# Elastic strings in solids: Thermal nucleation

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(Received 24 July 1997)

Thermal nucleation of kink-antikink pairs in an elastic string subjected to a washboard potential is analyzed in the limit of low temperature and high damping. The pair nucleation rate is calculated *analytically* for any value of the tilt up to close the instability threshold. Finite-size effects are quantified by computing the relevant end-point nucleation rate under a wide class of boundary conditions. Applications to the physics of lattice dislocations and flux lines in type-II superconducting materials are outlined. [S0163-1829(98)04513-5]

## I. INTRODUCTION

Elastic strings provide the simplest solvable model of linear imperfections in solid-state physics. An early, prominent application of the elastic string paradigm dates back to the heyday of dislocation theory,[1–3] More recently, elastic string models have been shown to provide a full characterization of flux line dynamics in type-II superconductors.[4–6] In both instances, *transport* processes are experimentally accessible through macroscopic dissipation measurements, namely, internal friction for a strained crystal and finite resistivity for a type-II superconductor in a magnetic field. Stationary *currents* may be driven by either field gradients (generalized forces) or spatiotemporal asymmetries (ratchets[7]). The role of disorder in the aforementioned transport mechanisms has been assessed too, its relevance depending on the topology and the phase of the stringlike objects under consideration. Pointlike defects are likely to be as important in dislocation theory, as line or planar defects are to the dynamics of superconducting vortex arrays.[5]

One mechanism has been identified as central to string transport by subthreshold forces: in the presence of a weak bias a line imperfection can jump from a substrate potential trough into an adjacent one by *nucleating* kink-antikink pairs, which can be then pulled infinitely apart with almost no effort. Most notably, such a mechanism is *thermally* assisted, whence its clear-cut experimental signature.[8,9] In order to analyze in detail the nucleation process we focus here on the most tractable string model, namely, the overdamped sine-Gordon (SG) string in $1+1$ dimensions. Thermal equilibrium is attained by means of a local coupling to a Gaussian heat bath and disorder is neglected altogether. Such a model is intended to describe the motion of a dislocation in the Peierls potential[8] or a flux line in the crystal phase.[4]

In Sec. II we determine a piecewise analytical expression for the kink-antikink pair nucleation rate in an *infinite, classical* SG string subjected to a driving force of any intensity, from zero up to close to the instability threshold. The formulas for the nucleation rate in the limits of weak[10] and large tilts[11] are outlined in Secs. II A and II B, respectively. In Sec. II C the latter formula is improved to account for smaller values of the driving force and is thus proven to match the former one. In Sec. III we consider the case of a *semi-infinite* SG string by imposing appropriate boundary conditions. Thermal nucleation turns out to be favored at the string end point. In Sec. III A we determine the rate for inhomogeneous nucleation in a SG string and in Sec. III B we summarize the role of the different length scales introduced thus far. Finally, in Sec. IV we discuss the possibility of extending the present approach to nucleation in underdamped strings.

## II. NUCLEATION IN AN INFINITE STRING

The *perturbed* SG equation[12,13]

$$\phi_{tt}-c_0^2\phi_{xx}+\omega_0^2 \sin \phi=-\alpha\phi_t+F+\zeta(x,t) \tag{1}$$

provides an ideal model to study nucleation processes in a variety of periodic physical systems at thermal equilibrium. The coupling of the classical SG field $\phi(x,t)$ to the heat bath at temperature $T$ is described by a viscous term $-\alpha\phi_t$ and a zero-mean Gaussian noise source $\zeta(x,t)$. The damping constant $\alpha$ and the noise intensity are related through the noise autocorrelation function

$$\langle\zeta(x,t)\zeta(x',t')\rangle=2\alpha kT\delta(t-t')\delta(x-x'). \tag{2}$$

The constant force $F$ represents an external drive, or bias, which breaks the $\phi\rightarrow-\phi$ symmetry of the SG equation, thus making the nucleation process possible.[8] Correspondingly, the SG potential $V[\phi]=\omega_0^2(1-\cos\phi)$ gets tilted by the bias term $-F\phi$: The resulting washboard potential retains a multistable structure for $|F|<F_3\equiv\omega_0^2$. In the overdamped limit $\alpha\gg\omega_0$, $F_3$ coincides with the static threshold for the locked-unlocked transition.[14]

