Scaling solution of the overscreened multichannel Kondo model

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(Received 4 December 1996; revised manuscript received 9 May 1997)

In the Kondo problem the overscreened multichannel model is the only one that can be assessed perturbatively. However, until now there are still problems remaining in studies by means of a perturbative renormalization group due to the absence of a proper treatment on the constraint of the pseudofermion. In this paper we try to settle these problems. We studied the general overscreened multichannel Kondo problem by means of a field-theory renormalization group with the technique of treating the pseudofermion constraint as developed recently. We calculated the critical exponent, specific heat, magnetic susceptibility, residue entropy, and Wilson ratio for the overscreened multichannel Kondo model. From these results we conclude that all overscreened multichannel Kondo models with the same channel number K but different S have the same fixed point properties. Therefore, they belong to the same universal class, which agrees with the result obtained by Bethe ansatz and conformal field theory. [S0163-1829(98)01713-5]

I. INTRODUCTION

In spite of its long history, the overscreened multichannel Kondo problem[1] is still of current interest in condensed matter physics. This is because the overscreened Kondo model provides the simplest version of non-Fermi-liquid systems, and up to now we only have known a few systems in high spatial dimension ($d \geqslant 2$) which have non-Fermi-liquid fixed points: moreover, the overscreened Kondo model is related to a variety of physical systems, such as heavy fermion compounds,[2,3] mesoscopic quantum boxes,[4] or two-level systems.[5,6]

In the past ten years there have been intensive studies in this region by a variety of methods, such as the Bethe ansatz,[7–9] conformal field,[10,11] numerical renormalization group (NRG),[12–14] and perturbative renormalization group (PRG).[15–17] A fairly complete understanding of this model has been achieved.

Even so, there are still some problems remaining. It is well known in the Kondo problem that the overscreened multichannel model is the only one which can be assessed perturbatively. But until now consistency between the results obtained in the framework of the PRG and the results of the Bethe ansatz and conformal field theory has not been achieved. A question arises: What is the origin of these problems? Before answering this question, one should be reminded that when studying the Kondo model in the framework of field theory, the constraint of pseudofermions should be dealt with carefully.

In the Kondo model the local spin operator $\mathbf{S}$ does not satisfy the fermion or the boson algebra, and thus cannot be treated in the field theory framework. To surmount this difficulty one often uses the pseudofermion representation of the local spin operator $\mathbf{S}$, introduced by Abrikosov,[18]

$$\mathbf{S}=\sum_{\alpha,\beta} \mathbf{S}_{\alpha,\beta} a_{\alpha}^{\dagger} a_{\beta}, \tag{1}$$

where $a_{\alpha}^{\dagger}$ and a_{α} are fermion created and annihilated operators, respectively.

There is a constraint for the pseudofermion operator,

$$\sum_{\alpha} a_{\alpha}^{\dagger} a_{\alpha}=1, \tag{2}$$

which is very hard to deal with.

A widely used method of dealing with this constraint is Abrikosov's method[18] or its more sophisticated form developed by Barnes[19] and Coleman.[20] In this method we take the pseudofermion energy $\lambda \rightarrow \infty$ to freeze out the contribution of unphysical states. However, there are some troubles remaining in this way of eliminating the unphysical contributions.[21,16] Because of the absence of a proper way to use the pseudofermion representation, some problems have been brought in the study of the multichannel Kondo model by means of the PRG.

In Ref. 16, Grunberg and Keiter obtain that the critical exponent Δ of overscreened Kondo model is related to both the channel number K and the impurity spin S (an early result about Δ related to S can be seen in Ref. 22) which is contradict the exact result,

$$\Delta=\frac{2}{2+K}. \tag{3}$$

In a recent work,[23] Fabrizio and Zarand use Abrikosov's method to study the overscreened Kondo model with a high-conductive electron spin. In the framework of the PRG and up to K^{-3} of the large K expansion, they obtain that the

exponent Δ is not related to the local spin S and consistent with the exact result in the sense of the $1/K$ expansion. But there are still some things needed to be further clarified in their work, such as if their results can still be arrived at when the calculation of the large K expansion is up to K^{-2} or K^{-4}.

Beside Abrikosov's method there are also other methods to deal with the constraint. In the cases of spin $S=1/2$ and $S=1$, the Hilbert space spanned by the pseudofermion operator has some specific properties, and allows one to freeze out the contribution of the unphysical states while the useful linked cluster theorem is preserved.[24–26] The constraint problem for general spin cases has also been settled recently.[27]

In this paper we use the method developed in Ref. 27 to study the general overscreened Kondo model in the framework of the PRG. For high-spin ($S>1/2$) cases, we decompose the local spin S into spin 1/2. In this procedure, the partition function of the original system is separated into the proper weighed partition functions of subsystems. Therefore one can study these subsystems separately. Because all M subsystems, induced by the overscreened Kondo model in the procedure of the decomposition, can be assessed perturbatively, in this paper we only study the overscreened multichannel Kondo model. We first study the statistical average of physical observables of the M subsystem, then consider its weight, and finally we obtain the statistical average of the physical observables of the original Kondo system.

Because of the proper treatment of the pseudofermion constraint, we obtained[27(a)] the correct critical exponent Δ of the overscreened Kondo model, which is only related to K, and agree with the results of Bethe ansatz and conformal field solutions in the sense of the $1/K$ expansion when $K\gg 1$. The results of resistivity, specific heat, magnetic susceptibility, residue entropy, and Wilson ratio are all satisfactory.

The paper is organized as follows. In Sec. II we introduce the method of treating the constraint. Section III is devoted to defining the statistical average of physical observables under the pseudofermion representation. In Sec. IV, we discuss some problems about the application of the field-theory technique to the M subsystem. In Sec. V, we discuss the perturbative character of the M subsystem. Sections VI and VII are the calculations of the critical exponent Δ, thermodynamic and transport properties of the overscreened Kondo model. In Sec. VIII we give our conclusions.

II. SETTLING THE CONSTRAINT PROBLEM IN THE PSEUDOFERMION REPRESENTATION

The constraint condition (2) can be treated in the sense of a statistical average: that is, we discuss eliminating the contribution of unphysical states in a statistical average.

The key point is that for spin 1/2 and spin 1, the Hilbert spaces spanned by pseudofermions have some specific properties. For the case of a local spin $S=\frac{1}{2}$, the unphysical states satisfy

$$\mathbf{S}|\text{unphy}\rangle=0, \tag{4}$$

and for the case of a local spin $S=1$ the unphysical states are divided into two parts: one part satisfies Eq. (4), and another (redundant states) is isomorphic to the physical subspace. From these properties, the contribution of unphysical states can be freezed out while the useful linked cluster theorem is preserved.[24–26] But for high-spin cases ($S>1$), there are no such properties, and the constraint (2) is very hard to deal with.

Recently the constraint problem for high-spin cases has been settled.[27] From decomposing the high-spin S into spin 1/2, we can utilize Eq. (4) and the Popov-Fedotov imaginary chemical potential μ_a to eliminate the contribution of false states in a statistical average.

For any systems which contain a high-impurity spin S (original system), its partition function is separated into many parts (subsystems) because of the decomposition of the local spin operator. The recursion formulas of the partition functions are,[27]

$$\mathrm{Tr}^{(S)}[e^{-\beta H}]=i\ \mathrm{Tr}^{(S-1/2)}_{(1/2)}[e^{-\beta H}]-\mathrm{Tr}^{(S-1)}[e^{-\beta H}],$$

$$\mathrm{Tr}^{(S)}_{(1/2)}[e^{-\beta H}]=i\ \mathrm{Tr}^{(S-1/2)}_{(1/2,1/2)}[e^{-\beta H}]-\mathrm{Tr}^{(S-1)}_{(1/2)}[e^{-\beta H}].$$

From these recursion formulas, the partition function of the original system (under an ordinary spin representation) can be transformed into the sum of partition functions of the subsystem (under a pseudofermion representation) and the expansion coefficient M subsystems can be obtained from the above recursion formulas. For integer spin S, the result is

$$Z^{(S)}=\sum_{m=0}^{S} C_1[S,m]Z_{(2m,1/2)}, \tag{5}$$

$$C_1[S,m]=(-1)^S\frac{(S+m)!}{(2m)!(S-m)!}, \tag{6}$$

and for half-odd spin S is

$$Z^{(S)}=\sum_{m=0}^{S-1/2} C_2[S,m]Z_{(2m+1,1/2)}, \tag{7}$$

$$C_2[S,m]=(i)^{2S}\frac{(S+0.5+m)!}{(2m+1)!(S-0.5-m)!}. \tag{8}$$

In Eqs. (5) and (7), $Z_{(M,1/2)}$ is the partition function of the M subsystem,

$$Z_{(M,1/2)}=\mathrm{Tr}_{(M,1/2)}[e^{-\beta(H-\mu_a N_a}], \tag{9}$$

where μ_a is the imaginary chemical potential, $\mu_a=i\pi/2\beta$, and N_a is the number operator of pseudofermions,

$$N_a=\sum_{i=1}^{M} N_i=\sum_{\alpha=1}^{2}\sum_{i=1}^{M} a^{\dagger}_{\alpha,i}a_{\alpha,i},$$

where N_i is the pseudofermion number operator of ith local spin 1/2.

