Geometric equivalence of an integrable discrete Heisenberg spin chain

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We formulate and propose a procedure for mapping the dynamics of discrete (lattice) classical Heisenberg ferromagnetic-spin-chain models onto the differential-difference nonlinear Schrödinger family of equations which help in isolating integrable discrete ferromagnetic spin models. [S0163-1829(98)07801-1]

Since the discovery of solitons by Zabusky and Kruskal,[1] many nonlinear differential equations have been found to admit a soliton solution and a large number of physically interesting integrable models have been identified.[2] Among the different physical models, the one-dimensional classical continuum Heisenberg ferromagnetic spin chain with different magnetic interactions has been identified as one of the interesting class of nonlinear dynamical systems in condensed matter exhibiting soliton spin excitations. The spin dynamics of the above system is governed by the Landau-Lifshitz (LL) equation[3] and found to be integrable in many cases. As the LL equation is a highly nontrivial vector nonlinear partial differential equation, on many occasions, it has been treated, after rewriting it in an equivalent representation. Notable among them are the method of gauge equivalence put forward by Zakharov and Takhtajan[4] and the geometric equivalence (space curve formalism) proposed by Lakshmanan.[5,6] In both the cases the LL equations representing the spin dynamics can be mapped onto the family of the completely integrable nonlinear Schrödinger (NLS) equation and its generalizations.[4,7] For example, the simplest LL equation $\partial \mathbf{S}/\partial t=\mathbf{S}\wedge\partial^2\mathbf{S}/\partial x^2$ representing the dynamics of the one-dimensional classical continuum isotropic ferromagnetic spin chain described by the Heisenberg Hamiltonian $H=-\Sigma_n\mathbf{S}_n\cdot\mathbf{S}_{n+1}$ is connected to the completely integrable cubic NLS equation $i(\partial q/\partial t)+(\partial^2 q/\partial x^2)+2|q|^2q=0$ via the geometric and gauge equivalence methods.[4,6] These methods have been used extensively to identify several integrable classical continuum ferromagnetic spin models (see for example Ref. 7, and references therein). Simultaneously, over the years, there was a development entirely on a different perspective proving that the motion of a space curve can pick up integrable dynamics and identify themselves with integrable and soliton possessing nonlinear partial differential equations. (For details see Refs. 8,9, and references therein).

The LL equation representing the nonlinear dynamics of various classical continuum ferromagnetic spin systems have been obtained as the long wavelength and low-temperature limit of the lattice spin models. However, in nature, real ferromagnetic crystals are characterized by lattice spin models. Therefore it is more realistic to understand the spin dynamics of ferromagnetic systems at the discrete (lattice) level rather than in the continuum limit. Advantageously, a few years after the discovery of the soliton, discretization of the integrable nonlinear differential equations started and Ablowitz and Ladik[10] constructed an integrable discrete analog of the Ablowitz-Kaup-Newell-Segur hierarchy.[11] Though a number of integrable continuum ferromagnetic-spin models have been isolated, not much is known about the integrability of discrete ferromagnetic spin systems. In this direction, an integrable ferromagnetic-spin model was proposed by Ishimori[12] through the method of gauge equivalence and expressed by the classical discrete equation of motion

$$\frac{d\mathbf{S}_n(t)}{dt}=2\mathbf{S}_n\wedge\left[\frac{\mathbf{S}_{n+1}}{1+\mathbf{S}_n\cdot\mathbf{S}_{n+1}}+\frac{\mathbf{S}_{n-1}}{1+\mathbf{S}_n\cdot\mathbf{S}_{n-1}}\right], \tag{1}$$

in which the spins $\mathbf{S}_n$ are treated as classical three-dimensional vectors, $\mathbf{S}_n=(S_n^x,S_n^y,S_n^z)$. The structure of Eq. (1) demands that the length of the spin vector does not change with time, i.e., $\mathbf{S}_n^2=\text{const.}$ Ishimori[12] obtained Eq. (1) from the integrable differential-difference NLS equation

$$i\frac{dq_n}{dt}=(1+|q_n|^2)[q_{n+1}+q_{n-1}]-2q_n, \tag{2}$$

which can be expressed as the compatibility condition $dL_n/dt=M_{n+1}L_n-L_nM_n$ for the linear eigenvalue problem $\varphi_{n+1}=L_n\varphi_n$, $d\varphi_n/dt=M_n\varphi_n$, where the Lax pair (L_n,M_n) is given by

