Superfluid mass current induced by chaotic vortex lines in turbulent He II

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Using the Gaussian model of a chaotic vortex tangle in counterflowing superfluid turbulent He II, we calculate superfluid mass current induced by vortex filaments. This additional superfluid current directed against the external superfluid flow is shown to appear due to the nonzero average polarization of the vortex loops. In macroscopic descriptions, in particular in hydrodynamic experiments, the induced superfluid current displays itself as a suppression of the superfluid density. The temperature dependent relative change of the superfluid density $-\Delta\rho_s/\rho_s$ turns out to be several percent. Some experimental consequences such as an additional pressure drop and a decrease of the velocity of the second sound propagating in the superfluid turbulent He II are discussed. [S0163-1829(98)04809-7]

I. INTRODUCTION AND SCIENTIFIC BACKGROUND

The idea that chaotic vortex loops comprising the vortex tangle in superfluid turbulent He II create a nonzero average superfluid velocity has been suggested by many scientists (see, e.g., Ref. 1). Indeed, since the vortex loops have nonzero mean polarization (see Fig. 5 in Ref. 2), it seems to be natural that they induce a nonzero mean superfluid velocity as demonstrated in Fig. 1. Furthermore, since the mean polarization is directed along the external relative velocity $\mathbf{V}_{ns}$ (created, e.g., by a heater), it should be expected that the induced superfluid current is also orientated along $\mathbf{V}_{ns}$. Thus it will partly cancel the superfluid current $\rho_s\mathbf{V}_s$ due to the external source. On the macroscopic level, in particular in hydrodynamic experiments, the partial cancellation of the external superfluid current can be described as a suppression of the ''bare'' superfluid density. Apart from some quantitative corrections, the study of this effect is essentially important for a further understanding of the nature and structure of the vortex tangle as well as the laws governing stochastic dynamics of vortex filaments. Another motivation of the interest in the discussed effect is connected with the famous Kosterlitz-Thouless theory.[3] Indeed, as is well known, in the two-dimensional (2D) case the spontaneously created vortex pairs in helium film induce a superfluid mass current, comparable with the one due to other excitations. This effect renormalizes the ''bare'' superfluid density and leads to a specific phase transition. There exists a number of works asserting that the bulk λ transition has the same nature (see, e.g., Ref. 4). Therefore the study of $\mathbf{J}_V$ created by vortices in 3D superfluid turbulent helium seems to also be attractive from this point of view. Furthermore, a consideration of superfluid mass current induced by chaotic vortex filaments is of crucial importance for hydrodynamics of superfluid turbulence.[5] It was convincingly demonstrated in a series of works made by Geurst (see Refs. 6,7), where he constructed the hydrodynamics of superfluid turbulence in which the impulse of the vortex tangle was introduced as a new independent variable. It was shown that this modified hydrodynamics includes a number of effects, for instance it explained such classical problems of the theory of superfluid turbulence as a problem of the slow decay of the free vortex tangle.

To our knowledge the only attempt to calculate the superfluid current induced by the vortex tangle was undertaken by Geurst, in the papers cited above,[6,7] in a phenomenological way. Using the variational principle as well as the Onsager reciprocity relation and also some dimensional speculations, Geurst derived a closed set of governing equations (including an equation for the impulse of the vortex tangle) which included some empirical constants. These constants were specified from the comparison of the expressions for some quantities (e.g., mutual friction force or the parameters of the Vinen equation) with the ones obtained by Schwarz in a numerical simulation of the vortex tangle dynamics.[8] In this way Geurst concluded that the impulse of the vortex tangle decreases as the polarization of the vortex tangle increases which seems to be strange.

Thus the problem of the proper calculation of superfluid current induced by vortex filaments remains open in spite of its obvious importance. Of course the main obstacle was the lack of any advanced stochastic theory of the vortex tangle which would allow one to evaluate various averages over vortex line configurations. The approach developed in the previous paper[2] provides the means to realize this task. In Sec. II of this paper we calculate the superfluid current density with the help of the characteristic functional introduced in Ref. 2. As it was predicted from a qualitative consider-