Equations (5) and (7) are exact. On the left-hand side of Eqs. (5) and (7), partition functions are in the ordinary spin representation, and on the right-hand side of Eqs. (5) and (7), partition functions are all in a pseudofermion representation. Therefore Eqs. (5) and (7) give a correct way for using the pseudofermion representation for high-spin cases.

III. STATISTICAL AVERAGE

The method of using a pseudofermion representation for high-spin cases developed in Ref. 27 leads to some new characters in the statistical average of physical observables.

Let R be the operator which does not contain pseudofermion operators: for the same reasons as in deducing Eqs. (5) or (7), we have

$$\langle R\rangle_{(S)}=\frac{\mathrm{Tr}^{(S)}[Re^{-\beta H}]}{\mathrm{Tr}^{(S)}[e^{-\beta H}]}=\frac{\Sigma_M C_i[S,M]\mathrm{Tr}_{(M,1/2)}[Re^{-\beta(H+\mu_a N_a)}]}{\mathrm{Tr}^{(S)}[e^{-\beta H}]}=\sum_M D[S,M]\langle R\rangle_{(M,1/2)}, \tag{10}$$

where

$$D[S,M]=C_i[S,M]\ \frac{\mathrm{Tr}_{(M,1/2)}[e^{-\beta(H+\mu_a N_a)}]}{\mathrm{Tr}^{(S)}[e^{-\beta H}]}, \tag{11}$$

$$\langle R\rangle_{(M,1/2)}=\frac{\mathrm{Tr}_{(M,1/2)}[Re^{-\beta(H+\mu_a N_a)}]}{\mathrm{Tr}_{(M,1/2)}[e^{-\beta(H+\mu_a N_a)}]}. \tag{12}$$

In Eqs. (10) and (11) the index $i=1,2$ corresponds to S being even and half odd, respectively. For simplicity, in the discussions below we shall omit it.

From Eqs. (10)–(12), we see that under a pseudofermion representation the statistical average of R can be treated as double averages: The first is the M-subsystem ensemble average defined by Eq. (12), and the second is for the subsystems; $D[S,M]$ is treated as the weight of the M subsystem. It is obvious that $D[S,M]$ satisfies

$$\sum_M D[S,M]=1.$$

We first study the conductivity under a pseudofermion representation. Consider the linear response of current,

$$j_\nu=-e\sum_i \frac{dx_{\nu i}}{dt},\quad \nu=x,y,z,$$

to the external field

$$V[t]=e\sum_i \mathbf{r}_i\cdot\mathbf{E}[t].$$

From linear response theory, conductivity is defined as[28]

$$\sigma_{\nu\nu'}[w,t]=\lim_{s\to\infty}\int_0^\infty\int_0^\beta\langle j_\nu[0]j_{\nu'}[t+i\lambda]\rangle_{(S)}e^{-i(w+s)t}d\lambda\,dt, \tag{13}$$

where

$$\langle\cdots\rangle_{(S)}=\frac{\mathrm{Tr}^{(S)}[\cdots e^{-\beta H}]}{\mathrm{Tr}^{(S)}[e^{-\beta H}]}$$

means the statistical average (original system) under an ordinary spin representation. From Eq. (10), we can rewrite Eq. (13) as

$$\sigma_{\nu\nu'}[w,t]=\sum_M D[S,M]\lim_{s\to\infty}\int_0^\infty\int_0^\beta\langle j_\nu[0]j_{\nu'}[t+i\lambda]\rangle_{(M,1/2)}\times e^{-i(w+s)t}d\lambda\,dt=\sum_M D[S,M]\sigma^{\nu\nu'}_{(M,1/2)}[w,t], \tag{14}$$

where $\langle\cdots\rangle_{(M,1/2)}$ is defined by Eq. (12), and $\sigma^{\nu\nu'}_{(M,1/2)}[w,t]$ is the conductivity of the M subsystem. From Eq. (14), under a pseudofermion representation, the calculation of the conductivity is separated into two parts. The first part is the calculation of the M-subsystem average of conductivity under a pseudofermion representation, and the second part is about the weight $D[S,M]$; from Eq. (14) we obtain the total average of the conductivity.

Now we consider the calculation of the thermodynamic potential under a pseudofermion representation, for any system which contains impurity spin S; the thermodynamic potential is written as

$$\Omega^{(S)}=-\beta^{-1}\ln[Z^{(S)}]. \tag{15}$$

From Eq. (5) or (7), we have

$$\Omega^{(S)}=-\beta^{-1}\ln\left[\sum_M C[S,M]Z_{(M,1/2)}\right]=-\beta^{-1}\ln\left[\sum_M C[S,M]e^{-\beta\Omega_{(M,1/2)}}\right], \tag{16}$$

where the thermodynamic potential of the M subsystem is defined as

$$\Omega_{(M,1/2)}=-\beta^{-1}\ln[Z_{(M,1/2)}]. \tag{17}$$

From Eq. (16), the average of the number of conduction electrons can be written as

$$\langle N_C\rangle_{(S)}=\frac{\partial\Omega^{(S)}}{\partial\mu}=\sum_M D[S,M]\langle N_C\rangle_{(M,1/2)},$$

$$\langle N_C\rangle_{(M,1/2)}=\frac{\partial\Omega^{(M,1/2)}}{\partial\mu}. \tag{18}$$

The magnetization $M_{(S)}$ of the system can be written as

$$M_{(S)}=-\frac{\partial\Omega^{(S)}}{\partial B}=\sum_M D[S,M]M_{(M,1/2)},$$

$$M_{(M,1/2)}=-\frac{\partial\Omega_{(M,1/2)}}{\partial B}. \tag{19}$$

From Eq. (19), the magnetic susceptibility can be calculated under the pseudofermion representation

$$\chi_{(S)}=\frac{\partial M_{(S)}}{\partial B}.$$

The detailed deducing procedure for entropy and specific heat are given in Appendix B; here we give the results

$$S(T)=\Delta F^{(S)}+\sum_M D(S,M)S_{(M,1/2)}(T), \tag{20}$$

$$C(T)\cong\sum_M D(S,M)C_{(M,1/2)}(T), \tag{21}$$

where $S_{(M,1/2)}(T)$, and $C_{(M,1/2)}(T)$ are the entropy and the specific heat of the M subsystem, respectively, and

$$S_{(M,1/2)}=-\frac{\partial\Omega_{(M,1/2)}}{\partial T},$$

$$C_{(M,1/2)}=-T\frac{\partial S_{(M,1/2)}}{\partial T}.$$

From the above discussions we see that the M subsystems can be studied separately; thus these subsystems are nearly statistically independent.

For the overscreened multichannel Kondo model, the partition function of the M subsystem is

$$Z_{(M,1/2)}=\mathrm{Tr}_{(M,1/2)}[e^{-\beta(H+\mu_a N_a)}], \tag{22}$$

and the corresponding Hamiltonian is

$$H=H_0+H_I,$$

$$H_0=\sum_{n=1}^{K}\sum_{k,\alpha}\varepsilon[k]c^{\dagger}_{n\alpha}[k]c_{n\alpha}[k],$$

$$H_I=\frac{J}{N}\sum_{n=1}^{K}\sum_{i=1}^{M}\sum_{k,k',\alpha,\beta,\gamma,\sigma}(c^{\dagger}_{n\alpha}[k]\vec{\sigma}_{\alpha\beta}\times c_{n\beta}[k'])\cdot(a^{\dagger}_{i\gamma}\vec{\sigma}_{\gamma\sigma}a_{i\sigma}), \tag{23}$$

where $\vec{\sigma}_{\alpha\beta}$ is the Pauli matrix.

