Universality of the spin-wave frequency in ferromagnets below T_c

H. Schinz
sd&m software design & management GmbH & Co. KG, Thomas-Dehler-Straße 27, D-81737 München, Germany

F. Schwabl
Institut für Theoretische Physik, Technische Universität München, James-Franck Straße, D-85747 Garching, Germany
(Received 17 July 1997)

In this article, we deal with the scaling function of the spin-wave frequency of ferromagnets. This quantity enters the mode-coupling theory for ferromagnets below the Curie temperature. To obtain quantitative results, we discuss the universal properties of the spin-wave frequency and calculate the value of the amplitude factor. We compare the amplitude factors with experimentally obtained values for the materials Fe, Ni, Co, EuO, and EuS. The agreement is quite convincing, given the experimental uncertainties. The theoretical value which has been used earlier and which stems from a Green-function calculation is substantially smaller and does not agree with the measurements. Finally, we demonstrate how the agreement between recent experiments on the longitudinal linewidth in Ni and theory can be improved substantially, if one uses the correct value for the amplitude factor. [S0163-1829(98)02713-1]

I. INTRODUCTION

In contrast to the very satisfactory situation above T_c, where quantitative agreement between experiment and theory of the critical dynamics has been achieved (see, e.g., Refs. 1, 2 and references therein), the situation in the ferromagnetic phase is far less clear. Even in the isotropic case there are open questions. The theory, which was able to explain the experimental findings on the critical dynamics quantitatively, has been mode-coupling theory. One important issue, now, is the universality of the quantities which enter this theory.

From renormalization-group theory, it is clear that the scaling functions are universal. This is especially true for the scaling function of the spin-wave frequency. On the other hand, for quantitative calculations, mode-coupling theory is far superior to renormalization-group calculations. However, this theory is a microscopic theory and it is not obvious *a priori*, that the quantities in question are still universal.

The purpose of this paper is, therefore, to deal with the universal properties of the scaling function of the spin-wave frequency, based on mode-coupling theory. We want to present two lines of argument, proving that for the isotropic Heisenberg ferromagnet, the eigenvalues of the frequency matrix are universal, i.e., the scaling function for the spin-wave frequency is universal. Furthermore, we will obtain an explicit formula for the amplitude factor of this quantity, which allows us to calculate its value. It will then be compared with a result obtained earlier, based on Green functions and the random-phase approximation (RPA). We also discuss some experiments and compare the experimental results with our findings.

There are five important conclusions from our results: First, we show that the amplitude factor of the spin-wave frequency is universal also in the mode-coupling theory. Second, our value improves the old RPA value used in earlier mode-coupling calculations. Third, our value compares much better with the experimental values. Fourth, this quantity influences even qualitatively the shape of the scaling functions. With this value we therefore get, finally, better agreement between our theoretical mode-coupling calculations and the experiments. A brief account of the essential results has been given in Ref. 3.

The outline of this paper is as follows. In Secs. II–IV we present theoretical calculations leading to a value for the amplitude factor of the spin-wave frequency. In Sec. V, we show how experimental values for this quantity can be derived. We extract values for Fe, Ni, Co, EuO, and EuS from different earlier experiments. In Sec. VI, these findings are discussed and conclusions for mode-coupling calculations are drawn.

II. UNIVERSALITY OF THE SPIN-WAVE FREQUENCY—THEORETICAL RESULTS

In the beginning of this section we summarize the results on static critical exponents and amplitude ratios which are needed in the following. It is well known that there exist critical exponents the values of which do not depend on the particular model investigated or on other specific details but rather are universal. Within one universality class, all members of this class exhibit the same critical behavior and therefore possess the same critical exponents. Dealing with static properties, 12 different exponents can be defined[4] but since there exist 10 scaling relations between them, only 2 are really independent. For example, the exponents ν, β, and γ can be expressed by the two exponents α and η according to

$$\nu=\frac{2-\alpha}{d},\qquad \beta/\nu=\frac{d}{2}-1+\frac{\eta}{2},\qquad \gamma/\nu=2-\eta. \tag{1}$$

Considering dynamics, a further exponent, the dynamic critical exponent z has to be added, which is also universal. There are far more dynamic universality classes, since models belonging to the same static universality class can nevertheless belong to different dynamic universality classes. They have been classified in Ref. 5 and for model J, which describes isotropic Heisenberg ferromagnets, we have

$$z=\frac{d}{2}+1-\frac{\eta}{2}. \qquad (2)$$

In our case, the dimension is $d=3$ and the exponent η can approximately be set to zero. [For example, $\eta=0.04$ for the isotropic n-vector model in $d=n=3$ in $O(\epsilon^2)$ (Ref. 6).] If we additionally set $\alpha=0$ [e.g., $\alpha=-0.10$ for the isotropic n-vector model in $d=n=3$ in $O(\epsilon^2)$ (Ref. 6)], we get a rough estimate also for the static critical exponents

$$\nu=2/3, \quad \beta=1/3, \quad \gamma=4/3, \quad z=5/2. \qquad (3)$$

The amplitudes in the scaling laws set the scales of the quantities in question, and can therefore not be universal. Analogous to the critical exponents, 12 amplitudes can be defined for static critical properties.[4] There exist, however, 10 combinations of these nonuniversal amplitudes, which are themselves universal such that only 2 of the 12 amplitudes are really nonuniversal and model dependent (two-scale-factor universality).[4] Important examples for these universal amplitude ratios are the following:

$$R_c=\frac{C_0\tilde{\chi}_0}{M_0^2}k_BT_c,$$

$$R_\xi^+=\xi_0 C_0^{1/d},$$

$$R_\xi^T=\xi_0^T(C_0^-)^{1/d},$$

$$\frac{\xi_0}{\xi_0^T},\frac{C_0}{C_0^-}. \qquad (4)$$

(Different from our notation in Ref. 4 additional normalization constants for the amplitudes have been introduced.) The corresponding amplitudes appear in the scaling laws for the static isotropic susceptibility χ, correlation length ξ, magnetization M, and specific heat C

$$\chi(q=0)=\tilde{\chi}_0|\tau|^{-\gamma},$$

$$\xi=\xi_0|\tau|^{-\nu} \quad (\tau<0:\xi_0^T),$$

$$M=M_0|\tau|^{\beta},$$

$$C=C_0\frac{|\tau|^{-\alpha}}{\alpha}+C' \quad (\tau<0:C_0^-). \qquad (5)$$

In these formulas, q is the wave vector, τ the reduced temperature $\tau=(T-T_c)/T_c$ relative to the Curie temperature T_c, and α, β, γ, and ν are the usual critical exponents. Below T_c, we introduced factors ξ_0^T and C_0^- for the correlation length and the specific heat, respectively.

