Images and nonlocal vortex pinning in thin superfluid films

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For thin films of superfluid adsorbed on a disordered substrate, we derive within a mean field (Hartree) description of the condensate the equation of motion for a vortex in the presence of a random potential. The compressible nature of the condensate leads to an effective pinning potential experienced by the vortex which is nonlocal, with a long-range tail that smooths out the random potential coupling the condensate to the substrate. We interpret this nonlocality in terms of images, and relate the effective potential governing the dynamics to the pinning energy arising from the expectation value of the Hamiltonian with respect to the vortex wave function. [S0163-1829(98)06813-1]

I. INTRODUCTION

Pinning of vortices in bulk helium has a long history (see, for example, Ref. 1) but the study of vortex pinning in thin films is at a much earlier stage of development, to our knowledge only one experimental study having been performed, Ref. 2, although there are some aspects addressed in Ref. 3. In this paper we will concentrate on two-dimensional pinning of idealized (in a sense defined presently) superfluid films which provides two simplifications compared to the situation in three dimensions: the ability to change the healing length as well as the obvious contrast between point vortices and extended vortex lines. The healing length is altered by changing the coverage of the helium film, although quantifying this relationship is not easy.

Ellis and Li[2] showed that by ‘‘swirling’’ a gold-plated Mylar substrate a remanent vorticity could be created, where the density of pinned vortices was 50 000 cm^{-2} so that the separation (45 μ) was of the order of 10^4 larger than the film thickness (3.2 nm). Putnam *et al.*[4] have studied the topography of gold films deposited under similar conditions to those used by Ellis and Li; they find that the films have a surface of ‘‘rolling hills’’ with a characteristic length scale of 300–3000 Å. Thus the separation was considerably larger than the topographic features which are presumably the pinning agents. In this paper we will concentrate on the pinning of single vortices (and their dynamics) which should be appropriate under these experimental conditions. However, we will confine ourselves to the simplest case of monolayer films.

We wish to derive from first principles the form of the pinning potential which a vortex experiences, given the potential in which the helium atoms move due to the substrate. We will see that these two quantities are not the same. We will describe the condensate at a mean field (Hartree) level using the hydrodynamic representation[5] (in terms of the two-dimensional density of the film and velocity potential of the flow) and of the resulting nonlinear Schrödinger equation (NLSE) which determines the condensate wave function.[6] We will allow for some effects of compressibility—for instance that the (two-dimensional) density is modified to ‘‘screen’’ the random potential[7]—and will find an equation of motion that bears some relation to the Magnus effect.

In this very thin film limit, there are two regimes in considering the behavior of a vortex in a random potential. First there is the region, near the center of the vortex, where the largest contributions to the energy density are the kinetic energy of the fluid and the change in the film density that is caused by the flow. However, at large distances, the dominant contribution to the energy density is the response of the density to the random (pinning) potential. Hence we must analyze the behavior in two regions and the manner in which to match the approximate solutions in those regions. The matching is performed by following the analysis of Neu[8] which was constructed to determine the motion of a many-vortex system in a compressible ideal fluid from first principles, reproducing the Kirchhoff result for widely separated vortices. There the two regions are where single-vortex effects are predominant (near the center of each vortex) and where many-vortex contributions are important (on longer length scales). Other expansions similar in spirit have been for compressible vortex rings[9] and a number of works following Neu, which may be traced from Ref. 10.

In the monolayer regime, there is an additional complication: the superfluid-insulator transition (see, for example, the recent publications in Ref. 11 and Ref. 12 and references therein). However, sufficiently far away (a small fraction of a monolayer) from the onset of superfluidity, one may describe the helium film as being an ‘‘inert,’’ nonsuperfluid, initial layer with a mobile, superfluid, film adsorbed on top.[13] The treatment in this paper will be of the latter part of the film, where the random potential is the residual one which includes any interaction with the inert layer.