From Eq. (23) we see that in the M subsystem there are M equivalent local spins 1/2 at the impurity site. Because of the local spin index i is a good quantum number, at the bare level these local spins 1/2 coupled to the K channel conductive electrons separately. For the system defined by Eqs. (23), one can use the standard field-theory technique conveniently.

IV. FUNCTIONAL INTEGRAL TECHNIQUE APPLIED TO THE *M* SUBSYSTEM

From the above discussions, we know, under a pseudofermion representation, that those subsystems defined by Eqs. (23) are nearly statistically independent, and can be studied separately. As the first step of the study of the overscreened Kondo model, in this section we apply the standard field-theory technique to the M subsystem defined by Eq. (23), and in the later Sec. VI we shall study the fixed point properties of the M subsystem by means of the PRG.

The generating functional of the M subsystem is written as[29]

$$Z_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}=\int D\bar{\psi}D\psi D\bar{\phi}D\phi\,Exp[S\{\bar{\eta},\eta,\bar{\zeta},\zeta\}],$$

$$S\{\bar{\eta},\eta,\bar{\zeta},\zeta\}=S_0+S_I+\frac{1}{\beta}\sum_w\Bigg\{\sum_{n,k,\alpha}(\bar{\eta}_{n\alpha}[k,w]\bar{\psi}_{n\alpha}[k,w]+\eta_{n\alpha}[k,w]\psi_{n\alpha}[k,w])+\sum_{i,\alpha}(\bar{\zeta}_{i\alpha}[w]\bar{\phi}_{i\alpha}[w]+\zeta_{i\alpha}[w]\phi_{i\alpha}[w])\Bigg\}, \tag{24}$$

$$S_0=\frac{1}{\beta}\sum_w\Bigg\{\sum_{n,k,\alpha}\bar{\psi}_{n\alpha}[k,w](iw-\varepsilon[k]-\mu)\psi_{n\alpha}[k,w]+\sum_{i,\alpha}\bar{\phi}_{i\alpha}[w](iw-\mu_a)\phi_{i\alpha}[w]\Bigg\},$$

$$S_I=\frac{J}{\beta^3N}\sum_{w_1\cdots w_4}\delta_{w_1+w_2,w_3+w_4}\sum_{n=1}^{K}\sum_{i=1}^{M}\sum_{k,k',\alpha,\beta,\gamma,\sigma}\times(\bar{\psi}_{n\alpha}[k,w_1]\vec{\sigma}_{\alpha\beta}\psi_{n\beta}[k',w_3])\cdot(\bar{\phi}_{i\gamma}[w_2]\vec{\sigma}_{\gamma\sigma}\phi_{i\sigma}[w_4]),$$

where the Grassmann variables $\psi,\bar{\psi}$ denote electron fields, $\phi,\bar{\phi}$ denote pseudofermion fields, and $\eta,\bar{\eta}$ and $\zeta,\bar{\zeta}$ are the source fields of the electron and pseudofermion, respectively.

The generating functional of connected diagrams is[29]

$$F_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}=\ln[Z_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}]. \tag{25}$$

The two-point function of the electron and pseudofermion of the M subsystem can be generated from Eq. (25).[29]

$$G^{(2,0)}_{(M,1/2)}[k,w]=\langle\bar{\psi}_{n\alpha}[k,w]\psi_{n'\alpha'}[k,w]\rangle_{(M,1/2)}=\frac{\delta^2}{\delta\bar{\eta}_{n\alpha}[k,w]\delta\eta_{n'\alpha'}[k,w]}\times F_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}|_{\bar{\eta}=\eta=\bar{\zeta}=\zeta=0}, \tag{26}$$

$$G^{(0,2)}_{(M,1/2)}[w]=\langle\bar{\phi}_{i\alpha}[w]\phi_{i'\alpha'}[w]\rangle_{(M,1/2)}=\frac{\delta^2}{\delta\bar{\zeta}_{i\alpha}[w]\delta\zeta_{i'\alpha'}[w]}\times F_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}|_{\bar{\eta}=\eta=\bar{\zeta}=\zeta=0}, \tag{27}$$

and the $(2m,2n)$-point Green function $G^{(2m,2n)}_{(M,1/2)}[k,w]$ can also be generated from Eq. (25) by the functional derivatives on the source fields $\bar{\eta}$, η, $\bar{\zeta}$, and ζ.

These Green functions can be calculated purterbatively by the standard diagram technique. Although in the Kondo model the spatial translation symmetry is broken, in each interacted vertex point the momentum k of the conduction electron is not conserved, but in the sense of a random average of impurity configurations, we still only need to consider one-particle-irreducible (1PI) diagrams in a perturbative expansion. According to the standard way in quantum field

theory (QFT), we introduce generating functional of 1PI diagrams through the Legendre transform[30,16]

$$\Gamma_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}+F_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}$$
$$=\sum\ (\bar{\eta}\langle\psi\rangle_{(M,1/2)}+\eta\langle\bar{\psi}\rangle_{(M,1/2)})$$
$$+\sum\ (\bar{\zeta}\langle\phi\rangle_{(M,1/2)}+\zeta\langle\bar{\phi}\rangle_{(M,1/2)}). \tag{28}$$

From $\Gamma_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}$, we can generate the $(2m,2n)$-point vertex function

$$\Gamma^{(2m,2n)}_{(M,1/2)}[k_1,\ldots,k_{2m};w_1,\ldots,w_{2m+2n}]$$
$$=\prod_{i=1}^{m}\frac{\delta}{\delta\langle\bar{\psi}_{n_i\alpha_i}[k_i,w_i]\rangle_{(M,1/2)}}\prod_{j=1}^{m}\frac{\delta}{\delta\langle\psi_{n_j\alpha_j}[k_j,w_j]\rangle_{(M,1/2)}}$$
$$\times\prod_{k=1}^{n}\frac{\delta}{\delta\langle\bar{\phi}_{i_k\alpha_k}[w_k]\rangle_{(M,1/2)}}\prod_{l=1}^{n}\frac{\delta}{\delta\langle\phi_{i_l\alpha_l}[w_l]\rangle_{(M,1/2)}}$$
$$\times\Gamma_{(M,1/2)}\{\bar{\eta},\eta,\bar{\zeta},\zeta\}|_{\bar{\eta}=\eta=\bar{\zeta}=\zeta=0}. \tag{29}$$

From Eqs. (25)–(29), one can prove that the $(2m,2n)$-point Green function can be written as

$$G^{(2m,2n)}_{(M,1/2)}[k_1,\ldots,k_{2m};w_1,\ldots,w_{2m+2n}]$$
$$=-G^{(2,0)}_{(M,1/2)}[k_1,w_1]\cdots G^{(2,0)}_{(M,1/2)}[k_{2m},w_{2m}]$$
$$\times G^{(0,2)}_{(M,1/2)}[w_{2m+1}]\cdots G^{(0,2)}_{(M,1/2)}[w_{2m+2n}]$$
$$\times\Gamma^{(2m,2n)}_{(M,1/2)}[k_1,\ldots,k_{2m};w_1,\ldots,w_{2m+2n}]$$
$$+Q^{(2m,2n)}_{(M,1/2)}, \tag{30}$$

where $Q^{(2m,2n)}_{(M,1/2)}$ represents the one-particle-reducible diagrams, and $\Gamma^{(2m,2n)}_{(M,1/2)}$ is the $(2m,2n)$-point vertex function.

Equation (30) gives the rule of calculating the $(2m,2n)$-point vertex function $\Gamma^{(2m,2n)}_{(M,1/2)}$ from the diagram rule about Green function $G^{(2m,2n)}_{(M,1/2)}$. Using the vertex function to study the renormalization of the interactive model has its advantage.[30]

For the electron two-point vertex function we have

$$\Gamma^{(2,0)}_{(M,1/2)}[k,w]=\frac{1}{G^{(2,0)}_{(M,1/2)}[k,w]}$$
$$=iw-\varepsilon[k]-\mu-\Sigma_{(M,1/2)}[w], \tag{31}$$

where $\Sigma_{(M,1/2)}[w]$ is the conductive electron self-energy of the M subsystem.

In the M subsystem we have used the Popov-Fedotov imaginary chemical potential to freeze out the contribution of unphysical states, and so the pseudofermion is deconstrained in the above QFT framework. The cost is that now we must study the M subsystem and its weight, and the procedure is more complex than the study of the original system.