The static scaling hypothesis for the static susceptibility including an external magnetic field, which leads to a nonvanishing magnetization M, can now be stated in the following form:[4]

$$\chi(q,T,M)=|\tau|^{-\gamma}\tilde{\chi}_0\hat{\chi}'\left(\tau\left|\frac{M}{M_0}\right|^{-1/\beta},x\right)$$
$$=q^{-2+\eta}\chi_0\hat{\chi}\left(\tau\left|\frac{M}{M_0}\right|^{-1/\beta},x\right). \qquad (6)$$

Here, we introduced the scaling variable $x=1/q\xi_+$ with the correlation length ξ_+ above T_c and the universal scaling functions $\hat{\chi}'$ and $\hat{\chi}$. From there, we get above T_c, without field $(M=0)$, and in an expansion[4] for $x\gg 1$

$$\hat{\chi}'(\infty,x)=1+O(x^{-2}), \quad \hat{\chi}(\infty,x)=\frac{x^\eta}{x^2+1+O(x^{-2})},$$

$$\chi(\tau,q=0,H=0)=\tilde{\chi}_0|\tau|^{-\gamma}, \quad \chi(\tau,q,H=0)\approx\frac{\chi_0\xi^{-\eta}}{q^2+\xi^{-2}}, \qquad (7)$$

which is the well known Ornstein-Zernike result. Below T_c we get the result

$$\hat{\chi}_T(-1,x)=Rx^\eta, \quad \hat{\chi}_T'(-1,x)=Rx^2,$$

$$\chi_T(\tau,q\to 0,H=0)\sim q^{-2}, \qquad (8)$$

which is in accordance with the Goldstone theorem[7] stating that for systems with a spontaneously broken continuous symmetry, the corresponding transverse susceptibility has to diverge in the limit of long wavelengths $(q\to 0)$. Here, R is a further universal amplitude ratio

$$R=\left(\frac{M_0^2\xi_0^d}{\tilde{\chi}_0k_BT_c}\right)\left(\frac{\xi_0^T}{\xi_0}\right)^{d-2}=\frac{(R_\xi^+)^d}{R_c}\left(\frac{\xi_0^T}{\xi_0}\right)^{d-2}. \qquad (9)$$

In Refs. 8, 9 a dynamical scaling hypothesis was proposed, i.e., it was proposed that analogously, there should be scaling laws also for frequency-dependent quantities near a critical point. For the dynamic susceptibility, this leads to the following scaling law:[5]

$$\chi(\vec{q},\omega)=\chi_{\vec{q}}Y\left(\frac{\omega}{Fq^z},\frac{1}{q\xi}\right),$$

$$\omega_\psi(\vec{q})=Fq^z\bar{\omega}\left(\frac{1}{q\xi}\right). \qquad (10)$$

$\chi_{\vec{q}}$ is the static susceptibility, ω_Ψ a characteristic frequency, Y and $\bar{\omega}$ are universal scaling functions. The characteristic frequency sets the time or frequency scale and contains the dynamic critical exponent z and also a further (nonuniversal) amplitude F. The corresponding scaling function $\bar{\omega}$ only depends on static scaling variables.

Now, based on these assumptions, we want to present two lines of arguments, which both show theoretically that the amplitude factor of the scaling function for the spin-wave frequency is universal. To this end, we take a closer look at the frequency matrix $\omega_{\vec{q}}$, which appears in mode-coupling theory below the Curie temperature T_c. It represents the characteristic energy for the fundamental excitations—the spin waves. In Ref. 10, the theory has been presented in detail and applied to ferromagnetic substances. There, an explicit formula for the calculation of the frequency matrix has been given [Eq. (26) of Ref. 10). Now, we note that the scaling law for the spin-wave frequency, given in Ref. 10, is of course in accordance with the above general considerations

$$\omega_{\vec{q}} = F q^z \hat{\omega}(x,y,\vartheta), \quad \hat{\omega}(x,y,\vartheta) = \hat{b}\sqrt{x(1+y^2\cos^2\vartheta)}. \tag{11}$$

Here, $x=1/q\xi_-$ is the scaling variable below T_c, $y=q_D/q$ is the scaling variable measuring the strength of the dipolar interaction, expressed by the dipolar wave vector q_D [cf. Eq. (25)], and $\hat{b}$ is a numerical constant. In the common range of validity, this result also agrees with spin-wave theory.[11,12] Proving the universality of the scaling function $\hat{\omega}$ is therefore equivalent to proving the universality of $\hat{b}$. Since we have to specify the value of the amplitude F, we need a theory which treats the dynamics. The theory we use will be the mode-coupling theory.[10] In this theory the central object for describing the dynamics is the so-called Kubo relaxation function ϕ. In the general case, this function is a matrix. It can be expressed through the dynamic susceptibility

$$\phi(\vec{q},\omega) = \frac{1}{i\omega}\{\chi(\vec{q},\omega) - \chi(\vec{q},0)\}. \tag{12}$$

III. THE HYDRODYNAMIC LIMIT OF THE MODE-COUPLING THEORY

Now, we are in a position to start with our first argument. We assume, that the static and dynamic scaling hypotheses are valid. It then follows from Eqs. (10) and (6) that we have a scaling law with a universal scaling function for ϕ. If we now cast ϕ into the form

$$\phi(\vec{q},\omega) = i\sqrt{\chi_{\vec{q}}}\,\frac{1}{\omega\mathbf{1} + \omega_{\vec{q}} + i\Gamma(\vec{q},\omega)}\,\sqrt{\chi_{\vec{q}}}, \tag{13}$$

where $\omega_{\vec{q}}$ is the frequency matrix, we also get a scaling law with universal scaling function for the quantity $\omega_{\vec{q}} + i\Gamma(\vec{q},\omega)$:

$$\Gamma(\vec{q},\omega) - i\omega_{\vec{q}} = Fq^z\left[G\left(\frac{\omega}{Fq^z},\frac{1}{q\xi}\right) - i\hat{\omega}\left(\frac{1}{q\xi}\right)\right]. \tag{14}$$

We thus conclude that the combined scaling function $G - i\hat{\omega}$, which is a matrix, is universal. If we know that G is universal, we would be finished, since then also $\hat{\omega}$ had to be universal. [Mode coupling theory tells us that—because of Eq. (13)—$\hat{\omega}$ in Eq. (14) is identical to $\hat{\omega}$ in Eq. (11).]

To address this last open question—the universality of G—we now turn to mode-coupling theory quantitatively. In principle, G can be calculated using mode-coupling theory and because $\hat{\omega}$ in Eq. (11) is involved in its calculation, the resulting scaling function will depend on $\hat{b}$. In the isotropic case and in the hydrodynamic limit ($y=0$ and $x\to\infty$), it can be shown analytically that the longitudinal scaling function G_L depends asymptotically as $1/\hat{b}$ on $\hat{b}$.

To this end, we study the mode-coupling equations for the scaling functions of the transverse and longitudinal damping G_T and G_L in the isotropic limit (no dipolar interaction). They can be written

$$G_L(x) = \frac{\hat{c}}{(4\pi)^2}\int_{-1}^{1} d\eta \int_0^\infty d\kappa\, I_L(x,\kappa,\eta),$$

$$G_T(x) = \frac{\hat{c}}{(4\pi)^2}\int_{-1}^{1} d\eta \int_0^\infty d\kappa\, I_T(x,\kappa,\eta), \tag{15}$$

with

$$I_L(x,\kappa,\eta) = \frac{(2\kappa\eta-1)^2}{\kappa_-^2}\,\frac{1}{\hat{\chi}_L(x)} \times 2\Re\left\{\frac{1}{\kappa^z g_T(x/\kappa) + \kappa_-^z g_T^*(x/\kappa_-)}\right\},$$

$$I_T(x,\kappa,\eta) = \frac{(2\kappa\eta-1)^2}{\kappa_-^2}\,\hat{\chi}_L(x/\kappa) \times \frac{2}{-i\overline{\nu}(x) + \kappa^z G_L(x/\kappa) + \kappa_-^z g_T(x/\kappa_-)},$$

$$g_T = G_T + i\overline{\nu}, \quad \overline{\nu}(x) = \hat{b}\sqrt{x}, \quad \kappa_-^2 = \kappa^2 - 2\kappa\eta + 1. \tag{16}$$

With $\hat{\chi}_L$ we denote the scaling function of the longitudinal static susceptibility and with $\overline{\nu}$ we denote the scaling function of the (isotropic) spin-wave frequency. Again, $\hat{c}$ is the normalization constant.