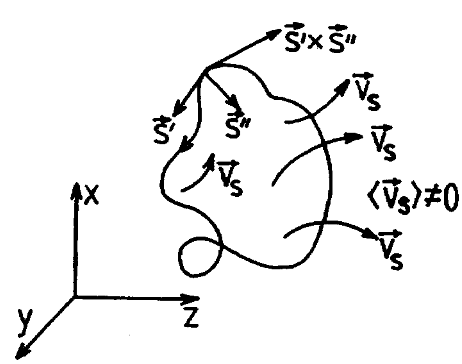

FIG. 1. The oriented vortex loop induces average superfluid velocity in the direction of the mean polarization.

ation the vortex tangle creates superfluid flow directed against external superfluid flow thereby decreasing the latter. The corresponding temperature dependent relative change of the superfluid density $-\Delta\rho_s/\rho_s$ turns out to be of several percent. We discuss both longitudinal and transverse cases. Section III is devoted to discussion of some experimental consequences such as a jump of the Bernoulli pressure and change of the second sound velocity due to suppression of superfluid density.

II. SUPERFLUID CURRENT CREATED BY THE VORTEX TANGLE

A. Longitudinal case

Given some particular configuration of the quantized vortex filaments $\{\mathbf{s}_j(\xi_j)\}=\cup_j \mathbf{s}_j(\xi_j)$, the superfluid velocity $\mathbf{v}_s^V(\mathbf{r})$ created by them is obtained from the Biot-Savart law (the notation throughout this paper corresponds to that of our previous paper[2])

$$\mathbf{v}_s^V(\mathbf{r})=\frac{\tilde{\kappa}}{4\pi}\sum_j \int \frac{\mathbf{s}_j'(\xi_j)\times[\mathbf{r}-\mathbf{s}_j(\xi_j)]}{|\mathbf{r}-\mathbf{s}_j(\xi_j)|^3}d\xi_j . \quad (1)$$

Accordingly a full momentum $\mathbf{P}_V$ of the additional superfluid motion connected to the presence of the vortex lines is

$$\mathbf{P}_V=\rho_s\int \mathbf{v}_s^V(\mathbf{r})d^3\mathbf{r}. \quad (2)$$

The direct use of Eqs. (1),(2) encounters a problem typical of vortex flows. The integral in Eq. (2) diverges both for small and for large $|\mathbf{r}-\mathbf{s}_j(\xi_j)|$ (see, e.g., Refs. 9,10). Therefore the question of the average velocity or of the full momentum generated by vortices cannot be resolved in straightforward way. On the other hand, it is known that in many respects the so-called Lamb impulse plays the role of momentum.[11] In general the Lamb impulse density $\mathbf{J}_V$ is defined by the relation

$$\mathbf{J}_V=\frac{\rho_s}{2\mathcal{V}}\int \mathbf{r}\times\omega(\mathbf{r})d^3\mathbf{r}, \quad (3)$$

where $\omega(\mathbf{r})$ is the distribution of the vorticity. The singular distribution of vorticity, viz., the vortex filament relation (3), can be rewritten as

$$\mathbf{J}_V=\frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \int \mathbf{s}_j(\xi_j)\times\mathbf{s}_j'(\xi_j)d\xi_j . \quad (4)$$

Thus the Lamb impulse depends on a particular configuration of the vorticity. In the case of superfluid turbulence we have to calculate the quantity $\mathbf{J}_V$ [Eq. (4)] averaged over the vortex loop configuration. We will accomplish the according averaging using the Gaussian model of the vortex tangle developed in the previous paper.[2] Let us start with the according calculation.

For convenience we perform the one-dimensional Fourier transform of the function $\mathbf{s}_j(\xi)$:

$$\mathbf{s}_j(\xi_j)=\sum_\kappa \mathbf{s}_j(\kappa)e^{i\kappa\xi_j}, \quad \kappa=2\pi n/L_j . \quad (5)$$

In κ space the average α component of the Lamb impulse (4) can be written as

$$\mathbf{J}_V^\alpha=\left\langle \frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \sum_{\kappa\neq 0} \epsilon^{\alpha\beta\gamma}L_j\left(\frac{1}{-i\kappa}\right)\mathbf{s}_{j\beta}(\kappa)\mathbf{s}_{j\gamma}(-\kappa)\right\rangle , \quad (6)$$

where the quantity $\epsilon^{\alpha\beta\gamma}$ is the unit antisymmetric tensor. Harmonic $\kappa=0$ should be excluded in the summation because of the closure of the vortex loops (see Ref. 2 for explanations).