Now there are some things needed to be further clarified. We use the imaginary chemical potential μ_a to freeze out the contribution of unphysical states, and this elimination is full in the exact sense (this corresponding to the fact that the imaginary parts induced by μ_a are all canceled in the partition function $Z_{(M,1/2)}$ exactly). However, in the perturbative treatment, this property of the canceling of each other is not realized in each order of the perturbative expansion. It is realized only after all orders of perturbative expansion terms have been summed. This does not lead to any difficulty, because we know that the imaginary parts induced by μ_a are unphysical and summing the perturbative expansion terms cannot mix the real part and the imaginary part; so throwing away the imaginary parts induced by μ_a in perturbative computations is reasonable.

From this we see that the imaginary chemical potential method provides a proper way to eliminate the unphysical contribution in perturbative calculations. Throwing away some terms in perturbative calculations is similar to Arbrikosov's method. However, in Appendix A we shall show that, at finite temperature, in the calculation of perturbative diagrams, these two different methods lead to different results.

V. PERTURBATIVE CHARACTER OF THE M SUBSYSTEM

In this paper we intend to study the overscreened multichannel Kondo model by means of the PRG. A question arises: Under which conditions can the M subsystem be assessed perturbatively?

Similar to Ref. 1, we give an argument on the type of the fixed point of the M subsystem. This argument is based on the exclusion principle and the following facts: For the spin exchange model, antiferromagnetic exchange has a strong-coupling fixed point, while ferromagnetic exchange has a weak-coupling fixed point. We must point out that we only use this argument to clarify the perturbative character of the M subsystem. The fixed point configurations of M subsystems are not directly related to the original system, because in our formalism one cannot use a single M subsystem to determine the properties of the original system [as an example see Eq. (10)]. However, there are also possibilities where the weight $D[S,M]$ is concentrated on a single M subsystem; for this case the fixed point configuration of the original system can be determined by this M subsystem. We shall discuss this issue in Sec. VII.

As discussed above the M subsystem can be treated as a system where at an impurity site there are M equivalent local spins ($S=1/2$). From Eqs. (23) we see that the index i (which denotes the local spins) is a good quantum number; thus at the bare level one can suppose that these local spins are coupled to the K channel conductive electrons separately. On the other hand, because of the exclusion principle, there are only K electrons coupled to these local spins at the first shell of the impurity site. We first assume a strong-coupling configuration for the M subsystem at a fixed point; that is, the impurity site traps one electron in each conductive electron channel. We separate the discussion into two steps.

First we discuss the stable configuration for local spins trapping the conductive electrons. We first assume for one of the local spin-1/2 traps all the K channel electrons and form a complex of spin $K-\frac{1}{2}$ as shown in Fig. 1(a) (because of the

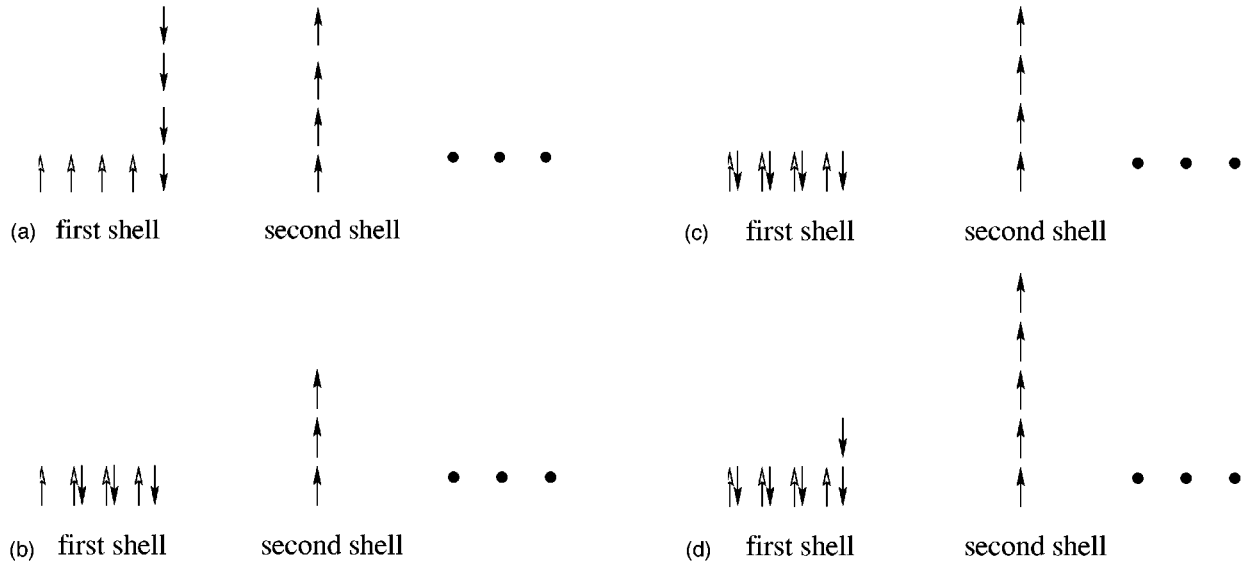

FIG. 1. Schematic strong-coupling configuration for the M subsystem. (a) The unstable configuration for local spins trapping conductive electrons. (b) The stable strong-coupling configuration for the case of $K<M$. (c) The stable strong-coupling configuration for the case of $K=M$. (d) The unstable strong-coupling configuration for the case of $K>M$.

exclusion principle the coupling between the other $M-1$ local spin-1/2 and the second-shell conductive electrons is ferromagnetic, and so the other $M-1$ local spin-1/2 are asymptotically free in relation to the second-shell electrons); then we consider the exchange coupling between this spin complex and the other local spin-1/2. Because the exchange coupling is still antiferromagnetic, the above configuration is unstable. For the same reason, one can see that the configurations, with one of the local spin-1/2 traps more than one electron while there are free local spin-1/2, are also unstable.

Second we consider the stability of the strong-coupling configuration. The situation is divided into three groups: (a) The first group corresponds to $K<M$ as shown in Fig. 1(b), (b) the second group corresponds to $K=M$ as shown in Fig. 1(c), and (c) the third group corresponds to $K>M$ as shown in Fig. 1(d).

For case (a) the coupling between the dressed impurity site (which has a residue $M-K$ local spin-1/2 trap) and the second-shell conductive electron is ferromagnetic; the first configuration shown in Fig. 1(b) is stable. It also clears that the second configuration shown in Fig. 1(c) is stable. For case (c), there are residue spins complexes at the dressed impurity site, and because of the exclusion principle, these spin complexes are coupled to second-shell conductive electrons antiferromagnetically. The third configuration shown in Fig. 1(d) is unstable. On the other hand, the weak-coupling configuration is also unstable.

From the above discussions we see that the M subsystems with $K\leqslant M$ have strong-coupling fixed points, while the M subsystems with $K>M$ have an intermediate fixed point. Therefore only the M subsystem with $K>M$ can be assessed perturbatively.

Under a pseudofermion representation the original system is decomposed into proper weighed M subsystems and from Eqs. (5) and (7) we know that the value of M ranges from 1 (0 for integer spin) to $2S$; there are always some M subsystems belonging to the case $K>M$. This is not the key point. Because in the decomposition if there is one M subsystem belonging to the case of $K\leqslant M$, then according to Eq. (5) or (7) the original system cannot be assessed perturbatively. So only the original system in the decomposition all the M subsystems that satisfy the condition $K>M$ can be assessed perturbatively. From Eq. (5) or (7) we see that this corresponds to the case of $K>2S$; the original system corresponds to the overscreened multichannel Kondo model.

From the above argument we arrive at the well-known result that only the overscreened multichannel Kondo model can be assessed perturbatively.

VI. RG EQUATIONS OF THE M SUBSYSTEM AND THE CALCULATION OF THE CRITICAL EXPONENT

In the Kondo model some vertex functions $\Gamma^{(2m,2n)}_{(M,1/2)}$ defined by Eq. (29) are divergent; from power counting[30,22] the divergence degree is given by

$$\delta_{(2m,2n)}=1-m. \tag{32}$$

From Eq. (32), we know that only the vertexes $\Gamma^{(0,2n)}_{(M,1/2)}$ and $\Gamma^{(2,2n)}_{(M,1/2)}$ are divergent. The divergence of $\Gamma^{(0,2)}_{(M,1/2)}$ and $\Gamma^{(2,2)}_{(M,1/2)}$ can be overcome by redefining J, and introducing the renormalization of the pseudofermion field[31,16(b)]

$$J\rightarrow J_R,$$

$$\phi\rightarrow\phi_R=Z^{-1/2}[J,\Lambda]\phi,$$

$$\bar{\phi}\rightarrow\bar{\phi}_R=Z^{-1/2}[J,\Lambda]\phi. \tag{33}$$

It is worth pointing out that in Eq. (33) there is no renormalization on the conductive electron field; therefore there is no possibility for a conductivity electron field to obtain an

anomalous dimension. In the Kondo problem if there is a non-Fermi-liquid fixed point, this non-Fermi liquid can only be local.