The unperturbed SG equation, obtained from Eq. (1) by setting its right-hand side to zero, has been derived from the covariant Hamiltonian density

$$H_{SG}[\phi]=\frac{\phi_t^2}{2}+c_0^2\frac{\phi_x^2}{2}+V[\phi] \tag{3}$$

 

and bears both extended (phonons) and localized solutions (solitons). Solitons can be regarded as an appropriate linear superposition of moving kinks $\phi_+$ and antikinks $\phi_-$ with

$$\phi_\pm(x,t)=4\arctan(\exp\{\pm\beta[x-X(t)]/d\}) \tag{4}$$

(mod $2\pi$), provided the separation between their centers of mass $X(t)\equiv X_0+ut$ is very large compared to their size $d\equiv c_0/\omega_0$ (*dilute gas* approximation). In this limit, the equilibrium kink (antikink) density in a SG theory at finite temperature and with natural boundary conditions $\phi(x\to\pm\infty,t)=0$ (mod $2\pi$) is[8]

$$n_\pm(T)=n_0(T)=\frac{1}{d}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{E_0}{kT}\right)^{1/2}\exp\left(-\frac{E_0}{kT}\right), \tag{5}$$

where $E_0=\int H_{SG}[\phi_\pm]dx=8\omega_0c_0$ is the rest energy and $M_0=E_0/c_0^2$ is the mass of $\phi_\pm$. It follows that the dilute gas approximation holds for $n_0^{-1}(T)\gg d$, i.e., at low temperature $kT\ll E_0$.

In the presence of weak perturbations (i.e., $kT\ll E_0$ and $F\ll\omega_0^2$) the *single* (anti)kink is stable, but undergoes a driven Brownian motion with the Langevin equation[15] (LE)

$$\dot{u}=-\alpha u\mp 2\pi F/M_0+\xi(t), \tag{6a}$$

where $\xi(t)$ is a zero-mean-valued Gaussian noise with autocorrelation function $\langle\xi(t)\xi(t')\rangle=2\alpha^2D\delta(t-t')$ and $D=kT/\alpha M_0$. To derive the LE (6a) it was assumed that at low temperature $kT\ll E_0$ the variance of the (anti)kink speed is much smaller than $c_0^2$, so that the relativistic boost factor $\beta\equiv(1-u^2/c_0^2)^{-1/2}$ in Eq. (4) may be approximated to unity (*nonrelativistic* approximation). As a matter of fact, one sees immediately from Eq. (6a) that the external bias pulls $\phi_\pm$ in opposite directions with average speed $u_F=\mp 2\pi F/\alpha M_0$ and variance $\langle(u-u_F)^2\rangle=kT/M_0$.

In the *overdamped* limit $\alpha\gg\omega_0$ the LE (6a) can be cast in the Smoluchowski form

$$\dot{X}=\mp 2\pi F/\alpha M_0+\eta(t), \tag{6b}$$

with $\eta(t)=\xi(t)/\alpha$. Moreover, the assumption of large damping $\alpha\gg\omega_0$ affords two major simplifications: (i) oscillating solutions of Eq. (1), such as breathers and phonons radiation, are damped out and therefore play no role in the nucleation process and (ii) kink-antikink collisions are always *destructive*. Indeed, the condition for kinks and antikinks to go through each other in the presence of damping,[16] $F/F_3\gg 2(2\alpha/\omega_0)^{3/2}$, is incompatible with the stability requirement $F<F_3$.

Finally, we notice that the (uncorrelated) drift of single kinks and antikinks determines a net string current

$$j=\overline{\phi_t}=(2\pi)2n_0(T)u_F, \tag{7}$$

whereas their spatial diffusion, with variance $\langle\Delta X^2(t)\rangle=\langle[X(t)-X_0]^2\rangle$, corresponds to the string diffusion

$$\overline{\Delta\phi^2}=(2\pi)^2 2n_0(T)\langle\Delta X^2(t)\rangle^{1/2}. \tag{8}$$

In Eqs. (7) and (8) overbars denote spatial averages, i.e. $\overline{(\cdots)}=\lim_{L\to\infty}\int_{-L/2}^{L/2}(\cdots)dx$.

