}$$

If we introduce a Fourier transformation for the site indices, we have

$$H=a\sum_{k,p=1}^{2M+1}(c_{k,p}^{+}c_{k,p+1}+\text{H.c.})+\frac{J}{2}\sum_{k,p}^{2M}[i\gamma_k(-1)^{p}c_{k,p}^{+}c_{k+1,2M+1-p}+\text{H.c.}], \tag{5}$$

where $a=(J'/2)$ $\gamma_k=-J\sin(k)$. The Hamiltonian then can be written in a form $H=\Sigma_k C_k^{+}h(k)C_k$ with $h(k)$

$$h(k)=\begin{pmatrix}
0 & a & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & i\gamma \\
a & 0 & a & 0 & .. & .. & .. & .. & .. & .. & .. & .. & -i\gamma & .. \\
0 & a & 0 & a & .. & .. & .. & .. & .. & .. & .. & i\gamma & .. & .. \\
.. & .. & a & 0 & a & .. & .. & .. & .. & .. & -i\gamma & .. & .. & .. \\
.. & .. & .. & a & .. & .. & .. & .. & .. & i\gamma & .. & .. & .. & .. \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
.. & .. & .. & .. & -i\gamma & .. & .. & .. & .. & .. & a & .. & .. & .. \\
.. & .. & .. & i\gamma & .. & .. & .. & .. & .. & a & 0 & a & .. & .. \\
.. & .. & -i\gamma & .. & .. & .. & .. & .. & .. & .. & a & 0 & a & .. \\
.. & i\gamma & .. & .. & .. & .. & .. & .. & .. & .. & .. & a & 0 & a \\
-i\gamma & .. & ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & a & 0
\end{pmatrix}.$$

The above matrix contains $2M$ eigenvalues for a given wave number k corresponding to the $2M$ individual bands separated by gaps. It can be proved straightforwardly that for a given wave number k, half of the eigenvalues are less than zero and other half are greater than zero. Furthermore we can also prove that zero is not an eigenvalue of the above matrix for any nonzero J'. We will prove the two statements in the Appendix. This result shows clearly that half of the energy bands of the spinless Fermions are below zero energy and another half is above zero. The energy gap between them is nonzero because zero is not the eigenvalue of the above matrix for nonzero J' and arbitrary k. Assuming there is no self-magnetization in one-dimensional systems, only the lower half of the states are occupied by spinless fermions in the ground state. Therefore a spinless fermion system is very similar to the traditional insulator in which the valence band is fully occupied and the conductive band is fully empty in the ground state. So a nonzero minimum energy is needed to excite the system from the ground state which indicates a spin-excitation energy gap. For spin ladders with an odd number of legs there exist odd numbers of energy bands. Since only half of the states is occupied in the ground state there must exist at least one band which is partially occupied. This picture is very similar to the traditional conductor in which there exists at least one partial occupied band (conduction band). Then the low-energy excitations are gapless for odd-numbered-leg ladders.

Moreover, we calculate the gap size of the 2,4,6,8, and 10-leg spin ladders, the results are shown in Fig. 2 together with the experimental value. Although our approach is quite rough, the result is in good agreement with the experimental value.[3,14,16,17]

The spin-excitation spectra can also be obtained from the above mean-field approach, the spin-wave dispersion is $\sqrt{(J'/2)^2+J^2\sin k^2}$ for the two-leg case. It has two energy minimums, one at 0 and another at π. The shape of the

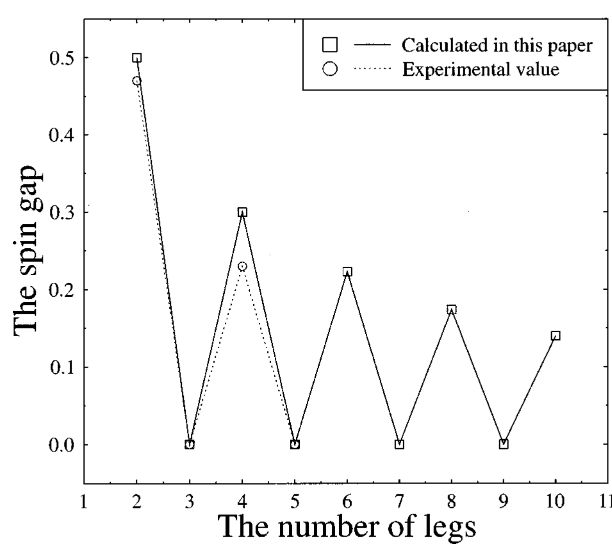

FIG. 2. The spin gap for n-leg spin ladders calculated in our mean-field approach which is compared by the experimental value (Refs. 14,17).