The procedure to derive the limiting behavior of G_L and G_T in the hydrodynamic ($x\to\infty$) and critical ($x\to 0$) limit from these coupled self consistency equations has been described in Ref. 24. The idea is to divide the integration interval for κ into three regions, extending from 0 to 1, from 1 to x and from x to ∞. (Here, we are only interested in the case $x \gg 1$.) For each region one applies specific approximations to the integrands which allows to calculate the integrals. This also involves assumptions on the behavior of the scaling functions to be determined, which have to be confirmed self-consistently afterwards. To leading order in x one thus obtains

$$G_T(x) \approx (B_0 + B_1 \ln x)x^{-3/2}, \quad x \gg 1. \tag{17}$$

To calculate the behavior of G_L one notices that the dominant contribution to G_L in Eq. (15) stems from the second region. I_L exhibits a maximum at $\kappa_{\max} \sim x^{2/3}$. Therefore, one approximates the integrand appropriately for this region, extends the integration from $[1,x]$ to $[0,\infty]$, and obtains

$$G_L(x) \sim \frac{1}{\hat{\chi}_L(x)\overline{\nu}(x)} \sim x^{1/2}, \quad x \gg 1. \tag{18}$$

Obviously, the behavior is only determined by the scaling function of the spin-wave frequency $\overline{\nu}$ and the asymptotic behavior of the static susceptibility $\hat{\chi}_L$. The transversal scaling function G_T does not influence the result. Assuming $\chi_L \sim 1/q$ one arrives at $\hat{\chi}_L \sim 1/x$. Together with $\overline{\nu(x)} \sim \sqrt{x}$, this yields $G_L \sim \sqrt{x}$. Concerning the dependence on $\hat{b}$ from Eq. (18) we conclude, therefore, that the longitudinal scaling function for the damping in the hydrodynamic limit ($x \to \infty$) is inversely proportional to the amplitude $\hat{b}$ of the spin-wave frequency.

Now, it is obvious that $G - i\hat{\omega}$ depends in a complicated manner on $\hat{b}$. Since we concluded that it has to be universal, $\hat{b}$ itself must be universal. Then, also the scaling function of the spin-wave frequency $\hat{\omega}$ and the relaxation function G are universal seperately. This closes our first proof.

IV. SCALING RELATIONS RESULTING FROM MODE-COUPLING THEORY

Now, we give a second derivation of the conclusion that $\hat{\omega}$ is universal. It will again demonstrate the universality of $\hat{b}$ and furthermore result in an explicit formula for $\hat{b}$ from which we can calculate its value. From mode-coupling theory, we know that the spin-wave frequency in the isotropic case, i.e., including exchange interaction only, can be written as[10]

$$\omega_{\vec{q}} = \frac{M}{\chi_{\vec{q}}^{T}} = \frac{M_0 |\tau|^{\beta}}{|\tau|^{-\gamma} \widetilde{\chi}_0 \hat{\chi}_T'(-1,x)} = \frac{M_0 \xi_0^z}{\widetilde{\chi}_0 R} \xi_+^{2-z} q^2. \tag{19}$$

The dynamical scaling hypothesis states that the following form should be valid:

$$\omega_{\vec{q}} = F q^z \hat{\omega}\left(\frac{1}{q\xi}\right). \tag{20}$$

Combining both expressions, we get

$$\hat{\omega}(x) = \hat{b}\left(\frac{1}{q\xi_-}\right)^{z-2} \quad \text{with} \quad \hat{b} = k_B T_c \left(\frac{\xi_0^{z-d}}{M_0 F}\right) \times \left(\frac{\xi_0}{\xi_0^T}\right)^{d-2} \left(\frac{\xi_-}{\xi_+}\right)^{z-2}. \tag{21}$$

To proceed, we have to specify the value of the dynamic amplitude F.

However, before we investigate the scaling behavior of the relaxation function, based on mode-coupling theory, we first want to define some physical quantities. Let us start with the Hamiltonian for the spin operators $\vec{S}_i$ located at lattice sites i with cartesian components $\alpha, \beta = x, y, z$ (Ref. 48):

$$H = -\sum_{i,j} U_{ij}^{\alpha\beta} S_i^{\alpha} S_j^{\beta}, \tag{22a}$$

$$U_{ij}^{\alpha\beta} = \frac{1}{2} \hat{J} \delta_{\alpha\beta} \delta_{j,i+\delta} - (1-\delta_{ij}) G \frac{\partial^2}{\partial R_i^{\alpha} \partial R_j^{\beta}} \frac{1}{|\vec{R}_i - \vec{R}_j|}. \tag{22b}$$

Here, the exchange interaction $\hat{J}$ is restricted to nearest neighbors $\delta = NN$ and the strength of the dipolar interaction is given by $G = 1/2(g_L \mu_B)^2$ with the Landé factor g_L and the Bohr magneton μ_B. Performing an Ewald summation of the dipolar interaction the Hamiltonian can be transformed into wave-vector space. Restricting ourselves to the leading order in the wave vector $\vec{q}$ we end up with the following expression:[48]

TABLE I. The configuration number c, the coefficient a_1, and the volume ratio b^2 for simple cubic (sc), body centered cubic (bcc), and face centered cubic (fcc) lattices. See text.

	sc	bcc	fcc
c	6	8	12
a_1	4π	$3\pi\sqrt{3}$	$4\pi\sqrt{2}$
$a^3/v =: b^2$	1	2	4

$$H = \int_{\vec{q}} U_{\vec{q}}^{\alpha\beta} S_{\vec{q}}^{\alpha} S_{-\vec{q}}^{\beta}, \tag{23a}$$

$$U_{\vec{q}}^{\alpha\beta} = -J_0 \delta^{\alpha\beta} + J\left\{ q^2 a^2 \delta^{\alpha\beta} + g \frac{q^{\alpha} q^{\beta}}{q^2} \right\}. \tag{23b}$$

The lattice constant of the conventional (cubic) unit cell is denoted by a, while v is the volume of the primitive unit cell (containing only one atom). For different Bravais lattices the number of atoms per cubic unit cell differs and thus $b = \sqrt{a^3/v}$ depends on the lattice type (see Table I and Ref. 13). Later on, we will measure lengths in units of $v^{1/3}$. The notation $\int_{\vec{q}}$ is an abbreviation for

$$\int_{\vec{q}} = \int v \frac{d^3 q}{(2\pi)^3}. \tag{24}$$

The expansion of the Fourier transformed exchange interaction leads to the coefficients J_0 and J. J is given by $J = (c/6)\hat{J}$, neglecting small corrections due to the dipolar interaction (we used $G/\hat{J}a^d \ll 1$). By c we denote the configuration number, i.e., the number of nearest neighbors (see Table I). Finally, we introduced the dimensionless quantity g related to the dipolar wave vector q_D as a measure for the relative strength of the dipolar interaction compared to the exchange interaction[1,13]

$$g = (q_D a)^2 = \frac{a_1 G}{J a^3} \left(\frac{c}{6}\right)^{3/2}. \tag{25}$$

The coefficient a_1 stems from the Ewald summation of the dipolar interaction and depends also on the lattice type (Table I and Ref. 48).