### A. The kinetic model

Let us consider a SG string with natural boundary conditions $\phi(x\to\pm\infty,t)=2\pi m$, $m=0,\pm1,\pm2,\dots$ [no geometrical (anti)kinks, $n_0=n_\pm$] and subjected to a weak external bias with $F>0$. The string will drift in the $F$ direction by nucleating kink-antikink pairs into the adjacent minimum $2\pi(m+1)$ of the $V[\phi]$ potential. Thermal equilibrium is achieved when, independently of the thermalization mechanism, the nucleation and the annihilation rates of the $\phi_\pm$ pairs coincide. Let $\Gamma$ denote the equilibrium nucleation rate per unit of string length; the (anti)kink lifetime $\tau$ is thus defined by

$$\Gamma=2n_0/\tau. \tag{9}$$

Following Ref. 10, we calculate $\tau$ by having recourse to the LE formalism (6)–(8). In the overdamped limit (6b) the mean-square displacement of $\phi_\pm$ is

$$\langle\Delta X^2(t)\rangle=u_F^2t^2+2Dt, \tag{10}$$

with $D=(kT/E_0)(c_0^2/\alpha)$. Moreover, we know that the collision between a kink and an antikink is always destructive, whereas two (anti)kinks bounce off one another almost elastically. The $\phi_\pm$ lifetime $\tau$ is then determined by the condition that $\langle\Delta X^2(\tau)\rangle$ equals the relevant mean-square free path $n_0(T)^{-2}$. A simple calculation yields

$$\Gamma=2Dn_0^3(T)[1+\sqrt{1+(F/F_c)^2}], \tag{11}$$

with $F_c=kTn_0(T)/2\pi$. The physical meaning of $F_c$ is discussed in Sec. II C. Two limits are of particular interest: for $F\ll F_c$

$$\Gamma_0=4Dn_0^3(T) \tag{12}$$

(*zero-bias* limit) and for $F\gg F_c$

$$\Gamma_1=2u_Fn_0^2(T) \tag{13}$$

(*weak-bias* limit).

We make now a few important remarks.

(i) In view of Eq. (5), $\Gamma_0$ and $\Gamma_1$ are Arrhenius rates with activation energies $3E_0$ and $2E_0$, respectively. While $\Gamma_1$ points to an underlying two-body nucleation mechanism (see Sec. II B), $\Gamma_0$ hints at a gas kinetics. In the absence of external bias the $\phi\to-\phi$ symmetry of the SG theory may be broken locally only: The presence of at least one (anti)kink spectator is required to make the decay of a subcritical nucleus possible.

(ii) Büttiker and Christen[17,18] criticized the zero-bias limit of Eq. (12) and in particular the predicted $3E_0$ activation energy, on the basis of phenomenological arguments. Although indirect numerical evidence[19] supports our viewpoint, a thorough simulation work on thermal pair nucleation would be highly desirable to assess the validity of the kinetic model.

(iii) The drift current corresponding to $\Gamma_1$ can be easily computed: Since the nucleated kink-antikink partners travel a relative distance $n_0^{-1}(T)$ under the action of the bias $F$, the resulting net current is $j=(2\pi)\Gamma_1/n_0$, whence results Eq. (7). More notably, in the weak-bias limit the string current turns out to be proportional to the driving force $F$, as expected in linear-response theory. Analogously, on making use of expression (12) for $\Gamma_0$, one recovers the string diffusion law (8). This proves the internal consistency of the kinetic model.

### B. The two-body model

Let us address now the question how a kink-antikink pair may be nucleated starting from a vacuum configuration, e.g., $\phi(x,t)=0$. Thermal fluctuations are expected to trigger the process by activating a critical nucleus,[8,20] the size of which is known to increase with decreasing $F$; see Eq. (18) below. Provided the critical nucleus size is small enough to ignore many-body effects on the length scale $n_0^{-1}(T)$ (see Sec. II A), we can describe the nucleation process as a *two-body* mechanism. The ensuing nucleation model can be treated as an escape process in a multidimensional system with one neutral equilibrium (or zero) mode.[21]

Thermal fluctuations may activate, with finite probability, a nucleus $\phi_N(x,X)$ of length $2X$ that encroaches upon the adjacent $V[\phi]$ minimum $2\pi$. For $X \gg d$, $\phi_N(x,X)$ is well described by the linear superposition of a kink and an antikink centered at $\mp X$, respectively,

$$\phi_N(x,X)=\phi_+(x+X,0)+\phi_-(x-X,0) = 4\arctan[\sinh(X/d)/\cosh(x/d)], \qquad (14)$$

where $\phi_\pm(x,t)$ is defined in Eq. (4) with $\beta=1$. [$\phi_N(x,X)$ has been centered at the origin for convenience.] The energy of the nucleus $\Delta E_N$ is a function of its size $2X$, namely,

$$\Delta E_N=\int H_{SG}[\phi_N(x,X)]dx = 2E_0\left[1-\frac{1}{\cosh(2X/d)+1}\left(1+\frac{2X/d}{\sinh(2X/d)}\right)\right]. \qquad (15)$$