The right-hand side of Eq. (6) belongs to a class of quantities which can be evaluated by the use of the characteristic functional introduced in Ref. 2. In terms of the characteristic functional the average Lamb impulse $\mathbf{J}_V^\alpha$ [Eq. (6)] is expressed as

$$\mathbf{J}_V^\alpha=\frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \sum_{\kappa\neq 0} \epsilon^{\alpha\beta\gamma}L_j\left(\frac{1}{-i\kappa}\right) \times \left.\frac{\delta^2 W}{iL_j\delta\mathbf{P}_j^\alpha(\kappa)iL_j\delta\mathbf{P}_j^\beta(-\kappa)}\right|_{\text{all }\mathbf{P}_j=0} . \quad (7)$$

Using the quantity $W\{\mathbf{P}_j^\alpha(\kappa)\}$ constructed in the previous paper we obtain that only the z component of the vector $J_{\mathbf{V}}^\alpha$ differs from zero and that it is equal to

$$\mathbf{J}_V^Z=\frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \sum_{n\neq 0} 2N^{xy}L_j\exp\left[-\left(\frac{2\pi\xi_0}{L_j}n\right)^2\right]. \quad (8)$$

Substituting the quantities N^{xy} and ξ_0 from Ref. 2 we finally have

$$\mathbf{J}_V^z=\frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \sum_{n\neq 0} 2\sqrt{\frac{\pi}{2}}\frac{I_l}{c_2^3\mathcal{L}_V}\exp\left[-\left(\frac{2\pi\xi_0}{L_j}n\right)^2\right] = \frac{\rho_s\tilde{\kappa}I_l\mathcal{L}_V^{1/2}}{2c_2^2}. \quad (9)$$

Evaluating Eq. (9) we changed $\Sigma_{n\neq 0}\to\int dn$ because $\xi_0 \ll L_j$, and used the fact that the total length of vortex lines per unit of volume $\Sigma_j(L_j/\mathcal{V})$ is nothing but the vortex line density $\mathcal{L}_V$.

In accordance with Schwarz's calculations[8] the ''equilibrium'' vortex line density $\mathcal{L}_V$ satisfies $\mathcal{L}_V^{1/2}=(I_l/\beta c_2^2)|\mathbf{V}_{ns}|$, where $\beta=(\tilde{\kappa}/4\pi)\ln(1/c_2\mathcal{L}_V^{1/2}a_0)$, ($a_0$ is the core radius). The weakly dependent on $\mathcal{L}_V$ combination $(1/4\pi)\ln(1/\mathcal{L}_V^{1/2}a_0)$ is close to unity for usual values of the vortex line density realized in the experiment. Using it as well as the relation $\mathbf{V}_s=-(\rho/\rho_n)\mathbf{V}_{ns}$ held in counterflow, formula (9) can be rewritten in terms of external superfluid velocity $\mathbf{V}_s$ as

$$\mathbf{J}_V^z=-\left[\frac{\rho I_l^2}{2\rho_n c_2^4}\right]\rho_s\mathbf{V}_s . \quad (10)$$

Relation (10) shows that the vortex tangle induces the superfluid current directed against the external superfluid current. Formally the connection between the nonzero polarization of the vortex tangle and the superfluid mass current induced by chaotic vortex lines is reflected in formula (9) by the fact that

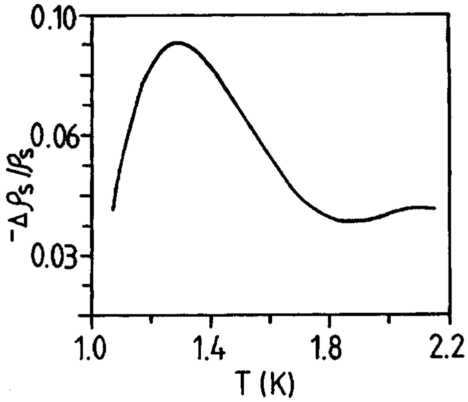

FIG. 2. The predicted behavior of the suppression of the superfluid density $\Delta\rho_s/\rho_s$ as a function of temperature.