Now, we turn to scaling considerations. Measuring all lengths in units of the lattice constant a, we define dimensionless quantities $\widetilde{H}$, $\widetilde{\beta}$, $\widetilde{\mathcal{H}}$ for the Hamiltonian H, the inverse temperature $\beta = 1/k_B T$ (k_B the Boltzmann constant), and an external field $\mathcal{H}$, coupling to the spins S, as follows:

$$H = J q^2 \widetilde{H}, \qquad \beta = \frac{1}{J q^2} \widetilde{\beta}, \qquad \mathcal{H} = J q^2 \widetilde{\mathcal{H}}. \tag{26}$$

For this transformation, we used the typical energy of this system Jq^2, where J is the exchange constant in Eq. (23). Furthermore, we introduced a magnetization M and an interaction vortex V according to

$$M = q^{\eta}\langle S_0^z\rangle, \tag{27a}$$

$$V_{ilk} = Jq^{2-\eta}v_{ilk}. \tag{27b}$$

For the transformation from the (microscopic) spin quantity S_z and from the (microscopic) interaction vortex v to the (macroscopic) expectation value M of the order parameter field and the (macroscopic) interaction vortex V, we included the exponent η to account for the effects of field renormalization. This is not rigorous here, but can be justified by a more sophisticated treatment.[24,49] In Ref. 24, a way to account for η in the mode-coupling equations is presented.

Assuming that field renormalization is represented through Eq. (27a), Eq. (27b) follows from the definition of V through an equation of motion for the microscopic spins S (see Ref. 10, $\hbar = 1$)

$$\dot{S}^i_{\vec{q}} = i[H, S^i_{\vec{q}}] = \frac{1}{2}\int_{\vec{q}\,'} V_{ilk}(\vec{q}\,', \vec{q}-\vec{q}\,')\{S^l_{\vec{q}\,'}, S^k_{\vec{q}-\vec{q}\,'}\}, \tag{28}$$

where {,} denotes the quantum-mechanical anticommutator.

The dynamic scaling hypothesis leads to the following scaling laws for the Kubo relaxation function ϕ and the time-dependent damping Γ [cf. Eqs. (13) and (14)]:

$$\phi(\xi, \vec{q}, t) = \chi_0 q^{-2+\eta}\varphi(x, \tau),$$

$$\Gamma(\xi, \vec{q}, t) = (Fq^z)^2 G(x, \tau). \tag{29}$$

Here, τ is the scaling variable belonging to the time t.

The static scaling hypothesis can be stated as follows [cf. Eq. (6)]:

$$\chi(\vec{q}, \xi) = q^{-2+\eta}\chi_0\hat{\chi}(x). \tag{30}$$

The amplitude χ_0 in front of the scaling function $\hat{\chi}$ of the static susceptibility χ can be written as $\chi_0 = \hat{a}/J$, where $\hat{a}$ is a numerical factor accounting for the normalization of the scaling function $\hat{\chi}$. Normalizing $\hat{\chi}$ to unity, we have $\hat{a} = 1/2$.

Now we want to derive an explicit formula for the amplitude F from mode coupling theory. The mode coupling equations can be written as[10]

$$\Gamma^{ij}(\xi, \vec{q}, t) \approx \frac{2k_BT}{\sqrt{\chi^i_{\vec{q}}\chi^j_{\vec{q}}}}\int_{\vec{q}\,'} v V_{ikl}(\vec{q}\,', \vec{q}-\vec{q}\,') \times V_{jmn}(\vec{q}\,', \vec{q}-\vec{q}\,')\phi^{km}(\vec{q}\,', t)\phi^{ln}(\vec{q}-\vec{q}\,', t). \tag{31}$$

Inserting the above scaling laws yields

$$(Fq^z)^2 G^{ij}(x, \tau) \approx v\frac{2k_BT}{\chi_0 q^{-2+\eta}}\frac{1}{\sqrt{\hat{\chi}^i_{\vec{q}}\hat{\chi}^j_{\vec{q}}}}q^d\int_{\vec{\kappa}}(Jq^{2-\eta})^2 v_{ikl}v_{jmn}(\chi_0 q^{-2+\eta})^2\varphi^{km}(x, a'\tau)\varphi^{ln}(x, a''\tau)$$

$$= \frac{8k_BT}{\hat{c}}\chi_0 q^{-2+\eta}J^2 q^{4-2\eta}q^d\frac{\hat{c}/4}{\sqrt{\hat{\chi}^i_{\vec{q}}\hat{\chi}^j_{\vec{q}}}}\int_{\vec{\kappa}} v_{ikl}v_{jmn}\varphi^{km}(x, a'\tau)\varphi^{ln}(x, a''\tau). \tag{32}$$

Collecting all factors, we get mode coupling equations for the (dimensionless) scaling functions, provided we choose

$$F = \sqrt{\frac{8\chi_0 J^2 k_BT}{\hat{c}}}, \quad z = d/2 + 1 - \eta/2. \tag{33}$$

In passing we note that different conventions for measuring lengths are used. Depending on whether one uses $v^{1/3}$ (where v is the volume of the primitve unit cell) or a (the lattice constant of the conventional cubic unit cell), different expressions for F result. Here, $v^{1/3}$ is used as a unit of length.

The quantity J denotes the exchange integral [its precise definition is given in Eqs. (22) and (23)] and $\hat{c}$ is a numerical constant which accounts for the normalization of the scaling functions. If we would normalize the scaling functions of the damping at T_c to unity, as in Ref. 13, we would have to choose $\hat{c} = 8\pi^4/(5.1326)^2$. Normalizing the relaxation-scaling functions as in Ref. 1, we will choose $\hat{c} = 8\pi^4$, however, in the following.

The nonuniversal dynamic amplitude F is thus completely determined by static quantitities. This is analogous to the dynamic critical exponent which, for model J, also contains only static exponents (besides the dimension d). The basic reason for this result is that the dynamic as well as the static properties are entirely ruled by the Hamiltonian H. We have not included any ''irreversible'' or ''dissipative'' terms which characterize, e.g., the equations of motion for model A or B. With this expression for F, we get the final result

$$\hat{b} = \left(\frac{T_c}{T}\frac{\hat{c}}{2}\right)^{1/2}\left(\frac{R_c}{(R_\xi^+)^d}\right)^{1/2}\left(\frac{\xi_0}{\xi_0^T}\right)^{d-2}\left(\frac{\xi_-}{\xi_+}\right)^{z-2}. \tag{34}$$

For $T \approx T_c$ the quantity $\hat{b}$ is obviously universal and its value can be calculated if we know the values for the static universal amplitude ratios.