From Eqs. (33), the renormalization vertex is defined as

$$\Gamma^{(2m,2n)}_{(M,1/2)}[\{k,w\},J_R,\Lambda]=Z^n[J,\Lambda]\Gamma^{(2m,2n)}_{(M,1/2)}[\{k,w\},J,\Lambda]. \tag{34}$$

From Eq. (34), the RG equations of Callen-Symanzyk type[31,16] can be written as

$$\left\{\Lambda\frac{\partial}{\partial\Lambda}+\beta[J]\ \frac{\partial}{\partial J}+n\,\eta[J]\right\}\Gamma^{(2m,2n)}_{(M,1/2)}[\{k,w\},J,\Lambda]=0, \tag{35}$$

and for $n=0$, we have

$$\left\{\Lambda\frac{\partial}{\partial\Lambda}+\beta[J]\ \frac{\partial}{\partial J}\right\}\Gamma^{(2m,0)}_{(M,1/2)}[\{k,w\},J,\Lambda]=0. \tag{36}$$

In Eqs. (35) and (36),

$$\beta[J]=\Lambda\frac{\partial}{\partial\Lambda}J|_R, \tag{37}$$

$$\eta[J]=\Lambda\frac{\partial}{\partial\Lambda}\ln\,Z[J,\Lambda]|_R, \tag{38}$$

where R means keeping the J_R fixed.

Focusing on the purpose of obtaining the critical exponent, we only need to study Eq. (36) for the case of $m=1$. We first calculate the vertex function $\Gamma^{(2,0)}_{(M,1/2)}[k,w,J,\Lambda]$ by means of the standard perturbative diagram technique. From Eq. (31) we know this corresponds to the calculation of the electron self-energy.

For the overscreened Kondo model K^{-1} is a small parameter ($K>2S$); thus using a large K expansion[15,30] is natural. In perturbative diagrams, J is counted as K^{-1}, and each loop of electrons contributes a factor K. From these we can select the diagrams which must be considered in the large K expansion. Similar to Ref. 15, we calculate the electron self-energy up to K^{-4}. For the M subsystem, the pseudofermion operator has two indices: α, the spin ($s=\frac{1}{2}$) index, and the i ($i=1,\ldots,M$) index; among them i is a good quantum number. So in diagrams, each loop of pseudofermions will contribute a factor M. On the other hand, after averaging the random impurity configurations, diagrams which contain L_a pseudofermion loops will contribute a factor $n_i^{L_a}$. In the dilute limit $n_i\ll 1$, we only need to consider those diagrams which contain one pseudofermion loop. The calculation of self-energy diagrams is too tedious, and we used a computer to calculate the sum of the internal spin indices and imaginary frequencies in the self-energy diagrams; then we performed the integrals of the internal electron energy ϵ_i at $T=0$. Apart from a factor M, the electron self-energy of the M subsystem is the same as the result of Ref. 15: the imaginary part of vertex function $\mathrm{Im}\,\Gamma^{(2,0)}_{(M,1/2)}[0,w]$ is written as

$$\begin{aligned}\mathrm{Im}\,\Gamma^{(2,0)}_{(M,1/2)}[0,w]=-M\Bigg\{\frac{3\pi}{16}n_i g^2\Bigg[1-2g\,\ln\frac{w}{D}+g^2K\,\ln\frac{w}{D}\\ +(\ln 2-1)g^2K+3g^2\,\ln^2\frac{w}{D}\\ -\frac{7}{2}g^3K\,\ln^2\frac{w}{D}+\left(5-\frac{5}{2}\ln 2\right)g^3K\,\ln\frac{w}{D}\\ +g^4K^2\,\ln^2\frac{w}{D}\\ +\left(2\ln 2-\frac{5}{2}\right)g^4K^2\,\ln\frac{w}{D}\Bigg]\Bigg\},\end{aligned} \tag{39}$$

where g is a dimensionless coupling constant, $g=\rho_f J$, and D is the bandwidth of the conductive electron,

$\Gamma'^{(2,0)}_{(M,1/2)}=\mathrm{Im}\,\Gamma^{(2,0)}_{(M,1/2)}$ also satisfies RG equation (36),

$$\left\{\Lambda\frac{\partial}{\partial\Lambda}+\beta[J]\ \frac{\partial}{\partial J}\right\}\Gamma'^{(2,0)}_{(M,1/2)}[k,J,\Lambda]=0, \tag{40}$$

where $\Lambda=D/w$.

The remaining calculations are same as those in Ref. 15. For the β function, we have

$$\beta[g]=-g^2+\frac{K}{2}g^3+\frac{1}{2}\left(1+\frac{3}{2}\ln 2\right)Kg^4-\frac{K^2}{4}g^5. \tag{41}$$

From $\beta[g^*]=0$, the fixed point coupling constant is

$$g^*=\frac{2}{K}\left(1-\frac{2}{K}\ln 2\right)+O[K^{-3}]. \tag{42}$$

From $\Delta=(\partial/\partial g)\,\beta|_{g^*}$, the critical exponent Δ of the M subsystem is

$$\Delta=\frac{2}{K}\left(1-\frac{2}{K}\right)+O[K^{-3}]. \tag{43}$$

From Eq. (43) we see that the critical exponent of the M subsystem is the same as the one of the spin $s=\frac{1}{2}$ system,[15] and agrees with Eq. (3) when $K\gg 1$.

The renormalized coupling constant g_R has the asymptotic form

$$g_R[w]=g^*-\mathfrak{s}\left(\frac{w}{T_K}\right)^{\Delta}, \tag{44}$$

$$\mathfrak{s}=(g^*-g)(g^*)^{K\Delta/2}e^{-\Delta/g^*}. \tag{45}$$

Equation (43) is the most important result of this paper, because Δ is not related to M; all the M subsystems ($K>M$) defined by Eq. (23) have the same critical exponent Δ.

VII. SCALING SOLUTION OF THE OVERSCREENED MULTICHANNEL KONDO MODEL

Under a pseudofermion representation the overscreened multichannel Kondo system is decomposed into M subsystems which are nearly statistically independent of each other. From the study in the last section we know that all

these subsystems belong to the same universal class with critical exponent $\Delta=2/(2+K)$. In this section we return to the original system, and study the scaling solution of physical observables.

From discussions in Sec. III the calculation of a statistical average of physical observables is separated into two steps. First we study the average of the M subsystem; second we calculate the weight $D[S,M]$ for the M subsystem: In the sense of a RG transformation, the partition function is dimensionless; thus $D[S,M]$ does not contribute an additional scaling exponent to the scaling solution of physical observables. Under the scaling approximation we only consider a zero-order partition function in the calculation of $D[S,M]$. From the discussions in Appendix B, we have

$$D[S,M]=\frac{C[S,M](-2i)^M}{\Sigma_M C[S,M](-2i)^M}. \tag{46}$$

We first study the resistivity $R_{(S)}$ of the OSMK system; from Eq. (14), the resistivity can be written as

$$R_{(S)}=\sum_M D[S,M]R_{(M,1/2)}. \tag{47}$$

In computing we only consider second-order electron self-energy diagrams of the M subsystem. $R_{(M,1/2)}$ can be determined by the imaginary part of the self-energy. The result is

$$R_{(M,1/2)}\simeq M\frac{3}{16}\frac{m_e}{n_e e^2\rho}n_i g^2. \tag{48}$$

Equation (48) is the result of a bare perturbative treatment; i.e., the energy scale is D. In the universal region we replace g in Eq. (48) by its renormalized form g_R given by Eq. (44), and so the scaling form of $R_{(M,1/2)}$ is

$$R_{(M,1/2)}=M\frac{3\pi^2}{4K^2}\frac{n_i}{n_e}\rho_0\left[1-K\mathfrak{s}\left(\frac{T}{T_K}\right)^{\Delta}\right], \tag{49}$$

where $\rho_0=4\pi/k_F e^2$.