spectra is not consistent with the numerical result which has the spectra minimum located only at wave number π and the maximum is near 0.356π. This is because the treatment for the phase term of Eq. (3) is too rough. In the next section we will propose a more careful treatment of the phase factor in Eq. (3). We can then obtain a more improved spin-excitation spectra, which is very close to the numerical results.

III. THE MEAN-FIELD THEORY OF THE TWO-LEG LADDER

For the two-leg case, we can introduce two bipartite lattices labeled α and β. Following the same procedure shown in the above section, the Hamiltonian for spinless fermions becomes

$$H=J'\sum_i\left(\frac{1}{2}-\alpha_i^+\alpha_i\right)\left(\frac{1}{2}-\beta_i^+\beta_i\right)+J\sum_i\left(\frac{1}{2}-\alpha_i^+\alpha_i\right)\left(\frac{1}{2}-\beta_{i+1}^+\beta_{i+1}\right)+J\sum_i\left(\frac{1}{2}-\alpha_{i+1}^+\alpha_{i+1}\right)\left(\frac{1}{2}-\beta_i^+\beta_i\right)$$
$$+\frac{J'}{2}\sum_i(\alpha_i^+\beta_i+\text{H.c.})+\frac{J}{2}\sum_i[\alpha_i^+\beta_{i+1}e^{i\pi(\beta_i^+\beta_i+\alpha_{i+1}^+\alpha_{i+1})}+\text{H.c.}]+\frac{J}{2}\sum_i(\beta_i^+\alpha_{i+1}+\text{H.c.}). \tag{6}$$

In our mean-field approach, different from the simple treatment used in the previous section, we first replace the phase factor in Eq. (6) by its average value:

$$\langle e^{i\pi(\beta_i^+\beta_i+\alpha_{i+1}^+\alpha_{i+1})}\rangle=\langle(1-2\beta_i^+\beta_i)(1-2\alpha_{i+1}^+\alpha_{i+1})\rangle=-4|\chi_2|^2,$$

where we define:

$$\chi_1=\langle\beta_i^+\alpha_{i+1}\rangle,\quad \chi_2=\langle\alpha_i^+\beta_{i+1}\rangle,\quad \chi_0=\langle\beta_i^+\alpha_i\rangle.$$

Then the fermion-fermion interacting term $(1/2-\alpha_i^+\alpha_i)(1/2-\beta_i^+\beta_i)$ can be factorized as

$$(1/2-\alpha_i^+\alpha_i)(1/2-\beta_i^+\beta_i)=\frac{1}{4}-\chi_0\beta_i^+\alpha_i-\chi_0^+\alpha_i^+\beta_i+\chi_0^+\chi_0.$$

We decouple the other two interacting terms in the same manner and obtain the following mean-field Hamiltonian of spinless Fermions:

$$H_{\text{MF}}=\sum_k\gamma_k\alpha_k^+\beta_k+\text{H.c.}, \tag{7}$$

where

$$\gamma_k=\left[\left(\frac{J'}{2}-J'\chi_0\right)+\left(\frac{J}{2}-2J|\chi_2|^2-J\chi_1-J\chi_2\right)\cos(k)\right]+i\sin(k)\left(J\chi_2-J\chi_1-2J|\chi_2|^2-\frac{J}{2}\right).$$