Universal amplitude ratios and their calculation based on renormalization-group theory are discussed in great detail in Ref. 4. As a reference we take the isotropic n-vector model in $d=3$ and with $n=3$ spin components. Using ε expansion around the upper critical dimension d_c for $\varepsilon=d_c-d=1$, we have

$$\begin{aligned} &R_c\approx 0.17 \quad (\text{Ref. 14}),\\ &R_\xi^+\approx 0.42 \quad (\text{Ref. 15}),\\ &\frac{\xi_0}{\xi_0^T}\approx 0.38 \quad (\text{Ref. 15}),\\ &R_\xi^T\approx 0.9\pm 0.2 \quad (\text{Ref. 15}),\\ &\frac{C_0}{C_0^-}\approx 1.521 \quad (\text{Ref. 16}). \end{aligned} \tag{35}$$

As a consistency check, we also calculated from these numbers a quantity which was introduced in Ref. 17 and which can be compared with the corresponding value of Eq. (37).

$$R_\xi^T\left(\frac{C_0}{C_0^-}\right)^{1/3}\frac{\xi_0}{\xi_0^T}\approx 0.39\pm 0.10. \tag{36}$$

Another method is calculating directly in $d=3$ and using a resummation of the perturbation series

$$\begin{aligned} &R_\xi^+\approx 0.4347(20) \quad (\text{Ref. 18}),\\ &\frac{\xi_0}{\xi_0^T}\approx 0.56 \quad (\text{Ref. 17}),\\ &R_\xi^T\approx 0.73 \quad (\text{Ref. 17}),\\ &\frac{C_0}{C_0^-}\approx 1.58 \quad (\text{Ref. 17}),\\ &R_\xi^T\left(\frac{C_0}{C_0^-}\right)^{1/3}\frac{\xi_0}{\xi_0^T}\approx 0.48 \quad (\text{Ref. 17}). \end{aligned} \tag{37}$$

Thirdly, one can use series expansions and Monte Carlo simulation to obtain

$$R_c\approx 0.165 \quad (\text{Ref. 14}). \tag{38}$$

In Ref. 14 this result is also compared with experiments. Depending on the method for data analysis (called A and B) the authors find

$$\begin{aligned} &\text{Ni:} \quad R_c=0.11(A) \ \text{or} \ 0.16(B) \quad (\text{Ref. 14}),\\ &\text{EuO:} \quad R_c=0.18(A) \ \text{or} \ 0.11(B) \quad (\text{Ref. 14}), \end{aligned} \tag{39}$$

from measurements on the ferromagnets Ni and EuO.

The ratio ξ_+/ξ_- of the correlation lengths above and below T_c can be calculated from the corresponding homogeneous static susceptibilities according to ($\kappa_1^\pm=1/\xi_\pm$)

$$\frac{\xi_+}{\xi_-}=\frac{\kappa_1^-}{\kappa_1^+}=\left(\frac{\tilde{\chi}_0^+}{\tilde{\chi}_0^-}\right)^{1/(2-\eta)}. \tag{40}$$

Using the expression

$$\frac{\tilde{\chi}_0^+}{\tilde{\chi}_0^-}=\frac{\gamma}{\beta}\left[\frac{(1-2\beta)\gamma}{2\beta(\gamma-1)}\right]^{(\gamma-1)} \tag{41}$$

from Ref. 19, $\beta=0.375$ from Ref. 20, and $\gamma=1.375$ and $\eta=0.043$ from Ref. 21, this yields

$$\frac{\kappa_1^-}{\kappa_1^+}=2.02. \tag{42}$$

Putting all this together, i.e., using

$$\begin{aligned} &R_\xi^+\approx 0.435\pm 0.01,\\ &R_c\approx 0.17,\\ &\frac{\xi_0}{\xi_0^T}\approx 0.47\pm 0.09,\\ &\sqrt{\xi_-/\xi_+}\approx 0.70, \end{aligned} \tag{43}$$

we get the final result $\hat{b}=9.5\pm 1.8$.

Since this derivation applies to the isotropic case, including exchange interaction only, we want to briefly discuss the possible influences of the dipolar interaction. The numerical values, of course, depend on the universality class, i.e., on the fixed point in the sense of the renormalization-group theory. Including now the dipolar interaction, the isotropic Heisenberg fixed point becomes unstable and is replaced by the dipolar fixed point. It is well known (see, e.g., Ref. 22) that the values for the exponents at these two fixed points do not differ very much. (We cannot rule out in principle, however, the possibility of nontrivial crossover phenomena between those two fixed points, i.e., effective exponents could reveal extremal values.) For the amplitude ratios this question is far less thoroughly investigated. We give only one result[23]

$$\begin{aligned} &\text{isotropic:} \quad \frac{C_0}{C_0^-}=1+O(\varepsilon), \quad Q_1^\delta=1-0.239\varepsilon,\\ &\text{dipolar:} \quad \frac{C_0}{C_0^-}=\frac{6}{5}+O(\varepsilon), \quad Q_1^\delta=1-0.250\varepsilon, \end{aligned} \tag{44}$$

which shows that the amplitudes also seem to have quite similar values in both cases. Therefore, it seems reasonable to assume that the value we calculated will not be very different from the value including the dipolar interaction.

Yet another way to find a numerical value for the amplitude $\hat{b}$ is described in Ref. 24. There, a sum rule is combined with results of Ref. 25 (spin-wave theory after Keffer and Loudon, and Dyson, respectively.) and Ref. 26 (Green function formalism after Bogolyubov and Tjablikov for the calculation of the spin-wave frequency and the static susceptibility) to obtain a value $\hat{b}=\pi^{3/2}\approx 5.57$. Summarizing, we gather the theoretical results in Table II. The quantity $W_-=\hat{b}/G(0)$ corresponds to a normalization of $\hat{b}$ by the scaling function G introduced in Eq. (14) and taken at $\omega=0=x$, $G(0)=5.1326$, which is convenient for compari-

TABLE II. Theoretical values for the spin-wave amplitude calculated by different approaches.

	Green-function method	n-vector model
$\hat{b}/G(0)=:W_-$	1.08	1.85(36)

son with experiment. The values in this table will later be compared with experimental results for W_-, given in Table III.

V. UNIVERSALITY OF THE SPIN-WAVE FREQUENCY—EXPERIMENTAL RESULTS

Now, we want to compare our theoretical results with findings based on experiments. The experimental determination of $\hat{b}$ is done according to the following procedure. We parametrize the spin-wave frequency ω_q, the critical damping $\Gamma_q(T_c)$ at T_c, and the inverse correlation length $\kappa_1^- = 1/\xi_-$ below T_c according to

$$\hbar\omega_q^- = D(T)q^2, \quad D(T) = 1.17D(0)|\tau|^{0.35} = D_0|\tau|^{0.35},$$

$$\hbar\Gamma_q^-(T=T_c) = Aq^{2.5},$$

$$\kappa_1^-(T) = \kappa_0^-|\tau|^{0.7}. \tag{45}$$

Here, we introduced the—temperature-dependent—spin-wave stiffness D. $\hbar$ is the Planck constant, A and κ_0 are constants. From this we derive

$$\frac{\omega_q}{\Gamma_q(T=T_c)} = \frac{D(T)}{A\sqrt{q}} = \frac{D(T)}{A\sqrt{\kappa_1(T)}}\,\frac{1}{\sqrt{q\xi_-}} = \frac{D_0}{A\sqrt{\kappa_0^-}}\,\frac{1}{\sqrt{q\xi_-}}. \tag{46}$$

Combining, on the other hand, the scaling laws, Eqs. (20), (21), and (14), at $\omega=0$, we obtain (at T_c, we have $\xi_- \to \infty$ and therefore $x = 1/q\xi_- = 0$)

$$\frac{\omega_q}{\Gamma_q(T=T_c)} = \frac{Fq^z\hat{\omega}(x)}{Fq^zG(0)} = \frac{\hat{b}}{G(0)}\,\sqrt{x}. \tag{47}$$

So we arrive at the final result

$$W_- = \frac{\hat{b}}{G(0)} = \frac{D_0}{A\sqrt{\kappa_0^-}}. \tag{48}$$

Of course, the above parametrization is only valid in the vicinity of the critical point. This condition has to be met for the experimental determination of the corresponding parameters. Also the experimental temperature and wave-vector dependencies of the different quantities should be such that they cancel in the final formula for W_-.