The components of a large nucleus experience two contrasting forces (Fig. 1): an attractive one with potential function $\pm 4E_0 \exp(-2X/d)$ [see Eq. (15)], due to the vicinity of the nucleating partner, and a repulsive one with effective potential $\pm 2\pi FX$, due to the external bias, which pulls the nucleus partners $\phi_\pm$ apart. The *critical* nucleus configuration $\phi_N(x,R_N/2)$ is attained for a distance $R_N(F)$ between $\phi_\pm$ such that the two competing forces balance each other. The critical nucleus $\phi_N(x,R_N/2)$ is thus the field saddle-point configuration in the escape process associated with the nucleation.

In the SG theory[12] $\phi_N(x,R_N/2)$ admits of one unstable mode only, with negative eigenvalue $\lambda_0^N$ and one neutrally stable translation mode with null eigenvalue (the so-called Goldstone mode). In the overdamped limit the decay of the critical nucleus is fully described by the reduced (or nucleus) coordinate $R=2X$ (Fig. 1). On adding the kink-antikink interaction term to the LE (6b) for $\phi_\pm$, we obtain

$$\dot{R}=-V_N'(R)+\eta_R(t), \qquad (16)$$

where for $R \gg 0$

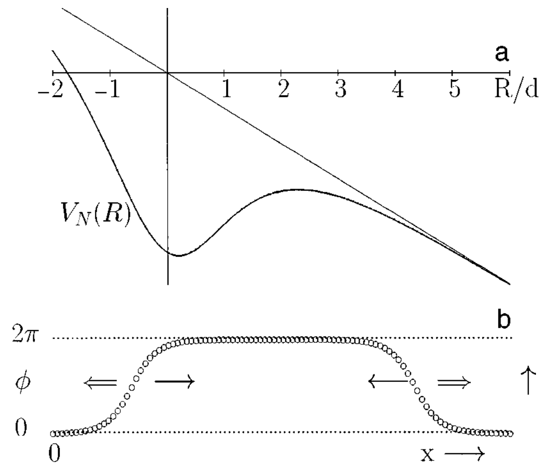


FIG. 1. (a) Critical nucleus potential $V_N(R)$ for arbitrary values of $\alpha M_R$. The straight line represents the bias potential $-2\pi FR/\alpha M_R$. For the kink-antikink potential we plotted the function $\Delta E_N(R)-2E_0$ with $R=2X$ and $\Delta E_N(R)$ given in Eq. (15). (b) Sketch of a critical nucleus. The attractive kink-antikink forces (inward arrows) and the bias pulling forces (double arrows) are marked for the readers convenience; the vertical arrow points in the direction of the string drift.

$$V_N(R)=-\frac{2\pi F}{\alpha M_R}R-\frac{4E_0}{\alpha M_R}e^{-R/d}. \qquad (17)$$

Here $M_R=M_0/2$ and $\eta_R(t)$ is the same as $\eta(t)$, but for the substitution of $D$ with $D_R=kT/\alpha M_R$ in its autocorrelation function. The size of the critical nucleus is set by the condition that $V_N'(R)|_{R_N}=0$, whence we obtain

$$R_N(F)=-d\,\ln\left(\frac{\pi}{16}\frac{F}{\omega_0^2}\right) \qquad (18)$$

and the negative eigenvalue

$$\lambda_0^N=V_N''(R_N)=-\frac{\pi}{2}\frac{F}{\alpha}. \qquad (19)$$

The two-body nucleation rate in Gaussian approximation[21] reads

$$\Gamma_2=\frac{|\lambda_0^N|}{2\pi L}\frac{Z_N'}{Z_0}e^{-\Delta E_N/kT}. \qquad (20)$$

Here the activation energy $\Delta E_N$ is given by Eq. (15) for $2X=R_N$; $Z_0$ and $Z_N'$ denote the (effective[21]) partition functions of a SG string with length $L\to\infty$ in its vacuum and saddle-point configuration, respectively. The entropy factor $Z_N'/Z_0$ can be factorized as

$$\frac{Z_N'}{Z_0}=L[4\pi kTM_0]^{1/2}\frac{2\pi kT}{(\alpha|\lambda_0^N|)^{1/2}}\left(\frac{Z_N}{Z_0}\right)_{ph}, \qquad (21)$$

where the contributions from the Goldstone mode, the unstable mode, and the phonon modes are clearly identifiable. On neglecting corrections of the second order in $F/\omega_0^2$ to the phonon spectrum, the factor $(Z_N/Z_0)_{ph}$ reduces to