$$L_n=\begin{pmatrix}\lambda & \lambda^{-1}q_n\\ -\lambda q_n^* & \lambda^{-1}\end{pmatrix}, \tag{3a}$$

$$M_n=i\begin{pmatrix}1-\lambda^2-q_nq_{n-1}^* & -q_n+\lambda^{-2}q_{n-1}\\ -q_n^*+\lambda^2q_{n-1}^* & -1+\lambda^{-2}+q_n^*q_{n-1}\end{pmatrix}, \tag{3b}$$

and using the gauge transformation

$$\hat{\varphi}_n=g_n^{-1}\varphi_n, \tag{4a}$$

$$\hat{L}_n=g_{n+1}^{-1}L_ng_n, \tag{4b}$$

$$\hat{M}_n=g_n^{-1}M_ng_n-g_n^{-1}\frac{dg_n}{dt}. \tag{4c}$$

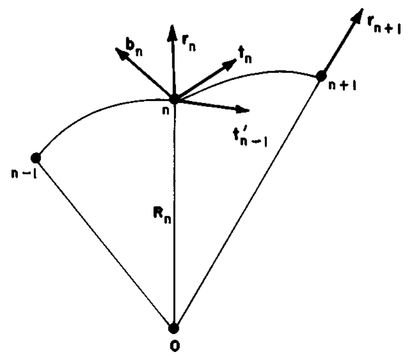

FIG. 1. Motion of a discrete space curve.

In Eq. (3b), λ is the eigenvalue parameter and in Eq. (4) g_n is an arbitrary matrix and φ_n is the eigenfunction. The classical differential-difference spin equation (1) can be generated from the generalized Hamiltonian

$$H=-2\sum_n \ln(1+\mathbf{S}_n\cdot\mathbf{S}_{n+1}). \tag{5}$$

It may be noted that the Heisenberg Hamiltonian given earlier can be obtained from Hamiltonian (5) when the angle between the nearest-neighboring spins are maintained small. Though several continuum Heisenberg spin-chain models have been mapped onto the NLS family of equations through the method of space curve formalism,[7] mapping of discrete spin-chain models to the discretized (differential-difference) NLS family of equations through this mechanism could not be achieved thus far. This is due to the fact that the theory of discrete moving space curves has not been well understood. However, recently, Doliwa and Santini[13] from the elementary geometric properties of discrete moving space curves, showed that the motion of it selects integrable dynamics of the Ablowitz-Ladik hierarchy of evolution equations. This has motivated us in this paper to develop a discretized mechanism of the space curve mapping procedure for the discrete spin chains and to see whether the lattice spin equation (1) can be mapped onto the differential-difference NLS equation (2) which are known to be connected by the gauge transformation.

We consider the dynamics of spins in a classical one-dimensional ferromagnetic lattice with N spins represented by the Hamiltonian (5) and governed by the classical lattice equation of motion (1). We now ask the question whether using a discretized mechanism of the theory of moving space curves, could the discrete spin equation (1) be mapped onto the differential-difference NLS equation (2). To find an answer for this question, we proceed as follows. We consider a discrete curve represented by a sequence of points and marked by the position vector $\mathbf{R}_n$. We then define a set of basis vectors $(\mathbf{r}_n,\mathbf{t}_n,\mathbf{b}_n)$ at the point n, on the discrete curve[13] as shown in Fig. 1. Here $\mathbf{r}_n$ is a unit vector in the direction of the position vector $\mathbf{R}_n$ at the nth point of the sequence, pointing from the center O of the sphere. $\mathbf{t}_n$ is the unit tangent vector at this point and $\mathbf{b}_n$ is the unit normal vector defined by $\mathbf{b}_n=\mathbf{r}_n\wedge\mathbf{t}_n$. The transition from the basis $(\mathbf{r}_n,\mathbf{t}_n,\mathbf{b}_n)$ to the basis $(\mathbf{r}_{n+1},\mathbf{t}_{n+1},\mathbf{b}_{n+1})$ at the next point $(n+1)$ can be obtained by the superposition of the following two transitions given by