Then the above Hamiltonian can be diagonalized as

$$H_{\text{MF}}=\sum_k E_k(\tilde{\alpha}_k^+\tilde{\alpha}_k-\tilde{\beta}_k^+\tilde{\beta}_k), \tag{8}$$

in which

$$E_k=\sqrt{\left[\left(\frac{J'}{2}-J'\chi_0\right)+\left(\frac{J}{2}-2J|\chi_2|^2-J\chi_1-J\chi_2\right)\cos(k)\right]^2+\sin^2(k)\left(J\chi_2-J\chi_1-2J|\chi_2|^2-\frac{J}{2}\right)^2}.$$

The three parameters χ_1, χ_2, and χ_0 are determined self-consistently. The gap size obtained within the present approach is shown in Fig. 3 and compared with the numerical results. Our results fit quite well to the numerical results in the parameter regime $J'/J>0.5$. For the case of $J'=J$, the three parameters are found to be $\chi_1=-0.188J$, $\chi_2=0.237J$, and $\chi_0=0.3867J$, and the gap is found to be $0.46J$ which is very close to the experimental value $(0.47\pm0.2)J$.[14] In the strong-coupling limit ($J'\gg J$) our result fits well with the result obtained by strong-coupling expansion,[11] which shows $\Delta/J'\to1$ when $J'/J\to\infty$. But in the regime $J'/J<0.5$ our results deviated from the numerical results, and a nonzero gap persists even at the case $J'=0$. So our mean-field approach is valid only in the intermediate- and strong-coupling regimes. In the weak-coupling regime our mean-field picture breaks down due to the overestimation of the interchain interaction. Since the phase factor in the hopping term in Eq. (6) is replaced by its average value, it makes the hopping term within one chain strongly modified by the motion of spinless fermions in the other chain, which is not valid for

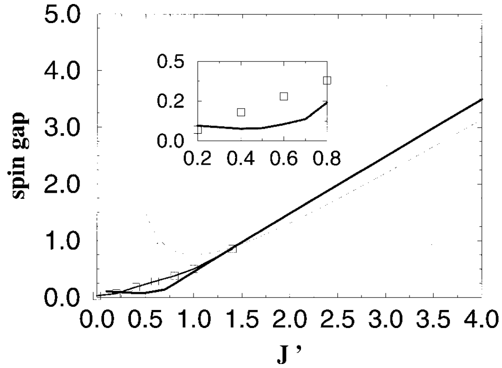

FIG. 3. The solid line shows the spin gap obtained by the self-consistent procedure for two-leg ladders as a function of interchain coupling J'. The dashed line is the result of strong-coupling expansion (Ref. 11). The squares show the numerical result of Ref. 4. The inset shows the results with J'/J less than 0.8.

the weak-coupling regime. We believe this approach is valid when the interchain coupling is on the order of unity, but is not valid for the weak-coupling case.

Another advantage of the present mean-field approach is that, in the case of $J'=J$, it gives the same spin-wave dispersion predicted by the numerical calculation as shown in Fig. 4. The minimum of the spectra is at the wave number π, and the maximum is at the wave number 0.356π. This result is in good agreement with the numerical result which has a minimum at π and maximum at 0.3π. We can also calculate the two-magnon continuum from our mean-field theory. The two magnon continuum is proportional to

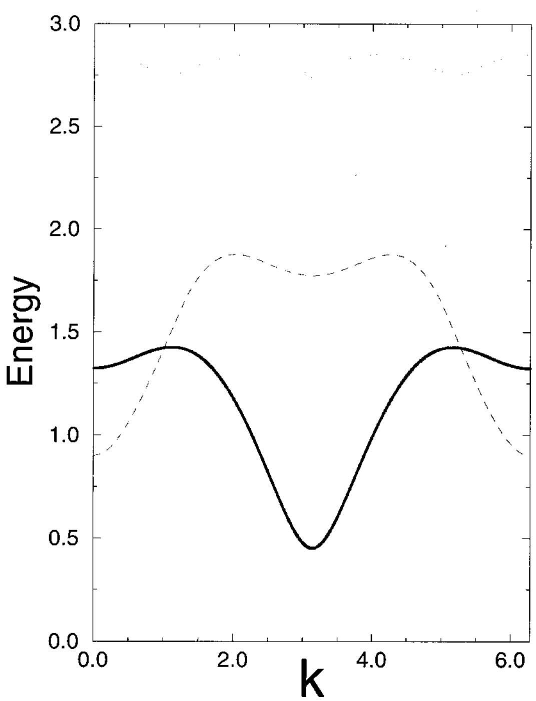

FIG. 4. The solid line is the dispersion of the spinless fermions for two-leg ladders calculated in our mean-field approach. The dashed (dotted) line is the bottom (top) of the two-magnon continuum.