$4\omega_0^2/(2\pi kT)^2$.[12] On making use of Eqs. (18), (19), and (21) we finally obtain for the nucleation rate (20) the well-known result[8,20,10]

$$\Gamma_2=\frac{2\omega_0}{\pi d}\left(\frac{|\lambda_0^N|}{2\pi\alpha}\right)^{1/2}\left(\frac{2E_0}{kT}\right)^{1/2}e^{-\Delta E_N/kT}. \tag{22}$$

The validity of this formula is restricted to $kT \ll Fd \ll E_0$. The upper bound corresponds to the nonrelativistic approximation $F \ll \omega_0^2$ and guarantees large critical nucleus sizes $R_N(F) \gg d$ and activation energies $\Delta E_N$ close to $2E_0$. The lower bound is required for the Gaussian approximation (20) at the saddle point of $V_N(R)$ to hold true.

### C. The crossover regime

Let us now compare the nucleation rates $\Gamma_1$ for the weak-bias limit and $\Gamma_2$ for the two-body model. For intermediate bias values, say, $kT \ll Fd \ll E_0$, the two determinations of the nucleation rates should coincide, at least in principle. Instead, as pointed out in Ref. 10, the ratio

$$\frac{\Gamma_1}{\Gamma_2}=2\pi\sqrt{\frac{Fd}{kT}} \tag{23}$$

(where $\Delta E_N$ has been approximated to $2E_0$) reveals a discrepancy that seems difficult to reconcile. We noted in Sec. II A that the rate $\Gamma_1$ is consistent with the prescriptions of linear-response theory; this led us to figure that the discrepancy in Eq. (23) should be caused by the assumptions introduced to derive $\Gamma_2$. Eventually, we concluded that the Gaussian approximation implied by Langer's formula (20) is inadequate to describe the decay process of the critical nucleus represented by the LE (16) for the nucleus coordinate $R$.

The approach of Sec. II B allows a simple estimate of the non-Gaussian corrections to $\Gamma_2$. Without entering the intricacies of Langer's formalism,[21] we remind the reader that the prefactor in Eq. (20) is *inverse* proportional to the Gaussian integral[22]

$$\int_{-\infty}^{\infty}\exp\left(-\frac{|\lambda_0^N|}{2D_R}R^2\right)dR. \tag{24}$$

Such an integral approximates the probability flow at the saddle point of $V_N(R)$. In the present case, however, the analytical form of the potential $V_N(R)$ is given explicitly in Eq. (17), so that instead the approximate factor (24) we could have used the exact factor

$$\int_0^{\infty}\exp\left(\frac{\overline{V}_N(R)}{D_R}\right)dR, \tag{25}$$

with $\overline{V}_N(R)=V_N(R)-V_N(R_N)$. Accordingly, the nucleation rate $\Gamma_2$ of Eq. (22) is underestimated by the factor

$$\kappa(F)=\int_{-\infty}^{\infty}\exp\left(-\frac{|\lambda_0^N|}{2D_R}R^2\right)dR \Bigg/ \int_0^{\infty}\exp\left(\frac{\overline{V}_N(R)}{D_R}\right)dR. \tag{26}$$

The reader can easily verify that $\kappa(F)\to 1$ for $Fd \gg kT$, whereas for smaller $F$ values the leading contribution from