The plan of the paper is as follows: In Sec. II we introduce the framework used to describe the vortex motion. Using a perturbation calculation, we derive the velocity of a vortex under the influence of an external potential, by matching the ‘‘inner’’ and ‘‘outer’’ solutions in Sec. III. In the next section, we establish that the dynamics are nonlocal in the random substrate potential and explore some of the consequences. In the following section we relate the motion to the gradient of the expectation value of the energy. In the final section we discuss several issues which emerge from the rest of the paper and conclude.

II. NOTATION AND FLUID REPRESENTATION OF THE NLSE

We wish to study the influence of a disordered substrate qualitatively on a superfluid film. We therefore pick a very simple interaction between the bosons, namely, a point interaction. Hence our starting point is the Hamiltonian

$$H=\sum_{i=1}^{N}-\frac{\hbar^2}{2m}\nabla_i^2+\frac{1}{2}\lambda\sum_{i\neq j}^{N}\delta(\mathbf{r}_i-\mathbf{r}_j)+\Delta\sum_{i=1}^{N}V(\mathbf{r}_i). \quad (1)$$

Here the coordinates and Laplacian are two dimensional, as we assume motion quantized normal to the substrate surface, with all atoms in the lowest state of that motion. Δ is the variance of the random two-dimensional potential V:

$$\Delta^2=\langle[V(\mathbf{r})]^2\rangle, \quad (2)$$

where the average is over the ensemble of potentials. Note that V itself is dimensionless.

We will treat the above Hamiltonian within the Hartree approximation; thus we will not include effects at the Bogoliubov approximation or beyond: for instance, any analog of the roton minimum. The Hartree approximation leads to the time-dependent NLSE which governs the condensate motion:[5,6]

$$-\frac{1}{2}\nabla^2\phi+|\phi|^2\phi+\sigma V\phi=i\frac{\partial\phi}{\partial t}, \quad (3)$$

where we are using appropriately scaled units: Lengths are measured in units of the healing length, $\ell_{\mathrm{h}}=(\hbar^2/\lambda nm)^{1/2}$, $\sigma=\Delta/n\lambda$ is the dimensionless measure of the strength of the external potential, and energy is measured in units of $n\lambda$, which is the Hartree energy or chemical potential. n is the average density of particles (equivalent to the condensate density in this Hartree case). ϕ is normalized to the size of the system, Ω,

$$\int_\Omega|\phi|^2\, dV=\Omega, \quad (4)$$

so that in the absence of a potential, $\phi=1$ everywhere within Ω. Note that the speed of sound in the condensate, $(n\lambda/m)^{1/2}$, is equal to unity with the above choice of units.

To describe the dynamics of the condensate possessing a vortex structure, it is natural to use the (Hartree) fluid, or Madelung, representation;[5] by setting $\phi=\sqrt{\rho}\, e^{iS}$, Eq. (3) is equivalent to the pair of equations

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\nabla S)=0 \quad (5)$$

and

$$\frac{\partial S}{\partial t}+\frac{1}{2}(\nabla S)^2+\rho+\sigma V-\frac{1}{2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}=0. \quad (6)$$

Equations (5) and (6) are, respectively, the continuity and Bernoulli equations describing the condensate flow. The density ρ is measured in units of n.

In the main body of this paper, we will be interested in the behavior of the flow field well beyond the core region, so that the rapid variations in the condensate density and velocity field near the core itself are not considered. Within this approximation we can neglect the ‘‘quantum pressure’’ term $\nabla^2\sqrt{\rho}/\sqrt{\rho}$ in the Bernoulli equation (6) (which we justify *a posteriori* presently) and use the set of equations

$$\frac{\partial\rho}{\partial t}+\nabla\rho\cdot\nabla S+\rho\nabla^2 S=0 \quad (7)$$

and

$$\frac{\partial S}{\partial t}+\frac{1}{2}(\nabla S)^2+\rho+\sigma V=0. \quad (8)$$

For this work, an important solution to these equations is that of a vortex in two dimensions:

$$S=\theta-t \qquad \rho=1-\frac{1}{2}\frac{1}{r^2}, \quad (9)$$

where θ is the polar angle. Note that the term $-t$ in the expression for S is the chemical potential, which cancels the term in the asymptotic density in the Bernoulli equation.