From Eqs. (47) and (49), we arrive at the scaling form of the resistivity of the overscreened Kondo model,

$$R_{(S)}[T]=\frac{3\pi^2}{4K^2}\frac{n_i}{n_e}\rho_0\left(\sum_M MD[S,M]\right)\left[1-K\mathfrak{s}\left(\frac{T}{T_K}\right)^{\Delta}\right]. \tag{50}$$

From Eq. (50) we see that the scaling exponent of the low-temperature resistivity of the overscreened multichannel Kondo model is independent of S. On the other hand, $\Sigma_M MD[S,M]$ depending on S, $R_{(S)}[0]$ does not approach the unitary limit. Therefore Eq. (50) satisfies the rule of charge neutrality.[32]

Second, we study the thermodynamic potential of the M subsystem. The thermodynamic potential of M subsystem is defined as

$$\Omega_{(M,1/2)}=-\beta^{-1}\ln Z_{(M<1/2)}. \tag{51}$$

From the linked cluster theorem,[33] $\Omega_{(M,1/2)}$ can be written as

$$\Omega_{(M,1/2)}=\Omega_0-\beta^{-1}\sum_{l=1}^{\infty}U^l_{(M,1/2)}, \tag{52}$$

where Ω_0 is the thermodynamic potential of the K channel noninteracting electron gas, with M free local spin-1/2 at impurity site, and

$$U^l_{(M,1/2)}=\frac{1}{l}[\text{sum of } l \text{ order connected vacuum diagrams}].$$

We only consider the contribution up to $1/K^2$; apart from a M factor, the calculation is the same as Ref. 15. The shift of the thermodynamic potential because of the presence of an impurity spin is

$$\delta\Omega_{(M,1/2)}=-TM\ln 2+M\frac{\pi^2}{4}n_i T\left(Kg^3-\frac{3}{8}K^2g^4\right), \tag{53}$$

and the scaling form of Eq. (53) is

$$\delta\Omega_{(M,1/2)}[T]=-TM\left(\ln 2-\frac{\pi^2}{2K^2}\right)-M\frac{3\pi^2}{4}T\mathfrak{s}^2\left(\frac{T}{T_K}\right)^{2\Delta}. \tag{54}$$

From Eq. (54), the zero-temperature entropy of the M subsystem is obtained:

$$S_{(M,1/2)}(0)=-\frac{\partial}{\partial T}\delta\Omega_{(M,1/2)}[T]|_{T=0}=M\left(\ln 2-\frac{\pi^2}{2K^2}\right). \tag{55}$$

From Eq. (53), the scaling form of the specific heat shift of the M subsystem is

$$\delta C_{(M,1/2)}[T]=-T\frac{\partial\delta S}{\partial T}=M\frac{3}{2}\pi^2\mathfrak{s}^2\Delta\left(\frac{T}{T_K}\right)^{2\Delta}. \tag{56}$$

From Eqs. (55) and (B5), we obtain the zero-temperature entropy of the overscreened multichannel Kondo model,

$$\delta S[0]=\Delta F^{(S)}+\sum_M D[S,M]S_{(M,1/2)}=\ln\left\{\sum_M C(S,M)(-2i)^M\right\}-\left\{\sum_M D(S,M)M\right\}\frac{\pi^2}{2K^2}. \tag{57}$$

It can be proved that there are two important relations: For integer spin we have

$$\sum_{m=0}^{S}C_1(S,m)(-2i)^{2m}=2S+1, \tag{58}$$

and for half-odd spin we have

$$\sum_{m=0}^{S-1/2} C_2(S,m)(-2i)^{2m+1}=2S+1. \tag{59}$$

So the residue entropy of the overscreened multichannel Kondo model is

$$\delta S[0]=\ln(2S+1)-\left\{\sum_M D(S,M)M\right\}\frac{\pi^2}{2K^2}. \tag{60}$$

From Eq. (60) we know that the ground state of the overscreened multichannel Kondo model is degenerate. It is well known that the fixed point coupling constant of the overscreened multichannel Kondo model is $J^*\sim 1/K$. When $K\rightarrow\infty$, the local spin is asymptotically free in low-energy physics, and the degeneracy degree of ground states is just the degeneracy degree of a free local spin S. In this paper we study the overscreened multichannel Kondo model through the M subsystem, where when $K\rightarrow\infty$ the ground state degeneracy degree of the M subsystem is 2^M. It is quite surprising that Eq. (60) satisfies the relation

$$\lim_{K\rightarrow\infty} S(0)=\ln(2S+1), \tag{61}$$

and we further compare our result and the result of conformal field theory.

The residue entropy $S(0)$ obtained by a conformal field is[10]

$$S_{\rm con}(0)=\ln\left(\frac{\sin[\pi(2S+1)/(2+K)]}{\sin[\pi/(2+K)]}\right). \tag{62}$$

In Fig. 2(a) and Fig. 2(b), we plot $S(0)$-K curves obtained by the PRG and conformal field theory for the case of $S=3,9/2$, respectively. From Fig. 2(a) and Fig. 2(b) we see that for large K the residue entropy obtained by the PRG is consistent with the results of conformal field theory.

From Eqs. (21) and (56) the scaling form of the specific heat shift is written as

$$\delta C_{(S)}[T]=\sum_M D[S,M]\delta C_{(M,1/2)}$$
$$=\left(\sum_M MD[S,M]\right)\frac{3}{2}\pi^2\varsigma^2\Delta\left(\frac{T}{T_K}\right)^{2\Delta}. \tag{63}$$

Finally, we consider the magnetic susceptibility of the M subsystem. Apart from a factor M coming from the pseudofermion loop, the result of Ref. 15 can be used:

$$\delta\chi_{(M,1/2)}[T]=\left.\frac{\partial\delta M_{(M,1/2)}[T]}{\partial B}\right|_{B=0}=M\left(\frac{K}{2}\varsigma\right)^2\frac{1}{T}\left(\frac{T}{T_K}\right)^{2\Delta}. \tag{64}$$

From Eqs. (19) and (57), we have

$$\delta\chi_{(S)}[T]\simeq\sum_M D[S,M]\delta\chi_{(M,1/2)}[T]$$
$$=\left(\sum_M MD[S,M]\right)\left(\frac{K}{2}\varsigma\right)^2\frac{1}{T}\left(\frac{T}{T_K}\right)^{2\Delta}. \tag{65}$$

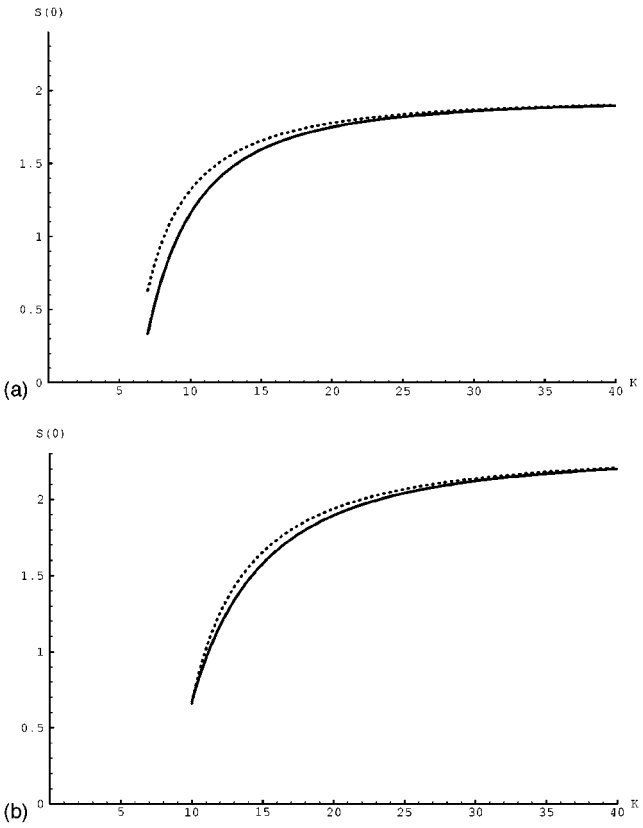

FIG. 2. The $S(0)$-K curves of the overscreened multichannel Kondo model obtained by the PRG and conformal field theory, respectively. Dotted lines denote the results of a conformal field, and the solid lines denote the results of the PRG. (a) For the case of $S=3$, and (b) for the case of $S=9/2$.