$\mathbf{J}_V^z$ increases as the quantity I_l increases. For this reason the result that $\mathbf{J}_V^z \propto I_l^{-1}$ obtained by Geurst[6,7] and discussed in the Introduction seems to be unclear. The power 2 with which the polarization I_l enters Eq. (10) is the result of the fact that the quantity $\mathcal{L}_V^{1/2}$ is also proportional to quantity I_l.

From a macroscopic point of view, in particular in hydrodynamic experiments, the additional superfluid mass current which cancels a part of the applied external velocity should display itself as a suppression of the superfluid density. It can be said that this effect is the 3D analog of the Kosterlitz-Thouless effect, except the distribution of vortices has not been derived theoretically, but in fact has been taken from experimental data (see for comments the previous paper[2]). Suppression of the superfluid density $\Delta\rho_s$ being defined as a response of the system to the applied infinitesimal superfluid velocity $\delta\mathbf{V}_s$,

$$\Delta\rho_s = \frac{\delta\mathbf{J}_V}{\delta(\delta\mathbf{V}_s)}, \tag{11}$$

has a tensor nature. Applying Eq. (11) to relation (10) one concludes that the longitudinal relative suppression $\Delta\rho_s/\rho_s$ is

$$\frac{\Delta\rho_s}{\rho_s} = -\left[\frac{\rho I_l^2}{2\rho_n c_2^4}\right]. \tag{12}$$

Thus we expressed the suppression of the superfluid component $\Delta\rho_s/\rho_s$ via structure parameters of the vortex tangle. Using the known values for the structure parameters (see Ref. 8) one can evaluate the suppression $\Delta\rho_s/\rho_s$ as a function of temperature. The result depicted in Fig. 2 shows that the suppression of the superfluid component $\Delta\rho_s/\rho_s$ as a function of the temperature is of the order of several percent. We think it is a pretty large effect deserving experimental study.

As was mentioned in the Introduction there are a number of works where authors explain the bulk λ transition in He II by the spontaneous appearance of oriented vortex loops which create a mean superfluid flow cancelling the applied superfluid flow (see, e.g., Ref. 4). According to Ref. 4 this cancellation can result in the vanishing of the superfluid component, i.e., to be the reason of the λ transition. In this connection it seems interesting to study the behavior of the quantity $\Delta\rho_s/\rho_s$ near T_λ obtained in our work. Strictly speaking our approach is not applied directly for temperatures near T_λ. Indeed, due to the divergence of coupling constants α and α' entering the equation of motion of the vortex line and due to the fact that they became the strong functions of both the relative velocity and frequency (see, e.g., Ref. 1), the dynamics of vortex lines is rather different from the one in the low-temperature region. Therefore both the experimental results on the superfluid turbulence and numerical simulations of the vortex tangle dynamics performed in the low-temperature region which were the bases for our formalism are hardly valid for the T_λ vicinity. However, the extremely simple form of relation (12) as well as its clear physical meaning give some hope to at least follow the tendency of the $\Delta\rho_s/\rho_s$ behavior while approaching T_λ.

Let us note first of all that the expression (I_l^2/c_2^4) entering relations (10),(12) can be extracted from the experimental data on heat transfer in He II near the λ point. Indeed, while applying a supercritical heat flux q, there appears a gradient of the temperature ∇T proportional to the mutual force $\mathbf{F}_{ns}$ between normal and superfluid components. In turn the mutual force $\mathbf{F}_{ns}$ is expressed via the Gorter-Mellink constant $A(T)$. Thus the former can be obtained analyzing experiments on heat transfer. On the other hand, from a microscopic point of view the quantity $\mathbf{F}_{ns}$ can be evaluated via the structure parameters (see Ref. 8) and is proportional to I_l^2/c_2^4. Elaborating this scheme one concludes that the singular part of the suppression of the superfluid density is

$$\frac{\Delta\rho_s}{\rho_s} = -\frac{A(T)\rho^2\tilde{\kappa}}{\rho_n B(T)}. \tag{13}$$