III. DYNAMICS DUE TO THE SUBSTRATE POTENTIAL

In the Magnus equation for incompressible point vortices, the presence of a potential coupling to the vortex will change its motion from being determined completely by the value of the velocity field at the vortex center. (The latter result, due to Kelvin and Helmholtz, has been derived in the context of Landau-Ginzburg theory by Neu.[8]) In this section we will derive an approximate solution for a vortex in the presence of a potential which couples to the helium atoms in the film. This will allow us to derive in the next section the form of the potential which applies to the *vortex*, as against the helium atoms, and hence appears in the equation of motion of the vortex. We apply Neu's[8] method of matched asymptotic expansions in this section.

We will perform a linear response calculation using the fluid (Madelung) representation of the condensate wave function. This is preferable to using linear response theory directly—in terms of the condensate wave function and the Bogoliubov excitations in the presence of the vortex. In the latter approach we would need to calculate the matrix elements of, for instance, the random potential between the ground state and the excitations. Since the wave functions of the excitations around a vortex are only known asymptotically, this would not be straightforward.

Before embarking on any calculation, we wish to divide the problem into one of ‘‘inner’’ and ‘‘outer’’ solutions. In the inner region, the vortex motion is dominant, in terms of both density and velocity field, and in the outer region, the random potential is dominant. We can estimate the boundary between these two regions to be where the perturbation in the density due to flow around the vortex is equal to the perturbation to the density due to the random potential, namely, at to be a distance r_c where

$$\frac{1}{r_c^2}\sim\sigma\Rightarrow r_c\sim\sigma^{-1/2} \tag{10}$$

and r_c is measured in units of the healing length.

The inner solution can be determined by applying perturbation theory to Eqs. (7) and (8) in powers of σ, since the random potential is assumed to be weak.

For the outer solution, we want to rescale the position and time variables $\mathbf{r}$ and t to make the dominance of σ at large distances manifest. We make the substitutions

$$\mathbf{r}\rightarrow\mathbf{r}'=\sigma\mathbf{r},\quad t\rightarrow t'=\sigma^2 t. \tag{11}$$

In terms of these new variables, the continuity and Bernoulli equations of (7) and (8) become (dropping primes):

$$\frac{\partial\rho}{\partial t}+\nabla\rho\cdot\nabla S+\rho\nabla^2 S=0 \tag{12}$$

and

$$\sigma^2\frac{\partial S}{\partial t}+\sigma^2\frac{1}{2}(\nabla S)^2+\rho+\sigma V=0. \tag{13}$$

The continuity equation remains unchanged but the Bernoulli equation is modified—we may neglect the time dependence of S [apart from the term $-t/\sigma^2$ which provides the chemical potential, as mentioned below Eq. (9)]. In the outer region we expand the density and velocity potential as

$$\rho=1+\sigma\rho_1(\mathbf{r}),\quad S=S_0(\mathbf{r}-\mathbf{R})+\sigma S_1(\mathbf{r}). \tag{14}$$

Note that the density does not depend on the position of the vortex, as the terms in Eq. (13) which depend on $\mathbf{R}$ are second order in σ; similarly we may neglect the time dependence of ρ as time does not enter Eq. (13) to first order in σ. In Eq. (14) the zeroth-order flow field retains a vortex flow, so that $\oint\nabla S_0\cdot d\mathbf{r}=2\pi$, despite a distortion of the flow as a whole due to the linear addition of ∇S_1.

Substituting the expansion (14) into Eqs. (12) and (13) we have to $O(\sigma)$

$$\rho_1=-V,$$

$$\nabla^2 S_1+\nabla S_0\cdot\nabla\rho_1=0. \tag{15}$$

Solving for S_1 and denoting the outer solution by $S_1^>$, we find

$$S_1^>(\mathbf{r})=\int G(\mathbf{r}-\mathbf{r}')\boldsymbol{\gamma}\cdot\nabla V(\mathbf{r}')d\mathbf{r}', \tag{16}$$

where

$$G(\mathbf{r}-\mathbf{r}')=\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'| \tag{17}$$

and we have introduced the more compact notation $\boldsymbol{\gamma}\equiv\nabla S_0(\mathbf{r}-\mathbf{R})$ for the vortex field. It is important to note that the variables here are the *scaled* ones defined in Eq. (11).