Based on these considerations, we now want to discuss several experiments on the ferromagnets Fe, Ni, Co, EuO, and EuS. The different parameters often have been measured by different groups and different values for them thus emerge. This also leads to a range of values for W_-. We start with experiments on Fe.

A. Fe

The spin-wave stiffness D was measured in 1969 by Collins and co-workers.[27] In the critical region $[0.005<\tau<0.2$, with the reduced temperature $\tau=(T_c-T)/T_c]$, they obtained $D(\tau)\sim\tau^{0.37(3)}$ and at $\tau=0.1$ they measured the value $D(\tau=0.1)=142(8)$ meV Å^2. Employing the formula

$$D(T) = 1.17D(0)\tau^{0.37}, \tag{49}$$

this yields $D(0)=285(16)$ meV Å^2. Using the same formula at 80%T_c yields $D(\tau=0.2)=184(10)$ meV Å^2. Later in 1984, Wicksted *et al.*[28] determined a value $D(\tau=0.2)=175$ meV Å^2.

For the critical damping at T_c, Collins *et al.* found a law $\hbar\Gamma = Aq^{2.7(3)}$ with $\hbar\Gamma=0.439(27)$ meV at $q=0.1$ Å^{-1}. This then results in the value $A=139(9)$ meV Å$^{2.5}$. Wicksted *et al.* measured $A=142.3$ meV Å$^{2.5}$, while in 1982, Mezei *et al.*[29] found $A=130$ meV Å$^{2.5}$.

The correlation length in Fe has only been determined above T_c with the result[27] $\kappa_1^+ = \kappa_0^+\tau^{0.65(3)}$. At a distance from T_c of 14.1 and 4.2 K above $T_c=1043$ K, they mea-

TABLE III. Experimentally determined values for various quantities entering the formula for calculating the amplitude factor W_- of the scaling function for the spin-wave frequency and the resulting values of W_-. Also included are the values for the dipolar wave vector (after Ref. 13, and references therein).

	Fe	Ni	Co	EuO	EuS
q_D (Å^{-1})	0.045/0.033	0.013	0.025	0.147	0.245
D_0 (meV Å^2)	317(16)	480(5)	677(56)	13.3	3.2(1)
		580(18)		14(1)	4.1
A (meV Å$^{2.5}$)	135(5)	350(15)	300(30)	8.5(9)	2.1(3)
κ_0^+ (Å^{-1})	0.82(3)	0.80(3)	0.94(8)	0.64(4)	0.55(3)
	1.22(12)	0.64			
κ_0^-/κ_0^+	2.02[a]	1	2.02[a]	2.4	2.02[a]
W_-	1.496(119)	1.533(73)	1.638(224)	1.263(139)	1.446(215)
	1.824(119)	1.714(76)		1.329(175)	1.852(269)
		1.853(89)			
		2.072(91)			

[a]Theoretically expected value.

sured $\kappa_1^+ = 0.052$ Å^{-1} and $\kappa_1^+ = 0.022$ Å^{-1}, respectively. This then yields $\kappa_0^+ = 0.82(3)$ Å^{-1}. In 1976 Als-Nielsen *et al.*[30] found the law $\kappa_1^+ = \kappa_0^+ \tau^{0.69(2)}$ with $\kappa_0^+ = 1.22(12)$ Å^{-1} (see also Ref. 29). Wicksted *et al.* obtained $\kappa_1^+ = \kappa_0^+ \cdot \tau^{0.7}$ with $\kappa_0^+ = 1.05$ Å^{-1}. To calculate from these results the correlation length below T_c we use the theoretically expected ratio of $\kappa_-/\kappa_+ = 2.02$.

B. Ni

In 1969 Minkiewicz *et al.*[31] obtained $D = 620(100)$ meV Å$^2 \tau^{0.39(4)}$ for the spin-wave stiffness, which at 80%T_c results in $D(\tau = 0.2) = 331(53)$ meV Å^2. Böni and Shirane in 1985 (Ref. 32) found $D(\tau = 0.2) = 330$ meV Å^2 while in the same year Martinez *et al.*[33] found $D(\tau = 0.2) = 342(8)$ meV Å^2. In 1991 Böni *et al.*[34] obtained $D(\tau) \sim \tau^{0.37}$ and $D(\tau = 0.2) = 265(3)$ meV Å^2.

The damping has been determined in the following three experiments: $A = 350$ meV Å$^{2.5}$ in Ref. 32, $A = 350(12)$ meV Å$^{2.5}$ in Ref. 33, and $A = 367(18)$ meV Å$^{2.5}$ in Ref. 35.

Anders and Stierstadt[36] give $\kappa_1^+ = 0.79(2)$ Å$^{-1}|\tau|^{0.702}$ for the correlation length above T_c. Steinsvoll *et al.*[37] and Martinez *et al.*[33] have $\kappa_1^+ = 0.09$ Å^{-1} at $1.06T_c$. From this, one obtains $\kappa_0^+ = 0.64$ Å^{-1}. Böni *et al.*[34] measured $\kappa_0^\pm = 0.81(4)$ Å$^{-1}|\tau|^{0.701}$ above as well as below T_c. This ratio of $\kappa_-/\kappa_+ = 1.0$ is in contrast to the theoretically expected ratio of $\kappa_-/\kappa_+ = 2.02$. In the same paper, they also observed a quite substantial deviation from the theoretically expected value for the universal amplitude ratio R_c.

C. Co

The only measurements on Co were done in 1977 by Glinka *et al.*[38] The law for the spin-wave stiffness was obtained as $D(T) = D_0|\tau|^{0.39(5)}$ and from $D(\tau = 0.02) = 60(5)$ meV Å^2 one calculates $D(\tau = 0.2) = 361(30)$ meV Å^2. The damping could be described by $\hbar\Gamma = Aq^{2.4(2)}$ with $A = 300(30)$ meV Å$^{2.5}$ and for the correlation length above T_c they obtained $\kappa_1^+ = F|\tau|^\nu/a_{nn}$ with $F = 2.4(2)$, $\nu = 0.65(4)$ and the nearest-neighbor distance $a_{NN} = 2.55$ Å. From this, one calculates $\kappa_0^+ = 0.94(8)$ Å^{-1}. Again, the correlation length below T_c has to be determined using the theoretically expected ratio of $\kappa_-/\kappa_+ = 2.02$.