From Eqs. (56) and (58), we can obtain the Wilson ratio of the OSMK system in general cases. The Wilson ratio is defined as

$$W_S=\frac{\delta\chi_{(S)}[T]}{\delta C_{(S)}[T]}\frac{C_b}{\chi_b}, \tag{66}$$

where C_b and χ_b are the bulk specific heat and magnetic susceptibility, respectively, and

$$C_b=\frac{2K\pi^2T\rho}{3},\quad \chi_b=2K\rho, \tag{67}$$

and we have

$$W_S=\frac{K^3}{36}. \tag{68}$$

The Wilson ratio obtained in Eq. (68) is the same as the one obtained in Ref. 15 which studied the case of spin $S=\frac{1}{2}$, and agrees with the result of the conformal field $W=\frac{1}{18}(2+K^2)(2+K/2)$ in the sense of $K\gg 1$.

VIII. DISCUSSION AND CONCLUSION

In this paper we study the scaling solution of the overscreened multichannel Kondo model by means of the PRG. The results we obtained are consistent with the results of the

Bethe ansatz or conformal field theory. But there are still some things that need more discussion.

The first thing one needs to discuss is the perturbative treatment of the M subsystem. One can say the reason that we arrive at Eq. (43) is the assumption of $n_i \ll 1$ which allows us to neglect interference effects. It is indeed so. However, we must point out that the physics of magnetic impurities in a Fermi sea are divided into two parts: (a) isolated magnetic impurity effects and (b) interference effects between impurities. Among them the isolated magnetic impurity effects are the typical many-body effects. Because the ground states with up impurity spin and down impurity spin are orthogonal,[34] the spin flipping scattering excites infinite particle-hole pairs. The isolated magnetic impurity in a Fermi sea leads to complex many-body effects (orthogonality catastrophe effect), and the Kondo model has no trivial infrared properties. In the study of the Kondo (s-d) model, we are mainly concerned with isolated magnetic impurity effects. From this point of view, we say $n_i \ll 1$ is not only an assumption. In this paper we study the Kondo model through the M subsystem, and from the argument given in Sec. V, we see that only the M subsystem with $K > M$ can be assessed perturbatively. For the case of $K \leqslant M$, the perturbative treatment is ineffective, but we still can learn from Eq. (39). As discussed above we only study isolated magnetic impurity effects, and from Eq. (39) we conjecture that M is unimportant in determining the low-temperature universal properties of subsystems. It may be related to the well-known fact that the universal properties of the local Fermi liquid fixed point (underscreened or screened multichannel Kondo model) are independent of S.

Second we discuss the calculation of $D[S,M]$ and $\Delta F^{(S)}$. The condition $K > M$ is quite important. For overscreened multichannel Kondo models, all the M subsystems satisfy the condition $K > M$, and so can be assessed perturbatively; the perturbative computations of $D[S,M]$ and $\Delta F^{(S)}$ are reliable. We use a zero-order partition function of the M subsystem to calculate the weight $D[S,M]$ and $\Delta F^{(S)}$ in Appendix B, and this leads to the correct asymptotic form of the degenerate degree of ground states of the overscreened Kondo model when $K \to \infty$. So the perturbative approximation in the calculation of $D[S,M]$ and $\Delta F^{(S)}$ is correct quantitatively.

For underscreened and screened multichannel Kondo models, the subsystems are separated into two groups. The first group satisfies the condition $K > M$ which can be assessed perturbatively and has non-Fermi-liquid fixed points; the second group satisfies the condition $K \leqslant M$ which cannot be assessed perturbatively and has Fermi liquid fixed points. From Eq. (B3) we see, for computing $D[S,M]$ and $\Delta F^{(S)}$, one needs to compute all the subsystem partition functions. Although for the first group subsystems the perturbative partition functions are reliable, the computation of the partition functions in the second group subsystems goes beyond perturbative theory. Therefore from perturbative theory, we cannot deduce whether the second group subsystems dominate the low-energy physics. However, the logic structure of the decomposition formalism gives some insight. The screened multichannel Kondo model is the interesting case, where $K = 2S$, and in decomposition all the subsystems belong to the first group but one with $K = M$ belongs to the second group. From the discussions in Sec. V we know that for the first group subsystems the ground states are degenerate, while for the M subsystem with $K = M$ there is a singlet ground state. We conjecture (but cannot prove) for this case that the partition functions have an asymptotic form $Z_{(I,1/2)} \sim e^{-\beta\alpha_I}$, $I = M,K$, when $T \ll 1$ (where α_I may be related to the ground state energy of the M subsystem), and $\alpha_K < \alpha_M$. From Eq. (B3) we see that, when $T \to 0$, this leads to $D[S,K] = 1$, $D[S,M] = 0$ $(M \neq K)$, and $\Delta F^{(S)} = 0$. From Eq. (20) we see this leads to the residue entropy of the original system as the well-known result: $S(0) = 0$. From the formula of the static magnetic susceptibility [see Eq. (19) and below] we see that this also leads to $\chi_{(S)} = \chi_{(K)}$.

Now we give the summary. In this paper we used a method developed in Ref. 27 to deal with the constraint of pseudofermions. We decomposed the high-spin S into spin 1/2. In this procedure the partition function of the original system is decomposed into proper weighed partition functions of the subsystem. We studied M subsystems of the overscreened multichannel Kondo model in the framework of the PRG; then we calculated the weight of the M subsystem by a perturbative approximation. From this procedure, we have achieved the fine results of the residue entropy and the scaling solution of the resistivity, specific heat, and susceptibility. From the scaling form of resistivity, specific heat, and susceptibility we deduced that the critical exponent of overscreened multichannel Kondo model is

$$\Delta = \frac{2}{K}\left(1 - \frac{2}{K}\right) + O(K^{-3}),$$

which agrees with the results of the Bethe ansatz and conformal field theory when $K \gg 1$.

ACKNOWLEDGMENTS

M.S. would like to thank Professor Z. B. Su for encouragement and support in computer sources. M.S. also thanks L. F. Xu for carefully reading the manuscript.

APPENDIX A: THE ROLE OF THE IMAGINARY CHEMICAL POTENTIAL IN THE PERTURBATIVE EXPANSION

In the imaginary chemical potential method we introduce μ_a to eliminate the contribution of unphysical states, and have the relation

$$Z^{(1/2)} = iZ_{(1/2)} = i\ \mathrm{Tr}_{(1/2)}[e^{\beta(H-\mu_a)}] = iZ_{(1/2)}, \qquad \text{(A1)}$$

where $Z^{(1/2)}$ is in ordinary spin representation and $Z_{(1/2)}$ is in a pseudofermion representation.

The term on the left-hand side of Eq. (A1) is real, and on the right-hand side of Eq. (A1) imaginary parts cancel each other exactly. But when we calculate the partition function perturbatively this canceling is not realized in each order of the perturbative expansion.[27(b)] It does so when all order perturbative expansion terms have been summed. It is the same for the calculation of the statistical average of the operator R which does not contain pseudofermion operators,

$$\langle R\rangle_{(S)}=\sum_{M} D[S,M]\langle R\rangle_{(M,1/2)}. \tag{A2}$$

In $\langle R\rangle_{(M,1/2)}$ the imaginary parts induced by μ_a (not including the intrinsic imaginary part if there is one) canceled each other exactly.

We have pointed out in Sec. IV that this does not lead to any difficulty, and throwing away the imaginary parts induced by μ_a is reasonable. The imaginary chemical potential method provides a proper way to freeze out unphysical contributions in perturbative calculations. In this appendix we show this procedure concretely, and compare the results obtained with the imaginary chemical potential method and Abrikosov's method, respectively.

As an example we calculate a fourth-order conductive electron self-energy diagrams shown in Fig. 3. We consider the single-channel spin-1/2 Kondo model. The zero-order pseudofermion two-point function is written as

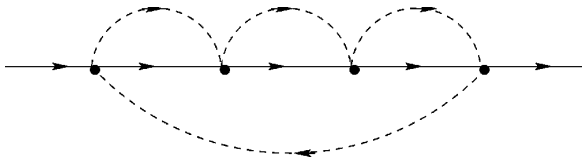

FIG. 3. Electron self-energy $\Sigma_4[\omega]$.

$$G_0^{(0,2)}[w]=\frac{1}{iw-\theta}, \tag{A3}$$

where θ in the imaginary chemical potential method is μ_a, and in Arbrikosove's method is λ.