Turning to the inner solution, we revert to the unscaled variables, and solve the original equations (7) and (8). We require the solution to be matched to the outer one at distances much greater than the core radius, i.e., at dimensionless distances $r\gg 1$. In the inner region we can write to $O(\sigma)$

$$\rho=\rho_0(\mathbf{r}-\mathbf{R})+\sigma\rho_1,\quad S=S_0(\mathbf{r}-\mathbf{R})+\sigma\nabla S_1,$$

with ρ_0 given by

$$\rho_0=1-\frac{1}{2|\mathbf{r}-\mathbf{R}|^2}=1-\delta\rho_0(\mathbf{r}). \tag{18}$$

So $\delta\rho_0$ is the same as the change in density due to the vortex in the absence of the random potential. The fluid equations taken to $O(\sigma)$ are then

$$\rho_1=-V+\boldsymbol{\gamma}\cdot(\dot{\mathbf{R}}-\nabla S_1),$$

$$\rho_0\nabla^2 S_1+\boldsymbol{\gamma}\cdot\nabla\rho_1+\nabla\rho_0\cdot(\nabla S_1-\dot{\mathbf{R}})=0. \tag{19}$$

The eventual aim is to find $\dot{\mathbf{R}}$ in terms of S_1; that is, we look for the change in flow caused by the potential, which then determines the vortex velocity through advection. Examining Eq. (19), we can inspect the individual terms to see which ones will dominate in the $\mathbf{r}\rightarrow\infty$ limit for the matching. Eliminating ρ_1 and omitting terms of order $1/r^3$ and higher, we are left with

$$\nabla^2 S_1=\boldsymbol{\gamma}\cdot\nabla V,$$

which can be solved $S_1^<(\mathbf{r})$ for to obtain

$$S_1^<(\mathbf{r})=\int G(\mathbf{r}-\mathbf{r}')\ \boldsymbol{\gamma}\cdot\nabla V(\mathbf{r}')\ d\mathbf{r}'+O(1/r),$$

where $S_1=S_1^<$ denotes the inner solution.

We will now match the inner and outer solutions at a distance $r\sim\sigma^{-\alpha}$, with $\alpha>0$ chosen to ensure that corrections are small. First, we must consider the *outer* solution back in terms of the *unscaled* variables. But from Eq. (16) it can be seen that there is in fact no change in its form, as lengths only enter $S_1^>$ and the scaling of $\boldsymbol{\gamma}\cdot\nabla$ and the element of integration $d\mathbf{r}$ cancel. Hence we can conclude that matching is *perfect* up to $O(\sigma)$. At $r\sim\sigma^{-\alpha}$, neglected terms in the equation determining the inner S_1 are $1/r$ or a factor of σ^{α} smaller; thus any inconsistency is negligible. We will make a more refined consistency check presently.

Note that (neglecting corrections) S_1 can be integrated by parts to give

$$S_1(\mathbf{r})=-\int \boldsymbol{\gamma}(\mathbf{r}')\cdot\nabla G(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\ d\mathbf{r}', \tag{20}$$

where we assume that the ''surface'' integral vanishes (and $\nabla\cdot\boldsymbol{\gamma}=0$). This alternate form for S_1 will be useful in the next section.

So far the velocity of the vortex, $\dot{\mathbf{R}}$, has not been determined. To do so, we Taylor expand the outer solution (16) about $\mathbf{r}'=\mathbf{R}'$ and check that corrections to this expansion are negligible in the matching region. This ensures that the Taylor expansion is consistent with the matching.[8] Thus we have

$$S^>(\mathbf{r}')=S^>(\mathbf{R}')+(\mathbf{r}'-\mathbf{R}')\cdot\nabla S_1^>(\mathbf{R}')+O(\mathbf{r}'^2).$$

In the second term on the right, $\nabla S_1^>$ appears as $\nabla S_0^>$ is perpendicular to $(\mathbf{r}'-\mathbf{R}')$.