D. EuO

This substance was examined very thoroughly in 1976 by Passell *et al.*[39] (together with the similar substance EuS). In this work, the exchange constants J_1 and J_2 of the Heisenberg interaction (nearest- and next-nearest-neighbor interactions) have been determined at low temperatures with the result $J_1(0) + J_2(0) = 0.725(6)$ Kk_B. Employing the formula $D(0) = 2Sa_1^2[J_1 + J_2]$, using $S = 7/2$, and $a_1 = \sqrt{2}a_{NN}$, valid for a fcc lattice, and inserting the nearest-neighbor distance $a_{NN} = 3.64$ Å, we get $D(0) = 11.6(1)$ meV Å^2 for the unrenormalized spin-wave stiffness. Using again Eq. (49) to include the effects of normalization at higher temperatures, this yields $D(\tau = 0.2) = 7.48(6)$ meV Å^2 at 80%T_c. Near T_c the law (49) was confirmed with a value $D(0) = 11.39$ meV Å^2. From this, we obtain $D(\tau = 0.2) = 7.35$ meV Å^2. In 1981 Mook[40] got the result $J_1(0) + J_2(0) = 0.750(61)$ Kk_B. Repeating the above calculations [and using Eq. (49)], we get $D(0) = 12(1)$ meV Å^2 and $D(\tau = 0.2) = 7.7(6)$ meV Å^2. Finally, Böni *et al.*[32] obtained the value $D(\tau = 0.2) = 7.4$ meV Å^2.

For the damping we have the following results: $A = 4.0(1)$ meV Å$^{2.5}$ in Ref. 39, $A = 8.7(7)$ meV Å$^{2.5}$ in Ref. 41, and $A = 8.3(7)$ meV Å$^{2.5}$ in Ref. 42. The early result, which is roughly a factor of 2 too small, appears to be erroneous, according to the two latter papers.

For the correlation lengths above and below T_c, Passell *et al.*[39] on the result $\kappa_1^+ = F^+|\tau|^\nu/a_{NN}$, $F^+ = 2.32(13)$, $\nu = 0.681(17)$, and $\kappa_-/\kappa_+ = 2.4$. This ratio is in a very fair agreement with the theoretically expected value. Using $a_{NN} = 3.64$ Å we calculate from there the value $\kappa_0^+ = 0.637(36)$ Å^{-1}. In 1986 Böni *et al.*[42] obtained $\kappa_0^+ = 0.64$ Å^{-1} above T_c with an exponent 0.7 for the temperature dependence.

E. EuS

Repeating the calculations done for EuO, we obtain with the values $J_1(0) + J_2(0) = 0.118(6)$ Kk_B and $a_{NN} = 4.22$ Å from Ref. 39 for the unrenormalized spin-wave stiffness $D(0) = 2.54(13)$ meV Å^2. At higher temperatures, we use Eq. (45) together with $T_c = 16.56$ K to get $D(T = 0.8T_c) = 1.69(9)$ meV Å^2 and $D(T = 10$ K$) = 2.15(11)$ meV Å^2, respectively. Bohn *et al.* in 1984 (Ref. 43) measured $D(T = 10$ K$) = 3$ meV Å^2. They also measured a reduction factor of 90% for the temperature renormalization which yields $D(0) = 3.3$ meV Å^2. In 1990 Rebelsky *et al.*[44] investigated the direction dependence of the spin-wave dispersion. Due to a different sign of the comparably large next nearest-neighbor interaction compared to the nearest-neighbor interaction, there is a larger direction dependence than usual. The quantitative results for the spin-wave stiffness were $D_{100}(T = 10$ K$) = 2.34$ meV Å^2 for the [100] direction and $D_{111}(T = 10$ K$) = 2.29$ meV Å^2 for the [111] direction. Applying again a reduction factor of 90%, this yields $D_{100}(0) = 2.6$ meV Å^2 and $D_{111}(0) = 2.54$ meV Å^2. Using a different experimental method, Kötzler *et al.*[45] in 1993 measured $J_1(0) + J_2(0) = 0.126$ Kk_B for the Heisenberg exchange constants. This yields $D(0) = 2.6$ meV Å^2.

For the damping Bohn *et al.*[43] obtain $\Gamma = 0.58(4)$ THz$(\text{Å}q)^{2.09(6)}$ which is equivalent to $\hbar\Gamma = 2.4(2)$ meV$(\text{Å}q)^{2.09(6)}$. Two further values can be found in Refs. 46 and 47: $\hbar\Gamma = 2.1(3)$ meV$(\text{Å}q)^{2.54(10)}$ and $\hbar\Gamma = 2.25$ meV$(\text{Å}q)^{2.5}$, respectively.

The correlation length above T_c has been measured by Passell *et al.*[39] They obtained $F^+ = 2.33(13)$ and $\nu = 0.702(22)$. Using $a_{NN} = 4.22$ Å, this yields $\kappa_0^+ = 0.552(31)$ Å^{-1}. The correlation length below T_c has to be determined using the theoretically expected ratio of $\kappa_0^-/\kappa_0^+ = 2.02$.

All these results are gathered in Table III. This table contains the values as used to calculate the amplitude factor W_- of the spin-wave frequency scaling function according to Eq. (48). For convenience, we also included the values for the dipolar wave vectors for these substances (after Ref. 13, and references therein). The values in this table will, in the next section, be compared with theoretical results for W_-, given in Table II.

VI. SUMMARY AND CONCLUSIONS

In this article, we have shown that the scaling function of the spin-wave frequency is universal. Furthermore, also the amplitude, determining the quantitative influence of the spin-wave frequency in scaling theories such as the mode-coupling theory is universal. For both arguments, we used mode-coupling theory as one possible theory for describing dynamic critical behavior.

The value of the amplitude factor has been determined and compared to the earlier used theoretical value. This value was obtained employing a Green-function method and is about a factor of 2 smaller. Then, a method for obtaining this amplitude factor from experiments is discussed and values have been derived for the classical ferromagnets Fe, Ni, and Co. We also calculated values for the dipolar ferromagnets EuO and EuS. They all lie quite close together, underlining the universality of this quantity also from an experimental point of view. The difference between the isotropic and the dipolar value is not significant experimentally. This is similar to other quantities when dealing with critical phenomena. The difference between isotropic and dipolar values in three dimensions is never very substantial.

The comparison of these experimental values with the theoretical ones, given in Table III and Table II, respectively, shows that the earlier used RPA value is too small, while the value calculated in this article agrees quite favorably with the experimentally deduced range. A further, indirect evidence for the achieved improvement in calculating the value of the amplitude factor is the comparison of dynamic measurements in Ni with mode-coupling theory. This comparison has already been carried through in Ref. 3 and is based on measurements described in Ref. 34. In Fig. 1 is shown the longitudinal line width g_L of Ni for different distances from the Curie temperature T_c. The dashed line shows the theoretical result, using the old value for the amplitude factor. The other two theoretical curves are obtained using values which lie in the range of experimentally obtained values and which are also close to the theoretical value obtained in this article. In Ref. 10 we discuss the same experiment, including the dipolar interaction.

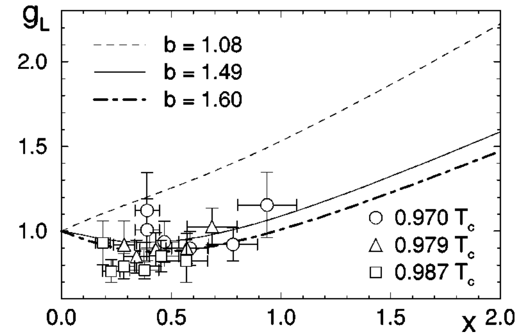

FIG. 1. Scaling function g_L vs $x=1/q\xi$. Experimental data from Ref. 34.