$$(\mathbf{r}_n,\mathbf{t}_n,\mathbf{b}_n)\xrightarrow{[A]}(\mathbf{r}_{n+1},\mathbf{t}'_n,\mathbf{b}_n)\xrightarrow{[B]}(\mathbf{r}_{n+1},\mathbf{t}_{n+1},\mathbf{b}_{n+1}),$$

where the matrices $[A]$ and $[B]$, respectively, take the form

$$[A]=\begin{pmatrix}\cos\theta_n & \sin\theta_n & 0\\ -\sin\theta_n & \cos\theta_n & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{6a}$$

$$[B]=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi_n & \sin\phi_n\\ 0 & -\sin\phi_n & \cos\phi_n\end{pmatrix}. \tag{6b}$$

Here θ_n is the angle between $\mathbf{r}_n$ and $\mathbf{r}_{n+1}$ and ϕ_n is the angle between $\mathbf{t}'_n$ and $\mathbf{t}_{n+1}$ as illustrated in Fig. 1.

Now, the full rotation giving the basis $(\mathbf{r}_{n+1},\mathbf{t}_{n+1},\mathbf{b}_{n+1})$ at the next point $(n+1)$ can be written using the matrix $[C]=[B][A]$ as

$$\begin{pmatrix}\mathbf{r}_{n+1}\\ \mathbf{t}_{n+1}\\ \mathbf{b}_{n+1}\end{pmatrix}=[C]\begin{pmatrix}\mathbf{r}_n\\ \mathbf{t}_n\\ \mathbf{b}_n\end{pmatrix}, \tag{7a}$$

where

$$[C]=\begin{pmatrix}\cos\theta_n & \sin\theta_n & 0\\ -\cos\phi_n\sin\theta_n & \cos\phi_n\cos\theta_n & \sin\phi_n\\ \sin\phi_n\sin\theta_n & -\sin\phi_n\cos\theta_n & \cos\phi_n\end{pmatrix}. \tag{7b}$$

In a similar way the basis $(\mathbf{r}_{n-1},\mathbf{t}_{n-1},\mathbf{b}_{n-1})$ can be obtained from $(\mathbf{r}_n,\mathbf{t}_n,\mathbf{b}_n)$ using the matrix $[C]^{-1}$ given by

$$\begin{pmatrix}\cos\theta_{n-1} & -\cos\phi_{n-1}\sin\theta_{n-1} & \sin\phi_{n-1}\sin\theta_{n-1}\\ \sin\theta_{n-1} & \cos\phi_{n-1}\cos\theta_{n-1} & -\sin\phi_{n-1}\cos\theta_{n-1}\\ 0 & \sin\phi_{n-1} & \cos\phi_{n-1}\end{pmatrix}. \tag{8a}$$

$$\begin{pmatrix}\mathbf{r}_{n-1}\\ \mathbf{t}_{n-1}\\ \mathbf{b}_{n-1}\end{pmatrix}=[C]^{-1}\begin{pmatrix}\mathbf{r}_n\\ \mathbf{t}_n\\ \mathbf{b}_n\end{pmatrix}. \tag{8b}$$

As the motion of the curve takes place on the surface of the sphere, the velocity field must always be tangent to the surface so that we can write the velocity field as

$$\frac{d\mathbf{r}_n}{dt}=V_n\mathbf{t}_n+U_n\mathbf{b}_n, \tag{9}$$

where V_n and U_n are the velocity field components parallel to $\mathbf{t}_n$ and $\mathbf{b}_n$, respectively. The evolution of the orthonormal frame $(\mathbf{r}_n,\mathbf{t}_n,\mathbf{b}_n)$ can then be described by the antisymmetric matrix equation

$$\frac{d}{dt}\begin{pmatrix}\mathbf{r}_n\\ \mathbf{t}_n\\ \mathbf{b}_n\end{pmatrix}=\begin{pmatrix}0 & V_n & U_n\\ -V_n & 0 & W_n\\ -U_n & -W_n & 0\end{pmatrix}\begin{pmatrix}\mathbf{r}_n\\ \mathbf{t}_n\\ \mathbf{b}_n\end{pmatrix}. \tag{10}$$