We now appeal to the Helmholtz theorem which states that the vorticity moves with the local velocity field (the ''dynamical boundary condition''). Using Eq. (16) this gives the result for $\dot{\mathbf{R}}$ as

$$\dot{\mathbf{R}}=\int \nabla_{\mathbf{R}}G(\mathbf{R}-\mathbf{r})\ \boldsymbol{\gamma}\cdot\nabla V(\mathbf{r})\ d\mathbf{r}, \tag{21}$$

where $\nabla_{\mathbf{R}}\equiv\partial/\partial\mathbf{R}$.

To check for consistency, we examine the size of $(\mathbf{r}'-\mathbf{R}')\cdot\dot{\mathbf{R}}=\sigma(\mathbf{r}-\mathbf{R})\cdot\dot{\mathbf{R}}$ compared with the corrections $1/r$ and σ^2r^2 [i.e., the $O(r'^2)$ in the Taylor expansion]. We require $\sigma r\gg 1/r$ and σ^2r^2 for $r\sim\sigma^{-\alpha}$. The first condition gives $\sigma^{(1-\alpha)}\gg\sigma^{\alpha}\Rightarrow\alpha>1/2$, while the second one gives $\sigma^{(1-\alpha)}\gg\sigma^{(2-2\alpha)}\Rightarrow\alpha<1$. Hence for both sets of corrections to be negligible, we need

$$1/\sigma^{1/2}<r<1/\sigma.$$

Since this region does exist, the expansion is consistent.

IV. INTERPRETATION OF VORTEX MOTION IN TERMS OF IMAGES

We have now deduced the form for $\dot{\mathbf{R}}$ which governs the vortex motion in an external potential, and in this section consider some of the consequences and the interpretation of Eq. (21). First, note that $\dot{\mathbf{R}}$ is clearly related to the potential in a *nonlocal* manner. Using Eq. (20), the expression for $\dot{\mathbf{R}}$ in Eq. (21) can be rewritten as

$$\dot{\mathbf{R}}=-\nabla_{\mathbf{R}}\int \boldsymbol{\gamma}(\mathbf{r})\cdot\nabla G(\mathbf{R}-\mathbf{r})V(\mathbf{r})\ d\mathbf{r}. \tag{22}$$

We can further manipulate this by using the identity $\boldsymbol{\theta}\cdot\nabla V=(\hat{\mathbf{z}}\times\nabla V)\cdot\hat{\mathbf{r}}$ where $\hat{\mathbf{z}}$ is perpendicular to the plane of motion, so that Eq. (22) becomes

$$\dot{\mathbf{R}}=-\hat{\mathbf{z}}\times\int (\boldsymbol{\mu}\cdot\nabla_{\mathbf{R}})\nabla_{\mathbf{R}}G(\mathbf{R}-\mathbf{r})V(\mathbf{r})\ d\mathbf{r},$$

with $\boldsymbol{\mu}\equiv(\mathbf{R}-\mathbf{r})/|\mathbf{R}-\mathbf{r}|^2$.

This implies

$$\hat{\mathbf{z}}\times\dot{\mathbf{R}}=\nabla_{\mathbf{R}}V_{\text{eff}},$$

where

$$V_{\text{eff}}(\mathbf{R})=\frac{1}{2\pi}\int\frac{V(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|^2}\ d\mathbf{r}. \tag{23}$$

This reproduces the form of the Magnus equation but with an effective potential V_{eff} which is nonlocal in the substrate potential with a $1/|\mathbf{R}-\mathbf{r}|^2$ kernel.