Here $B(T)$ is a singular part of the Hall-Vinen coefficient which behaves as $(T_\lambda - T)^{-0.33}$ in the vicinity of the λ point (see, e.g., Ref. 1). The situation with the behavior of the Gorter-Mellink constant $A(T)$ is much more involved. The gradient of the temperature ∇T is not proportional to V_{ns}^3, as expected from the relation $\mathbf{F}_{ns} \propto \mathcal{L}_V \mathbf{V}_{ns}$, but as was shown by Leiderer and Pobell[12] and by Ahlers,[13] it has a more complicated dependence $\nabla T\ (V_{ns})$ fitted by $\nabla T \propto V_{ns}^m$. To cure this problem and to retain the Vinen theory Leiderer and Pobell proposed that one should to take into account the influence of critical velocity $V_{ns,\mathrm{cr}}$. Fitting their data they concluded that $A(T) \propto (T_\lambda - T)^{-0.35}$, therefore $\Delta\rho_s/\rho_s$ has no singularity near T_λ. The same conclusion could be made from the direct scaling analysis of quantity (I_l/c_2^2) near T_λ made by Swanson and Donnelly.[14] On the other hand, Ahlers, neglecting the influence of the critical velocity $V_{ns,\mathrm{cr}}$ and straightforwardly fitting his data, concluded that m runs from 3 to 4 with the $A(T) \propto (T_\lambda - T)^{-0.23}$ when $m=3$, and $A(T) \propto (T_\lambda - T)^{-0.64}$ when $m=4$. It is easy to see that depending on the choice of quantity m, the suppression of the superfluid density $\Delta\rho_s/\rho_s$ as a function of $(T_\lambda - T)$ either converges $(m=3)$ or diverges $(m=4)$. Thus the question of whether the chaotic vortices are the cause of the full vanishing of the superfluid density remains open. We think, therefore, that experimental study of superfluid turbulence near the λ point is an important field having many applications. Moreover, we feel that the present problem of the λ transition in a heat flux attracts the attention of many researchers.

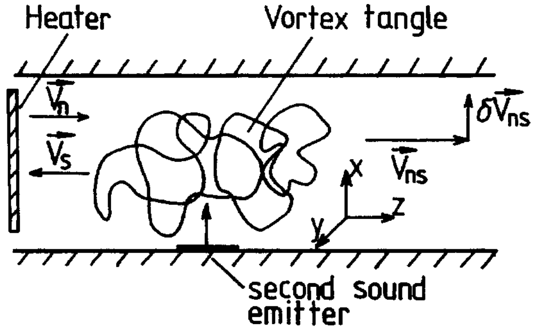

FIG. 3. Illustration of the transverse experiment. Small perpendicular deviation $\delta\mathbf{V}_{ns}$ of the counterflow velocity $\mathbf{V}_{ns}$ changes the orientation of the polarization of the vortex tangle, whereas the vortex line density and mean curvature do not change to the first order in $\delta\mathbf{V}_{ns}$.

B. Transverse case

With the goal of discussing the possibility to detect the effect of suppression of the superfluid component $\Delta\rho_s/\rho_s$ by the transverse second sound, let us study what would happen if one imposed a small transverse counterflow velocity $\delta\mathbf{V}_{ns}^x$ on the main flow (see Fig. 3). We would like to remind the reader that in Vinen theory the superimposing of the small transverse counterflow velocity changes nothing since both vortex line density $\mathcal{L}_v$ and its dynamics depend on only an absolute value of the counterflow velocity, but the latter does not change to first order. On the other hand, it is obvious that something should happen. From the point of view of the approach developed here, an application of the small transverse counterflow velocity $\delta\mathbf{V}_{ns}^x$ results in a slight change of the orientation of the polarization of the vortex tangle, whereas the mean curvature ξ_0 and vortex line density $\mathcal{L}_v$ do not change (to first order in $\delta\mathbf{V}_{ns}^x$). Apparently the change of the polarization direction should lead to the appearance of a transverse (x) component of superfluid mass current $\delta\mathbf{J}_V^x$ induced by the vortex tangle. Just as in the longitudinal case the additional current results in a suppression of superfluid density $\Delta\rho_s^x$ in the x direction defined by

$$\Delta\rho_s^x=\frac{\delta\mathbf{J}_V^x}{\delta(\delta\mathbf{V}_s^x)}. \tag{14}$$