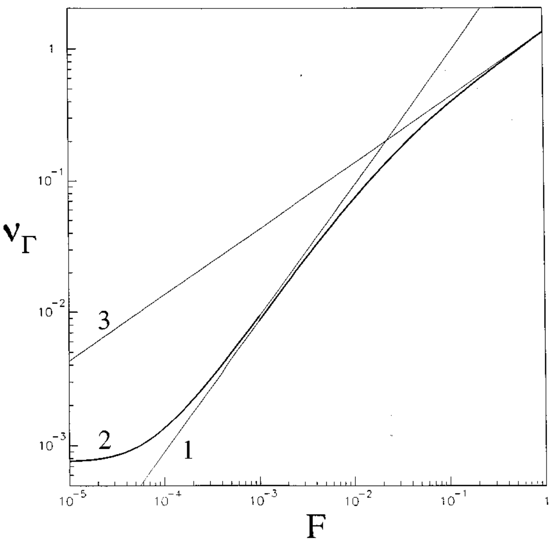


FIG. 2. Comparison of the different nucleation rates predicted in Sec. II. Curve 1, the corrected two-body rate $\kappa(F)\Gamma_2(F)$ [see Eq. (26)]; curve 2, $\Gamma(F)$ of Eq. (11); curve 3, $\Gamma_2(F)$ of Eq. (22). For the sake of simplicity all rates have been divided by the Arrhenius factor $\exp(-\Delta E_N/kT)$, that is, $\nu_\Gamma \equiv \Gamma\exp(\Delta_N/kT)$. The parameter values are $c_0^2=\omega_0^2=1$ and $E_0/kT=3$. For the readers' convenience, the interpolation of curves 1 and 2 is denoted by two solid lines.

the denominator is correctly estimated by approximating $\overline{V}_N(R)$ to $-(2\pi F/\alpha M_R)R$, whence we obtain

$$\kappa(F)=2\pi\sqrt{\frac{Fd}{kT}}. \tag{27}$$

The discrepancy of Eq. (23) is thus explained in close agreement with numerical evidence.[19] Thanks to the correcting factor (26), on decreasing $F$ the two-body rate $\Gamma_2$ goes continuously over into the weak-bias limit rate $\Gamma_1$; furthermore, $\Gamma_1$ and the zero-bias rate $\Gamma_0$ are analytically connected through Eq. (11). The relevant crossover $F$ values are $F_c=kTn_0(T)/2\pi$ from $\Gamma_0$ up to $\Gamma_1$ and $F_\kappa=kT/d$ from $\Gamma_1$ up to $\Gamma_2$ (see Fig. 2). Their physical interpretation is straightforward. For $F=F_c$ the thermal energy $kT$ coincides with the mechanical energy needed to pull a single (anti)kink through a distance of the order of its mean free path $n_0^{-1}(T)$. Thus, for values of $F$ smaller than $F_c$ the notion of critical nucleus becomes untenable. This is why the zero-bias rate $\Gamma_0$ cannot be reproduced through the two-body model of Sec. II B, no matter how accurately we handle it. For $F=F_\kappa$ the thermal energy $kT$ equals the mechanical work made by the external force to move an (anti)kink through a distance of the order of its size $d$. The linearization of the decay dynamics around the saddle point $R=R_N$ applies for $F \gg F_\kappa$.

## III. NUCLEATION IN A FINITE STRING

The effects due to the finite length $L$ of the string may become important. For instance, certain imperfections have the ability to reduce considerably the activation energy of the critical nucleus by introducing an additional *pinning* potential. Such a mechanism is particularly effective in the dislocation-induced internal friction at low temperature,[23]

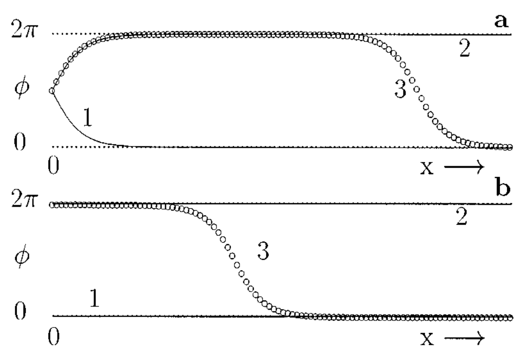


FIG. 3. Sketch of $\Phi_{-1}$ (curves 1), $\Phi_{+1}$ (curves 2), and $\Phi_N$ (curves 3) for a semi-infinite SG string with $c_0^2=\omega_0^2=1$ and (a) $\lambda \to 0+$ and (b) $\lambda \to \infty$. Note that we imposed $X_b(\lambda)>0$.

where the pinning action is exerted by pointlike defects. Another example is provided by the flux lines that thread through type-II superconducting films: Their length is necessarily limited by the thickness of the sample. The line end points, when not acting as tight pinning points, may ease the nucleation process depending on the choice of the boundary conditions. *Heterogeneous* nucleation processes may thus contribute appreciably[24] as either a bulk or a surface effect.