The most natural interpretation of this result is in terms of images. To indicate the relevance of images, note that if the potential were sufficiently large, then the density of the film would become zero. For simplicity assume that the boundary of the film is a straight line. In that case in a standard manner we may take into account the boundary condition that the velocity field perpendicular to the film boundary is zero on the boundary by adding the velocity field of an image vortex. It is plausible that when the potential causes changes in the density, but is not sufficient to drive the density to zero, there will still be features in the velocity field which may be attributed to images. Their presence ensures that the flow field obeys the continuity equation with terms involving $\nabla\rho$. We will now indicate, in terms of a simple example, how Eq. (23) is interpretable in such a manner.

We now give an example of vortex motion for the case of a repulsive δ function potential centered at $\mathbf{r}_a$:

$$V(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}_a). \tag{24}$$

The Green's function (17) satisfies the identity

$$\nabla\nabla G(\mathbf{r})=-\frac{1}{\pi r^2}\{\cos2\theta(\hat{\mathbf{e}}_x\hat{\mathbf{e}}_x-\hat{\mathbf{e}}_y\hat{\mathbf{e}}_y)+\sin2\theta(\hat{\mathbf{e}}_x\hat{\mathbf{e}}_y+\hat{\mathbf{e}}_y\hat{\mathbf{e}}_x)\}. \tag{25}$$

Using Eqs. (24) and (25), the vortex velocity [from Eq. (22)] is given by

$$\dot{\mathbf{R}}=\frac{\boldsymbol{\theta}_a}{|\mathbf{R}-\mathbf{r}_a|^3}, \tag{26}$$

where $\boldsymbol{\theta}_a$ is perpendicular to $\mathbf{r}_a-\mathbf{R}$.

To interpret this result, let us compare an image approach to the motion of a vortex in the presence of an impenetrable circular region centered at $\mathbf{r}_a$. Then the magnitude of the velocity experienced by the vortex due to the image is of $O(1/d^3)$ where d is the distance between the vortex and its image [this is because the magnitude of the velocity field due to the vortex itself at the circle is $o(1/d)$ and hence the dipole moment required to cancel out the normal component is of this magnitude; hence the velocity field back at the vortex will be $O(1/d\times1/d^2)$]. This has the same functional form as Eq. (26), leading to the interpretation in terms of a dipole image at the δ function potential, but with a strength which is not unity but proportional to the strength of the potential. This has some analogy with the difference between images in electrostatics in the case of metals and dielectrics. The direction of the motion of the vortex is also consistent: In the image picture the vortex would be advected around the circle, in agreement with the angular unit vector in Eq. (26). It is easy to check that a line of δ functions produces motion parallel to the line (of magnitude $1/d$), in analogy with the motion of a vortex in the presence of a wall.

We note parenthetically that the response of the condensate to a δ function perturbation without a vortex is of a much simpler form: a δ function change of the condensate. This is because the response of the condensate to a perturbation relaxes on a scale of the healing length; the fluid description only represents behavior on scales larger than the healing length, and hence the perturbation is of a δ function form within the fluid approximation. However, as we have seen, there is an interesting and nontrivial response on length scales in excess of the healing length in the presence of a vortex.

Having established that the vortex obeys a Magnus equation of motion and hence moves parallel to the equipotentials of the effective potential V_{eff}, we may immediately draw some additional conclusions using the work of Trugman and

Doniach.[14] A consequence of Eq. (23) is that the vortex trajectories must be closed, with V_{eff} instead of V as the underlying (local) potential. In other words, the vortex travels along the equipotential lines of the former. The exception to this is at the saddle points of the effective potential where the vortex may ''percolate,'' allowing in principle an extended orbit across the system. (One can think of ''hills'' and ''valleys'' representing the potential, and the percolation threshold corresponding to ''lakes'' which fill up the valleys, to connect.) However, the speed of the vortex goes to zero (due to the vanishing of the gradient of the potential) at the saddle points, so that the vortex transport is in fact pathological. These points have immediate parallels with the guiding center motion of charged particles in the quantum Hall effect,[15–17] where the existence of percolating paths is related to the mobility edge believed to occur in the middle of a Landau subband. Obviously the application of these ideas to vortices would only be appropriate when the vortex density was extremely low.