Here W_n is an unknown function to be determined. Now, defining a shift operator E along the curve giving $Eg_n=g_{n+1}$, the compatibility condition

$$E\left(\frac{d}{dt}\right)=\left(\frac{d}{dt}\right)E, \tag{11}$$

when implemented between Eqs. (7) and (10) or between Eqs. (8) and (10) gives the evolution equations for θ_n and ϕ_n

$$\frac{d\theta_n}{dt}=\cos\phi_n V_{n+1}-\sin\phi_n U_{n+1}-V_n, \tag{12a}$$

$$\frac{d\phi_n}{dt}=W_{n+1}-\cos\theta_n W_n+\sin\theta_n U_n, \tag{12b}$$

and specifies W_n in terms of the velocity as given below:

$$W_n=\frac{1}{\sin\theta_n}[\cos\phi_n U_{n+1}+\sin\phi_n V_{n+1}-\cos\theta_n U_n]. \tag{13}$$

We now map the one-dimensional classical discrete ferromagnetic spin chain described by the Hamiltonian (5) and the discrete equation of motion (1) onto the discrete curve by identifying the classical three-component spin vector $\mathbf{S}_n(t)$ with the unit vector $\mathbf{r}_n(t)$ of the discrete curve. In view of this identification, Eq. (1) can be rewritten as

$$\frac{d\mathbf{r}_n}{dt}=2\mathbf{r}_n\wedge\left[\frac{\mathbf{r}_{n+1}}{1+\mathbf{r}_n\cdot\mathbf{r}_{n+1}}+\frac{\mathbf{r}_{n-1}}{1+\mathbf{r}_n\cdot\mathbf{r}_{n-1}}\right]. \tag{14}$$

Using Eqs. (7) and (8) for $\mathbf{r}_{n+1}$ and $\mathbf{r}_{n-1}$, respectively, Eq. (14) can be rewritten as

$$\begin{aligned}\frac{d\mathbf{r}_n}{dt}=&-2\left(\tan\frac{\theta_{n-1}}{2}\sin\phi_{n-1}\right)\mathbf{t}_n\\&-2\left(\tan\frac{\theta_{n-1}}{2}\cos\phi_{n-1}-\tan\frac{\theta_n}{2}\right)\mathbf{b}_n.\end{aligned} \tag{15}$$

On comparing Eq. (15) with Eq. (9), we obtain

$$V_n=-2\tan\frac{\theta_{n-1}}{2}\sin\phi_{n-1}, \tag{16a}$$

$$U_n=2\left(\tan\frac{\theta_n}{2}-\tan\frac{\theta_{n-1}}{2}\cos\phi_{n-1}\right). \tag{16b}$$

Substituting Eqs. (16) in Eq. (12), we obtain

$$\frac{d\theta_n}{dt}=2\left(\tan\frac{\theta_{n-1}}{2}\sin\phi_{n-1}-\tan\frac{\theta_{n+1}}{2}\sin\phi_n\right), \tag{17}$$

$$\begin{aligned}\frac{d\phi_n}{dt}=&\frac{2}{\sin\theta_{n+1}}\left(\tan\frac{\theta_{n+2}}{2}\cos\phi_{n+1}+\tan\frac{\theta_n}{2}\cos\phi_n\right)\\&-\frac{2}{\sin\theta_n}\left(\tan\frac{\theta_{n+1}}{2}\cos\phi_n+\tan\frac{\theta_{n-1}}{2}\cos\phi_{n-1}\right).\end{aligned} \tag{18}$$

Thus the equation of motion (1) representing the dynamics of spins in a classical one-dimensional discrete ferromagnetic chain has been equivalently expressed in terms of a set of coupled equations (17) and (18) representing the evolution of the two angles θ_n and ϕ_n described in the motion of a discrete space curve on the surface of a sphere.