### A. End-point effects

For simplicity we assume that a long SG string is constrained at the origin $x=0$ with boundary conditions (BC's)

$$\lambda \phi_x + \sin \phi = 0. \tag{28}$$

Note that this equation is symmetric under the transformation $\lambda \to -\lambda$ and $x \to -x$. Natural BC's are assumed at $x=L$ with $L$ arbitrarily large. The limits $\lambda \to \pm\infty$ and $\lambda \to 0$ implement von Neumann's (free-end) and Dirichlet's (fixed-end) BC's, respectively. The Hamiltonian density

$$H[\phi] = H_{SG}[\phi] + \phi_x \sin \phi/\lambda \tag{29}$$

incorporates the BC (28) into the SG theory. Accordingly, the string energy is modified by an additional end-point term $-\cos \phi/\lambda|_{x=0}$.

In the presence of a weak external bias the two-body model of Sec. II B must be modified to account for the loss of translational invariance. Let us consider for simplicity a jump forward of the string, i.e., $\phi \to \phi + 2\pi$. In the vicinity of $x=0$, the shape and the position of a critical nucleus are affected by the constraint of Eq. (28) and so are the vacuum field configurations. The BC (28) amounts to fixing the position $X_b(\lambda)$ of the center of mass of the SG critical nucleus $\Phi_N(x,R_N/2) = \phi_+(x+R_N/2-X_b,0) + \phi_-(x-R_N/2-X_b,0)$. For simplicity we restrict ourselves to the solutions with $X_b(\lambda)>0$. The vacuum configurations can be regrouped into two classes $\Phi_{\pm 1}(x;\lambda)$: They are all represented by suitably truncated kink-antikink functions, namely, by $\phi_\pm(x,0)$ of Eq. (4) with $x>0$ and appropriate center of mass $X_0(\lambda)$, and related through the symmetry identity $\Phi_{+1}(x) + \Phi_{-1}(x) = 2\pi n$, with $n=0,\pm 1,\pm 2,\ldots$. Moreover, it can be easily proven that such vacuum solutions overlap with the $\phi_\pm$ components of a pinned critical nucleus $\Phi_N$. In Fig. 3 we show the pinned critical nucleus $\Phi_N$ and the vacuum solutions $\Phi_{\pm 1}$ for $\lambda \to 0$ and $\lambda \to \infty$.

An end-point nucleation process, say, from $\phi=0$ to $\phi=2\pi$, corresponds to a transition from $\Phi_{-1}(x;\lambda)$ to $\Phi_{+1}(x;\lambda)$ via the saddle-point configuration $\Phi_N(x,R_N/2)$. The activation energy of such a process is obtained by subtracting the vacuum energy of the symmetric solutions $\Phi_{\pm 1}$ from the energy of the critical nucleus $\Phi_N$, that is,

$$\Delta E_b(\lambda) = \Delta E_N(R_N) - E_0. \tag{30}$$

In the limit of small bias $F \ll \omega_0^2$ the activation energy $\Delta E_b(\lambda)$ is independent of $\lambda$ and tends to $E_0$.

The negative eigenvalue $\lambda_0^N$ of the pinned critical nucleus can be calculated following the LE approach of Sec. II B. Here the attractive force between the nucleus components $\phi_\pm$ is given by the variation of $\Delta E_b(\lambda)$ with respect to the coordinate of $\phi_-$ (being $\phi_+$ pinned at $x=0$). It follows immediately that $\lambda_0^N$ is one-half of the negative eigenvalue in the homogeneous case (19), $\lambda_0^N = -\pi F/4\alpha$.

In order to apply Langer's formula (20) to the present case we have to modify further the phonon spectrum according to the new BC. As the critical nucleus centered at $X_b(\lambda)$ is not invariant under translation, the Goldstone mode contribution to the entropy factor (21) drops out. Moreover, due to the constraint (28), the phonon modes ''dress'' $\Phi_{-1}$ and the nucleus component $\phi_+$ in the same manner, so that the total entropy factor $Z_N/Z_0$ reads $[2\pi kT/(\alpha|\lambda_0^N|)^{1/2}](2\omega_0/2\pi kT)$. The end-point nucleation rate, integrated over the entire string length, follows immediately from Eq. (20), that is,

$$r_2 = \frac{\omega_0^2}{2\alpha L}\left(\frac{F}{\pi\omega_0^2}\right)^{1/2} e^{-E_0/kT}. \tag{31}$$

Here the factor $1/L$ reminds us of the dishomogeneous nature of the nucleation process that occurs at the pinning point $x=0$. The total rate for any assigned distribution of pinners can be calculated immediately from Eq. (31).