V. PINNING ENERGY

So far the discussion has been in terms of the equations of motion. In this section we show that we may calculate the pinning energy and moreover that the force which occurs in the equation of motion of the vortex is the gradient of that potential.

The expectation value of the energy is

$$E=\frac{1}{2}\rho(\nabla S)^2+\frac{1}{2}\rho^2+\sigma V\rho.$$

Now, substituting in the expansions for the density and velocity potential to $O(\sigma)$, there are four terms which depend on the vortex position contributing to this energy [here $\delta\rho_0(\mathbf{r})$ is defined in Eq. (18)]:

$$\Sigma=\int d\mathbf{r}\left[\delta\rho_0 V+\rho_1\delta\rho_0+\rho_1\frac{1}{2}\gamma^2+\boldsymbol{\gamma}\cdot\nabla S_1\right], \tag{27}$$

where we have taken the vortex core to be situated at the origin.

We can interpret the separate terms on the right-hand side (RHS) of this equation as follows. The first term is just the pinning potential energy associated with the decreased density as the core is approached. The second term is the interaction of this decrease in the density with the distortion of the condensate due to the substrate potential. The third term comes from the change in the kinetic energy due to the substrate effect on the density. The final term comes from the change in the kinetic energy due to the distortion of the flow field due to the substrate (i.e., the consequence of the vortex flowing around ''obstacles'' in the substrate). Now the first and second terms cancel as $\rho_1=-V$. Thus we are left with the final two terms

$$\Sigma=\int d\mathbf{r}\left[-V\frac{1}{2}\gamma^2+\boldsymbol{\gamma}\cdot\nabla S_1\right]. \tag{28}$$

The (inverse) power law nature of $\delta\rho_0$ means that the pinning is *nonlocal* as to be expected from the discussion in Sec. IV. We may now rewrite the second term using the two-dimensional divergence theorem

$$\int d\mathbf{r}\;\boldsymbol{\gamma}\cdot\nabla S_1=\int\nabla\cdot(\boldsymbol{\gamma}S_1)-\int d\mathbf{r}\;\nabla\cdot\boldsymbol{\gamma}S_1=\int d\mathcal{S}\cdot\boldsymbol{\gamma}S_1-\int d\mathbf{r}\;\nabla\cdot\boldsymbol{\gamma}S_1=\int d\mathcal{S}\cdot\boldsymbol{\gamma}S_1-0, \tag{29}$$

as $\nabla\cdot\boldsymbol{\gamma}=0$.

The surface integral in Eq. (29) will be zero as the velocity field parallel to the boundary will vanish due to additional images (which we do not discuss explicitly). The final result is thus

$$E=-\int\frac{1}{2}\gamma^2V\,d\mathbf{r},$$

i.e., the interaction of the depletion due to the centrifugal forces interacting with the substrate potential.

It should be stressed that the core region $r<1$ will contribute to the pinning energy a term of comparable magnitude to the long-range part which we are explicitly discussing; however, the latter yields only a short-range correlated potential as against the long-range correlations which the above has. It is readily checked that minus the gradient of this potential is indeed the term on the RHS of the equation of motion of the vortex and hence is consistent with our considerations of the equation of motion.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have only discussed the motion of a single vortex in the presence of a random potential. Let us now briefly discuss the relevance of this to the more realistic case where there are many vortices present in the system. The vortices may be the remanent vorticity which is pinned by the random potential, or it may be that the substrate is rotating and hence there is some vorticity with a net sign.

We concentrate on the case where the disorder is strong compared to the interaction of the vortices at the average separation. Of course in the case of a rotating substrate the logarithmic nature of the interaction between the vortices will ensure that the density does not fluctuate too much; however, the local order in the vortices is determined by the higher Fourier components of the random potential, and the shear modulus of the vortex lattice which will decrease with density.