Now, in order to see whether the set of coupled differential-difference equations (17) and (18) are integrable, so that ultimately the spin excitations can be expressed in terms of solitons, we try to identify them with known integrable differential-difference equations. For this we define a new angle Ψ_n, and express ϕ_n as the difference between two successive values of Ψ_n, i.e.,

$$\phi_n=\Psi_{n-1}-\Psi_n. \tag{19}$$

In view of this, Eqs. (17) and (18) can be rewritten as

$$\begin{aligned}\frac{d\theta_n}{dt}=&-2\tan\frac{\theta_{n+1}}{2}\sin(\Psi_{n-1}-\Psi_n)\\&+2\tan\frac{\theta_{n-1}}{2}\sin(\Psi_{n-2}-\Psi_{n-1}),\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{d\Psi_n}{dt}=&2\left(1-\frac{1}{\sin\theta_{n+1}}\tan\frac{\theta_{n+2}}{2}\cos(\Psi_n-\Psi_{n+1})\right)\\&-\left(\frac{2}{\sin\theta_{n+1}}\tan\frac{\theta_n}{2}\cos(\Psi_{n-1}-\Psi_n)\right).\end{aligned} \tag{21}$$

Upon making the transformation

$$q_n=\tan\frac{\theta_n}{2}\exp(i\Psi_{n-1}), \tag{22}$$

Eqs. (20) and (21) are found to be equivalent to the following equation:

$$i\frac{dq_n}{dt}=(1+|q_n|^2)[q_{n+1}+q_{n-1}]-2q_n. \tag{23}$$

Equation (23) is the same integrable differential-difference NLS equation given as Eq. (2) which is also found to be gauge equivalent to the discrete spin equation (1). Equations (2) or (23) have been solved by the inverse scattering transform method and N-soliton solutions have been obtained.[12]

In order to verify the success and generality of the method we employ the above mapping procedure to the only other integrable discrete spin model available and which has been recently proved to admit N solitons and is gauge equivalent to the discrete Hirota equation.[14] We find that the integrable discrete spin model governed by the equation of motion

$$\begin{aligned}\frac{d\mathbf{S}_n}{dt}=&2J\mathbf{S}_n\wedge\left[\frac{\mathbf{S}_{n+1}}{1+\mathbf{S}_n\cdot\mathbf{S}_{n+1}}+\frac{\mathbf{S}_{n-1}}{1+\mathbf{S}_n\cdot\mathbf{S}_{n-1}}\right]\\&+2K\left[\frac{(\mathbf{S}_n\cdot\mathbf{S}_{n+1})\mathbf{S}_n-\mathbf{S}_{n+1}}{(1+\mathbf{S}_n\cdot\mathbf{S}_{n+1})}-\frac{(\mathbf{S}_n\cdot\mathbf{S}_{n-1})\mathbf{S}_n-\mathbf{S}_{n-1}}{(1+\mathbf{S}_n\cdot\mathbf{S}_{n-1})}\right],\end{aligned} \tag{24}$$

is geometrically equivalent to the discrete Hirota equation

$$\begin{aligned}i\frac{dq_n}{dt}=&J\{(1+|q_n|^2)[q_{n+1}+q_{n-1}]-2q_n\}\\&-iK\{(1+|q_n|^2)[q_{n+1}-q_{n-1}]\},\end{aligned} \tag{25}$$

upon using our discrete space curve mapping procedure developed here for which the velocity field components V_n and U_n are of the form

$$V_n = -2J\ \tan\frac{\theta_{n-1}}{2}\sin\phi_{n-1} - 2K\left(\tan\frac{\theta_n}{2} + \tan\frac{\theta_{n-1}}{2}\cos\phi_{n-1}\right) \tag{26a}$$

$$U_n = 2J\left(\tan\frac{\theta_n}{2} - \tan\frac{\theta_{n-1}}{2}\cos\phi_{n-1}\right) + 2K\tan\frac{\theta_{n-1}}{2}\sin\phi_{n-1}. \tag{26b}$$

Here J and K are constant parameters.

We conclude that the integrable discrete (lattice) spin equation (1) representing the dynamics of one-dimensional classical discrete ferromagnetic spin chain characterized by the Hamiltonian (5) and the higher-order integrable discrete spin model governed by the equation of motion (24) are geometrically equivalent to the integrable differential-difference NLS family of equations (2) and (25) namely the discretized cubic NLS and the discrete Hirota equations, respectively. The method of discrete space curve mapping proposed in this paper for classical discrete spin chain models help identifying equivalent integrable discrete soliton models in the NLS family. The procedure will immediately draw the attention for isolating integrable lattice spin models, if any, corresponding to the known integrable classical continuum ferromagnetic spin systems.

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