We calculate the diagram at finite temperature. After summing the internal spin indices and imaginary frequencies, we have

$$\Sigma_4[\omega]=\frac{21}{8}(J)^4 n_i\rho_f^3\int\int\int_{-D\leqslant\epsilon_i\leqslant D} d\epsilon_1 d\epsilon_2 d\epsilon_3\left\{-\frac{3}{4}\frac{(1+e^{\beta\theta})f[\epsilon_1](e^{\beta\theta}f[\epsilon_2]+f[-\epsilon_2])(e^{\beta\theta}f[\epsilon_3]+f[-\epsilon_3])}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)(i\omega-\epsilon_1)}\right.$$
$$\left.+\frac{1}{4}\frac{(e^{-\beta\theta}f[-\epsilon_1]+f[\epsilon_1])(e^{\beta\theta}f[\epsilon_3]+f[-\epsilon_3])(e^{\beta\theta}f[\epsilon_2]+f[-\epsilon_2])}{(i\omega-\epsilon_1)(i\omega-\epsilon_2)(i\omega-\epsilon_3)}\right\}. \tag{A4}$$

For the imaginary chemical method, $\theta=\mu_a$ and $e^{\beta\mu_a}=i$, and from Eq. (A4) we have

$$\Sigma_4[\omega]=\frac{21}{8}(J)^4 n_i\rho_f^3\int\int\int_{-D\leqslant\epsilon_i\leqslant D} d\epsilon_1 d\epsilon_2 d\epsilon_3\left\{-\frac{3}{4}\frac{(1+i)f[\epsilon_1](if[\epsilon_2]+f[-\epsilon_2])(if[\epsilon_3]+f[-\epsilon_3])}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)(i\omega-\epsilon_1)}\right.$$
$$\left.+\frac{1}{4}\frac{(-if[-\epsilon_1]+f[\epsilon_1])(if[\epsilon_3]+f[-\epsilon_3])(if[\epsilon_2]+f[-\epsilon_2])}{(i\omega-\epsilon_1)(i\omega-\epsilon_2)(i\omega-\epsilon_3)}\right\}. \tag{A5}$$

For the reasons discussed above, we further throw away the imaginary part induced by μ_a, and have

$$\Sigma_4[\omega]=\frac{21}{8}(J)^4 n_i\rho_f^3\int\int\int_{-D\leqslant\epsilon_i\leqslant D} d\epsilon_1 d\epsilon_2 d\epsilon_3$$
$$\times\left\{\frac{3}{4}\frac{f[\epsilon_1]f[\epsilon_2]f[\epsilon_3]+f[\epsilon_1]f[-\epsilon_2]f[\epsilon_3]+f[\epsilon_1]f[\epsilon_2]f[-\epsilon_3]-f[\epsilon_1]f[-\epsilon_2]f[-\epsilon_2]}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)(i\omega-\epsilon_1)}\right.$$
$$\left.+\frac{1}{4}\frac{-f[\epsilon_1]f[\epsilon_2]f[\epsilon_3]+f[-\epsilon_1]f[-\epsilon_2]f[\epsilon_3]+f[-\epsilon_1]f[\epsilon_2]f[-\epsilon_3]+f[\epsilon_1]f[-\epsilon_2]f[-\epsilon_2]}{(i\omega-\epsilon_1)(i\omega-\epsilon_2)(i\omega-\epsilon_3)}\right\}. \tag{A6}$$

Throwing away some terms in perturbative calculations is similar to Abrikosov's method, and we further compare these two methods. In Arbrikosov's method $\theta=\lambda$, and we take $\lambda\to\infty$ to freeze out the contribution of the unphysical states. For the electron self-energy $\Sigma[\omega]$ the operation is

$$\lim_{\lambda\to\infty}\frac{e^{\beta\lambda}}{2S+1}\Sigma_4[w],$$

and from Eq. (A4) we have

$$\Sigma_4[\omega]=\frac{21}{16}(J)^4 n_i\rho_f^3\int\int\int_{-D\leqslant\epsilon_i\leqslant D} d\epsilon_1 d\epsilon_2 d\epsilon_3$$
$$\times\left\{-3\frac{f[\epsilon_1]f[\epsilon_2]f[-\epsilon_3]}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)(i\omega-\epsilon_1)}\right.$$
$$\left.-\frac{f[-\epsilon_1]f[-\epsilon_2]f[-\epsilon_3]}{(i\omega-\epsilon_1)(i\omega-\epsilon_2)(i\omega-\epsilon_3)}\right\}. \tag{A7}$$

Comparing Eqs. (A6) and (A7), we see that the two different methods lead to different results. The existence of the

differences between the results obtained by the two methods is general in the calculation of perturbative diagrams, and we conclude that at finite temperature the two different methods give different results.

APPENDIX B: THE FORMULAS OF THE ENTROPY AND SPECIFIC HEAT

In this appendix we give a detailed procedure of the calculation of the entropy and specific heat under a pseudofermion representation. The entropy of the original system is defined as

$$S[T]=-\frac{\partial}{\partial T}\Omega^{(S)}. \tag{B1}$$

From Eq. (16), we have

$$\begin{aligned} S(T)&=\ln\left[\sum_M C(S,M)e^{-\beta\Omega_{(M,1/2)}}\right] \\ &\quad+\beta\frac{\Sigma_M C(S,M)\Omega_{(M,1/2)}e^{-\beta\Omega_{(M,1/2)}}}{\Sigma_M C(S,M)e^{-\beta\Omega_{(M,1/2)}}} \\ &\quad+\sum_M D(S,M)S_{(M,1/2)} \\ &=\Delta F^{(S)}+\sum_M D(S,M)S_{(M,1/2)}(T), \end{aligned} \tag{B2}$$

where $S_{(M,1/2)}$ is the entropy of the M subsystem $S_{(M,1/2)}[T]=(-\partial/\partial T)\,\Omega_{(M,1/2)}$, and $D(S,M)$ is the weight of the M subsystem. $D(S,M)$ and $\Delta F^{(S)}$ can be written as

$$D[S,M]=\frac{C[S,M]Z_{(M,1/2)}}{\Sigma_M C(S,M)Z_{(M,1/2)}},$$

$$\begin{aligned} \Delta F^{(S)}&=\left\{\ln Z^{(S)}-\frac{d}{d\lambda}\ln\left[\sum_M C(S,M)e^{-\lambda\beta\Omega_{(M,1/2)}}\right]\right\}\Bigg|_{\lambda=1} \\ &=\left\{\frac{d}{d\lambda}\ln\frac{[\Sigma_M C(S,M)Z_{(M,1/2)}]^\lambda}{\Sigma_M C(S,M)[Z_{(M,1/2)}]^\lambda}\right\}\Bigg|_{\lambda=1}. \end{aligned} \tag{B3}$$

From Eq. (B3) we see that both $\Delta F^{(S)}$ and $D[S,M]$ have the form

$$\frac{\text{partition functions}}{\text{partition functions}}.$$

In this paper we seek the scaling solution of the overscreened Kondo model. In the sense of the scaling transformation the partition function is dimensionless. $\Delta F^{(S)}$ and $D[S,M]$ would not contribute extra scaling exponents in a scaling solution of physical observables. Furthermore, for the overscreened multichannel Kondo model all M subsystems can be assessed perturbatively. We use the zero-order partition function of the M subsystem,

$$Z^0_{(M,1/2)}=(-2i)^M\prod_k\,(1+e^{-\beta\epsilon(k)})^{2K}, \tag{B4}$$

to calculate $\Delta F^{(S)}$ and $D(s,M)$.

From Eqs. (B3) and (B4) we have

$$\begin{aligned} \Delta F^{(S)}&=\left\{\frac{d}{d\lambda}\ln\frac{[\Sigma_M C(S,M)(-2i)^M]^\lambda}{\Sigma_M C(S,M)(-2i)^{M\lambda}}\right\}\Bigg|_{\lambda=1} \\ &=\ln\left\{\sum_M C(S,M)(2i)^M\right\}-\left\{\sum_M D(S,M)M\right\}\ln 2, \end{aligned} \tag{B5}$$

$$D(S,M)=\frac{C(S,M)(-2i)^M}{\Sigma_M C(S,M)(-2i)^M}. \tag{B6}$$

From the above discussions, it is clear from the underscaling approximation that the specific heat can be written as

$$C_{(S)}(T)\cong\sum_M D(S,M)C_{(M,1/2)}(T). \tag{B7}$$

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