Because of the long-range interactions, we must consider the possibility of an analog of a Coulomb gap which occurs for electronic excitations in a highly disordered system.[18–20] The ''gap'' refers to the density of states for adding an extra vortex to the system. In the case of vortices with a logarithmic interaction, the density of states, $n(\epsilon)$, as the energy ϵ tends to zero, behaves as $n(\epsilon)\sim\epsilon^3$. However, the situation with excitations where a vortex is moved within the system is more complicated as one is creating a dipole and the density of these excitations does not tend to zero at zero energy. The above results all assume that the particles (vortices in our case) reside on a lattice; however, if the potential is smooth, then as well as excitations where the vortex is moved between local minima in the overall potential (due to the combination of the underlying potential and the effect of the other vortices) there are excitations where it remains in the minimum which it started in. It is those excitations with which we have been concerned in this paper.

These excitations would (at least in principle) contribute to the damping of a torsion balance. This may be seen readily by considering a vortex in an approximately harmonic potential well $[V_{\rm eff}(\mathbf{r}) \simeq (1/2)Kr^2]$ experiencing a superflow with amplitude A and frequency Ω in the direction $\hat{\mathbf{x}}$. The equation of motion for the vortex with position $\mathbf{r}$ and unit circulation is then

$$(\dot{\mathbf{r}} - A\cos\Omega t \hat{\mathbf{x}}) \times \hat{\mathbf{z}} = -K\mathbf{r}.$$

The effect of the superflow is to give a response in terms of the motion of the vortex which is resonant if $\Omega = \omega_0 = K$ (that is, if the period of the superflow is the same as the frequency of the vortex moving around the well). For instance, the x component of the vortex position is

$$x = \alpha\cos\omega_0 t + \beta\sin\omega_0 t + \frac{A\Omega}{\Omega^2 - \omega_0^2}\sin\Omega t.$$

So those vortices which have associated natural frequencies which are the same as that of the torsion balance will be excited significantly. Of course this frequency is rather low on a microscopic scale and so it may be insignificant in practice.

The extent to which the excitations are single vortex in their nature is hard to estimate in general. However, one may start from the assumption of only a single vortex being excited and then examine whether the amplitude spreads to other vortices. The single-vortex excitation has a characteristic frequency involved in precessing around an equipotential of $V_{\rm eff}$. Whether this may then excite neighboring vortices depends on whether the natural frequencies of the neighboring vortices are sufficiently similar for a ''resonant'' process. If the random potential has a sufficiently large variance, as is assumed here, this is very unlikely, and so most excitations will only involve one vortex. Hence the results of this paper will be relevant.

We have not considered here the effect of the vortices having a nonzero inertial mass. The effect of this may be deduced by analogy with the case of a charged particle in a magnetic field—which is very similar upon the addition of an inertial mass term. As well as the ''guiding center'' motion of the vortex along the equipotentials, there is now some (fast) ''cyclotron motion'' (a discussion of this effect in classical hydrodynamics is given by Lamb[21]). However, the value of the inertial mass (and hence the magnitude of the cyclotron motion) has been a controversial issue. The suggestion of Baym and Chandler[22] was that it was associated with the virtual mass due to the backflow around a cylinder of radius approximately the coherence length. However, more recently Duan[23] has argued that the mass is larger by a factor of 20–30 due to the finite compressibility of helium. In terms of the analog of the ''cyclotron radius,'' $\ell_{\rm m} = (\rho/\mu)^{-1/2}$, where μ is the mass of the vortex, will only change from being roughly an interatomic distance to being 4–5 times that size.

In this paper we have nothing to say about corrections to the mean field picture. This has been addressed in part by Niu *et al.*,[24] where the magnitude of the vortex effective mass has been considered from a quantum point of view. The relationship between these calculations and the effects due to backflow is not very clear and we will make no other comment on these matters.

In summary we have derived the effective potential that a vortex experiences due to a potential coupled to the underlying bosons. The former is much smoother than the latter, with long-range tails. We showed that the tails may be interpreted as due to ''images'' caused by variations in the condensate density. Moreover, the effective potential was equal to the pinning energy in the expectation value of the Hamiltonian with a Hartree wave function representing the vortex.

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