The rate $r_2$ in Eq. (32) has been computed in the Gaussian approximation $Fd \gg kT$. In order to extend its validity to the weak-bias range $F_c \ll F \ll kT/d$, we must multiply $r_2$ by the correcting factor $\kappa(F)$ of Eq. (27). A few simple algebraic passages yield

$$r_1 = u_F n_0(T)/L. \tag{32}$$

This result is remarkable indeed. On comparing the rates $r_1$ and $\Gamma_1$ (or $r_2$ and $\Gamma_2$) we conclude that heterogeneous nucleation dominates over homogeneous nucleation for $2L \leq n_0^{-1}(T)$, independently of the bias intensity regime. Moreover, the kinetic model of Sec. II A provides a suggestive interpretation of our result for $r_1$: If the annihilation process takes place at the pinner, the (anti)kink lifetime is determined by the equation $\langle \Delta X^2(\tau)\rangle = 4L^2$; the solution for $FL \gg kT$ reads $\tau = 1/u_F$ and the relevant rate $r_1 = 2n_0/\tau$ coincides with Eq. (32).

### B. Nucleation length scales

We conclude this section by summarizing the role of the three length scales that characterize the nucleation process in a finite SG string with BC (28) (possibly at both end points): its length $L$, the critical nucleus size $R_N(F)$, and the $\phi_\pm$ mean free path $n_0^{-1}(T)$. The temperature $T$ and the intensity of the external bias $F$ define three different regimes.

(i) $R_N(F) < n_0^{-1}(T), L$. The nucleation process occurs through the two-body mechanism of Secs. II B and III A. Homogeneous (or bulk) and heterogeneous (or end-point) nucleation coexist, the relative importance being determined by the ratio of $\Gamma_2$ to $r_2$. For $n_0^{-1}(T) \ll L$ the string accommodates an equilibrium gas of thermal kinks and antikinks; boundary effects become negligible.

(ii) $n_0^{-1}(T) < R_N(F), L$. Many-body effects due to the presence of an equilibrium (dilute) gas of kinks and antikinks control the nucleation process, as described by the kinetic model of Sec. II A. For vanishingly small $F$ values $R_N(F)$ grows larger than $L$ and the zero bias rate $\Gamma_0$ is eventually attained.

(iii) $L < n_0^{-1}(T), R_N(F)$. This limit has not been addressed explicitly in the present section. However, the reader will sure recognize that this corresponds to the short-junction limit of Ref. 13: The string segment behaves like a *rigid* rod. The standard Kramers theory for the diffusion of a Brownian particle in a washboard potential with barriers $\omega_0^2 L$ provides a good description of the process for $R_N(F) \gg L$.

## IV. FINAL REMARKS

We conclude by hinting at a few affordable extensions of our theory of nucleation in one-dimensional strings. First of all, we notice that both the kinetic and the two-body model can be implemented for two more soliton-bearing strings of wide use in physics, namely, the $\phi^4$ and the double-quadratic strings.[12] A detailed analysis of the relevant nucleation process requires the knowledge of the kink (antikink) shape and the phonon spectrum in their presence; constraints in the kink (antikink) statistics must be introduced to account for the lack of transport currents. Analytical expressions for the nucleation rates in the overdamped limit may be easily obtained following the procedure of Secs. II and III.

As mentioned in Sec. I, there is no *a priori* reason why the nucleation process should be restricted to the overdamped limit. Indeed, important applications of the SG string model to dislocations[3,2] and long Josephson junctions[13] seem to work in the opposite limit $\alpha \ll \omega_0$. The nucleation rate in a SG string with zero bias $F=0$ and vanishingly small damping $\alpha \to 0$ has been calculated in the framework of the kinetic model in Ref. 25 and then applied successfully to the theory of the Bordoni peak.[9] In the same damping limit, however, the stability of a *biased* SG string against depinning becomes a delicate matter: A sudden transition from the locked to the running mode, and vice versa, takes place any time $F$ crosses a certain threshold value proportional to $\alpha$.[14] This shrinks the bias range where thermal nucleation may be appreciable to an arbitrarily small neighborhood of $F=0$ (described by the kinetic model). Furthermore, at even lower $\alpha$ values quantum tunneling effects become increasingly important[26] and eventually dominate over the thermal nucleation mechanism investigated in the present work.[27]

## ACKNOWLEDGMENTS

This work was supported in part by the Istituto Nazionale di Fisica Nucleare, VIRGO Project. F.M. wishes to thank D. K. Campbell and A.V. Granato for encouragement and useful discussions during his stay at the Loomis Laboratory of Physics of the University of Illinois.

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