We see that the agreement between experiment and theory is improved considerably. If one accounts for the fact that the experimental values are normalized to unity for $x=0$ and that they have to increase monotonically for large x (i.e., in the hydrodynamic limit), the data seem to indicate a minimum. The theoretical curve using the old value does not even qualitatively reproduce this finding, while with the new value, now the theoretical curve also exhibits a minimum.

In further investigations, it would be interesting to deal with the influence of the dipolar interaction. Are the values for the isotropic and dipolar ferromagnet really close together? Also, one could try to improve the quality of the Green-function calculation and see whether the value moves towards the result obtained here. Finally, it would be worthwhile to repeat the experiments on the ferromagnets and reduce the size of the error bars.

ACKNOWLEDGMENTS

The authors would like to thank E. Frey for valuable discussions in connection with the calculations of Sec. IV. They also would like to thank P. Böni for providing them with the numerical values of the measurements from Ref. 34 and for some valuable discussions. This work was supported by the German Federal Ministry for Research and Technology (BMFT) under Contract Nos. 03-SC3TUM and 03-SC4TUM2.

[1] E. Frey and F. Schwabl, Z. Phys. B **71**, 355 (1988); **76**, 139(E) (1989).
[2] F. Mezei, J. Phys. (Paris), Colloq. **49**, C8-1537 (1988); D. Görlitz, J. Kötzler, F. J. Bermejo, P. Böni, and J. L. Martínez, Physica B **180&181**, 214 (1992).
[3] H. Schinz and F. Schwabl, J. Magn. Magn. Mater. **140-144**, 1527 (1995).
[4] V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1991), Vol. 14.
[5] P. C. Hohenberg and B. J. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[6] S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, MA, 1976).
[7] J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[8] R. A. Ferrell, N. Menyhárd, H. Schmidt, F. Schwabl, and P. Szépfalusy, Phys. Rev. Lett. **18**, 891 (1967).
[9] B. J. Halperin and P. C. Hohenberg, Phys. Rev. Lett. **19**, 700 (1967).
[10] H. Schinz and F. Schwabl, preceding paper, Phys. Rev B **57**, 8438 (1998).
[11] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[12] F. Keffer, *Spin Waves in S. Flügge*, *Handbuch der Physik*, Bd. XVIII/2, *Ferromagnetismus* (Springer, Berlin, 1966).
[13] A. Yaouanc, P. Dalmas de Réotier, and E. Frey, Phys. Rev. B **47**, 796 (1993).
[14] A. Aharony and P. C. Hohenberg, Phys. Rev. B **13**, 3081 (1976).
[15] C. Bervillier, Phys. Rev. B **14**, 4964 (1976).

[16] C. Bervillier, Phys. Rev. B **34**, 8141 (1986).
[17] R. Schloms and V. Dohm, Phys. Rev. B **42**, 6142 (1990); **46**, 5883(E) (1992).
[18] C. Bagnuls and C. Bervillier, Phys. Rev. B **32**, 7209 (1985).
[19] P. Schofield, J. D. Litster, and J. T. Ho, Phys. Rev. Lett. **23**, 1098 (1969).
[20] R. L. Stephenson and P. J. Wood, J. Phys. C **3**, 90 (1970).
[21] D. S. Ritchie and M. E. Fisher, Phys. Rev. **5**, 2668 (1972).
[22] A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6, and references therein.
[23] A. Aharony and A. D. Bruce, Phys. Rev. Lett. **33**, 427 (1974).
[24] F. Schwabl, Z. Phys. **13**, 246 (1971).
[25] H. Mori and K. Kawasaki, Prog. Theor. Phys. **27**, 529 (1962); **28**, 690 (1962).
[26] K. Kawasaki and H. Mori, Prog. Theor. Phys. **28**, 690 (1962); K. Kawasaki, *ibid.* **38**, 1052 (1965).
[27] M. F. Collins, V. J. Minkiewicz, R. Nathans, L. Passell, and G. Shirane, Phys. Rev. **179**, 417 (1969).
[28] J. P. Wicksted, P. Böni, and G. Shirane, Phys. Rev. B **30**, 3655 (1984).
[29] F. Mezei, Phys. Rev. Lett. **49**, 1096 (1982); **49**, 1537(E) (1982).
[30] J. Als-Nielsen, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 5a, and references therein.
[31] V. J. Minkiewicz, M. F. Collins, R. Nathans, and G. Shirane, Phys. Rev. **182**, 624 (1969).
[32] P. Böni and G. Shirane, J. Appl. Phys. **57**, 3012 (1985).
[33] J. L. Martínez, P. Böni, and G. Shirane, Phys. Rev. B **32**, 7037 (1985).
[34] P. Böni, J. L. Martínez, and J. M. Tranquada, Phys. Rev. B **43**, 575 (1991).
[35] P. Böni, H. A. Mook, J. L. Martínez, and G. Shirane, Phys. Rev. B **47**, 3171 (1993).
[36] R. Anders and K. Stierstadt, Solid State Commun. **39**, 185 (1981).
[37] O. Steinsvoll, C. F. Majkrzak, G. Shirane, and J. P. Wicksted, Phys. Rev. Lett. **51**, 300 (1983); Phys. Rev. B **30**, 2377 (1984).
[38] C. J. Glinka, V. J. Minkiewicz, and L. Passell, Phys. Rev. B **16**, 4084 (1977).
[39] L. Passell, O. W. Dietrich, and J. Als-Nielsen, Phys. Rev. B **14**, 4897 (1976); J. Als-Nielsen, O. W. Dietrich, and L. Passell, *ibid.* **14**, 4908 (1976); O. W. Dietrich, J. Als-Nielsen, and L. Passell, *ibid.* **14**, 4923 (1976).
[40] H. A. Mook, Phys. Rev. Lett. **46**, 508 (1981).
[41] F. Mezei, J. Magn. Magn. Mater. **45**, 67 (1984).
[42] P. Böni and G. Shirane, Phys. Rev. B **33**, 3012 (1986).
[43] H. G. Bohn, A. Kollmar, and W. Zinn, Phys. Rev. B **30**, 6504 (1984).
[44] L. Rebelsky, P. Böni, S. M. Shapiro, H. G. Bohn, and W. Zinn, J. Magn. Magn. Mater. **84**, 201 (1990).
[45] M. W. Pieper, J. Kötzler, and K. Nehrke, Phys. Rev. B **47**, 11 962 (1993).
[46] P. Böni, G. Shirane, H. G. Bohn, and W. Zinn, J. Appl. Phys. **61**, 3397 (1987).
[47] P. Böni, G. Shirane, H. G. Bohn, and W. Zinn, J. Appl. Phys. **63**, 3089 (1988).
[48] M. E. Fisher and A. Aharony, Phys. Rev. Lett. **30**, 559 (1973); A. Aharony and M. E. Fisher, Phys. Rev. B **8**, 3323 (1973); A. Aharony, *ibid.* **8**, 3342 (1973).
[49] E. Frey and F. Schwabl, Adv. Phys. **43**, 577 (1994).