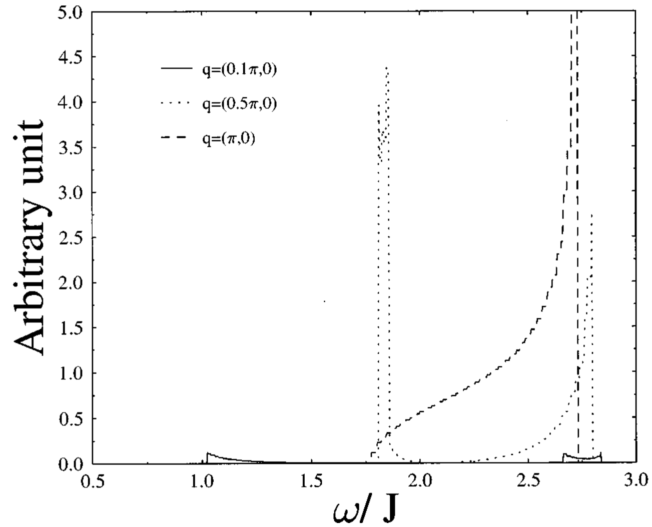

FIG. 5. The spectra of two-magnon excitation with $q=0.1\pi$ (dotted line), 0.5π (solid line), π (dashed line).

$$\int dt \langle S^z(q,t) S^z(-q,0)\rangle e^{-i\omega t},$$

which can be transformed into the density-density correlation of the spinless fermions by a Jordan-Wigner transformation. Then the two-magnon excitation can be viewed as a particle-hole excitation of the spinless fermions. The spectra of the two-magnon excitation with several specific q numbers is shown in Fig. 5. The bottom (top) of the two-magnon continuum is just the minimum (maximum) energy of hole-particle excitation of the spinless fermions for a given wave number. The result is shown by a dashed line (bottom) and a dotted line (top) in Fig. 4 and fits the numerical result quite well.[4,8] Compared to numerical methods such as density matrix renormalization group,[13] quantum Monte Carlo, and the Lanczos method,[4,8] our mean-field theory based on the Jordan-Wigner transformation gives a more transparent understanding of the gap formation in even-number-leg spin ladders and the low-energy spin excitation.

The spin susceptibility is also obtained by introducing a magnetic field in the original spinless fermion Hamiltonian, where this term acts like the chemical potential:

$$H=\sum_k \gamma_k \alpha_k^+ \beta_k + \text{H.c.} - \frac{1}{N}\sum_k (\alpha_k^+ \alpha_k + \beta_k^+ \beta_k) h.$$

The magnetization m then has the expression as

$$m=\frac{1}{N}\sum_i (1-\langle \alpha_i^+ \alpha_i\rangle - \langle \beta_i^+ \beta_i\rangle).$$

And the spin susceptibility could be derived as

$$\chi_s=\frac{\partial m}{\partial h}.$$

We calculate the spin susceptibility in the case of $J'=J$ in a wide range of temperatures showing the result in Fig. 6. Our result explains the temperature behavior of the spin susceptibility quite well, and it is again in good agreement with the numerical results which give a maximum at $T=0.8J$.

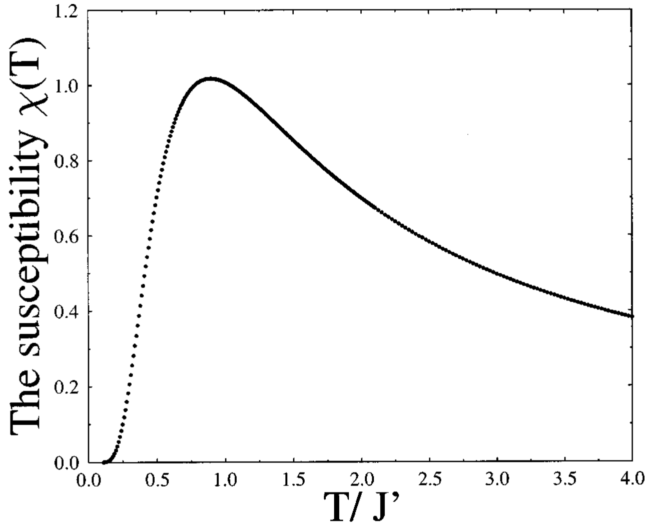

FIG. 6. The spin susceptibility of the two-leg ladder in the case of $J=J'$.

IV. CONCLUDING REMARKS

In this paper we propose a mean-field approach for spin ladders based on the Jordan-Wigner transformation along an elaborately chosen path defined above. We show that in the mean-field level that the spin gap is opened only in the even-numbered-leg ladders and vanishes in the odd-numbered-leg ladders. It gives a very simple picture of the formation and vanishing of the spin gap in the above-mentioned two types of spin ladders. The spin ladders with an even number of legs formed an insulatorlike band for spinless fermions, whereas in odd-number-leg ladders the band structure of the spinless fermions is metal-like.

Then we make a more careful study of the two-leg ladder. Particularly for the $J=J'$ case, the magnitude of the gap found in our approach is in good agreement with both the numerical result and the experimental result. Further the spin-excitation spectra and the uniform susceptibility are also calculated based on our mean-field treatment. The dispersion relation of the spin-excitation spectra obtained by our mean-field theory is very similar to the numerical result, which has its maximum locating between 0 and π. The uniform susceptibility is also consistent with the numerical results, which predict a maximum at $T=0.8J$.

APPENDIX

Here we prove that matrix $h(k)$ in Sec. II has the two following properties: (i) If λ is an eigenvalue of the matrix, $-\lambda$ is also an eigenvalue of it. (ii) Zero cannot be an eigenvalue of the matrix with any nonzero J'.

First we divide the Hermite matrix $h(k)$ into its real and imaginary part $h(k)=A+iB$, in which the matrices A,B are

$$A_{ij}=a\,\delta_{i,j+1}+a\,\delta_{i,j-1}, \quad B_{ij}=(-1)^{i+1}\delta_{i,2M+1-j}\gamma.$$

One can easily find that matrix A and B satisfy some relations

$$\text{(I)} \quad K\cdot h(k)\cdot K=-h(k),$$

$$\text{(II)} \quad A^T=A \quad B^T=-B \quad B^{-1}=\gamma^{-2}B \quad B^{-1}AB=-A,$$

in which $K_{ij}=\delta_{ij}(-1)^i$ and $K_{il}K_{lj}=\delta_{ij}$.

Based on the above equations we can prove the two properties straightforwardly. For (i) if we have $h(k)x=\lambda x$, we can multiply matrix K to both sides of the above equation: $K\cdot h(k)\cdot K\cdot Kx=\lambda Kx$, producing $h(k)(Kx)=-\lambda(Kx)$.

For property (ii), we can prove it as follows. First if $B=0$ the conclusion is obviously true because the determinant of matrix A is nonzero for $J'\neq 0$ and this means equation $Ax=0$ cannot be satisfied unless $x=0$. Next, when $B\neq 0$ we have

$$(A+iB)x=0 \quad \text{or equivalently } B^{-1}Ax=-ix.$$

For matrix $C=B^{-1}A$, we have

$$C^{+}=[B^{-1}A]^{+}=\gamma^{-2}A^{+}B^{+}=-\gamma^{-2}AB$$
$$=-(B^{-1})(\gamma^{-2}B)AB=(B^{-1})A=C.$$

Then the matrix C is Hermite, and it cannot have an imaginary eigenvalue, so the equation $(A+iB)\cdot x=0$ cannot be satisfied for nonzero x.

In the above proof we used relation (II). Then from the above paragraph we prove the two properties used in Sec. II.

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