The variation of the Lamb impulse $\delta\mathbf{J}_V$ while applying small arbitrary $\delta\mathbf{V}_{ns}$ should be evaluated from the following rule:

$$\delta\mathbf{J}_V=\frac{\rho_s\tilde{\kappa}}{2}\sum_j\left\{\int\langle\mathbf{s}_j(\xi_j)\times\mathbf{s}_j'(\xi_j)\rangle\Big|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}}d\xi_j-\int\langle\mathbf{s}_j(\xi_j)\times\mathbf{s}_j'(\xi_j)\rangle\Big|_{\mathbf{V}_{ns}}d\xi_j\right\}. \tag{15}$$

The index $\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}$ in the first integral points out that we have to calculate the average for the main flow plus a small additional flow, whereas the second average should be calculated for the main flow only. Imposing $\delta\mathbf{V}_s$ to be directed in the x direction and taking the x component $\delta\mathbf{J}_V^x$ of the vector $\delta\mathbf{J}_V$ one concludes that

$$\delta\mathbf{J}_V^x=\frac{\rho_s\tilde{\kappa}}{2}\sum_j\int\{\langle\mathbf{s}_{jy}(\xi_j)\times\mathbf{s}_{jz}'(\xi_j)\rangle|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}-\langle\mathbf{s}_{jz}(\xi_j)\times\mathbf{s}_{jy}'(\xi_j)\rangle|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}\}d\xi_j. \tag{16}$$

Note that it is not necessary to subtract the average on the main flow [the second term on the right-hand side of Eq. (15)] for the combination $\langle\mathbf{s}_j(\xi_j)\times\mathbf{s}_j'(\xi_j)\rangle|_{\mathbf{V}_{ns}}$ does not have a x component at all. Using prescription (7), relation (16) can be expressed in terms of the characteristic functional and can be rewritten via matrix $N^{\alpha\beta}(\kappa)$:

$$\delta\mathbf{J}_V^x=\frac{\rho_s\tilde{\kappa}}{2}\sum_j\sum_{\kappa\neq0}\left(\frac{2}{-i\kappa}\right)\{N^{yz}(\kappa)|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}-N^{zy}(\kappa)|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}\}. \tag{17}$$

Since in first order in $\delta\mathbf{V}_{ns}^x$ both the mean curvature ξ_0 and vortex line density $\mathcal{L}_v$ do not change, the distinction of the matrix $N^{\alpha\beta}(\kappa)|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}$ from $N^{\alpha\beta}(\kappa)|_{\mathbf{V}_{ns}}$ stems from the different orientations of the flows. Therefore the matrix $N^{\alpha\beta}(\kappa)|_{\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x}$ is to be obtained by the rotation of the matrix $N^{\alpha\beta}(\kappa)|_{\mathbf{V}_{ns}}$ through the small angle $\varphi=\arctan|\delta\mathbf{V}_s^x|/|\mathbf{V}_s|$ around the y axis (see Fig. 3). Multiplying the matrix $N^{\alpha\beta}(\kappa)|_{\mathbf{V}_{ns}}$ by the matrix of infinitesimal rotation

$$N^{\alpha\beta}(\kappa)|_{\mathbf{V}_s+\delta\mathbf{V}_s^x}=\begin{pmatrix}1&0&\varphi\\0&1&0\\-\varphi&0&1\end{pmatrix}N^{\alpha\beta}(\kappa)|_{\mathbf{V}_s}, \tag{18}$$

we have

$$N^{\alpha\beta}(\kappa)|_{\mathbf{V}_s+\delta\mathbf{V}_s^x}=\begin{pmatrix}N^{xx}(\kappa)&N^{xy}(\kappa)&\varphi N^{zz}(\kappa)\\N^{yx}(\kappa)&N^{yy}(\kappa)&0\\-\varphi N^{xx}(\kappa)&-\varphi N^{xy}(\kappa)&N^{zz}(\kappa)\end{pmatrix}, \tag{19}$$

where $N^{\alpha\beta}(\kappa)$ are the elements of the matrix N for the undisturbed counterflow (see Ref. 2). Substituting Eq. (19) into Eq. (17) we conclude that the superimposing of the small external transverse counterflow results in the appearance of transverse superfluid current directed against $\delta\mathbf{V}_s^x$ and equal to

$$\delta\mathbf{J}_V^x=-\left[\frac{\rho\tilde{\kappa}I_l\alpha_v}{4\rho_n c_2^2\beta_v}\right]\rho_s\delta\mathbf{V}_s^x. \tag{20}$$

Relation (20) implies that the transverse suppression of superfluid density $(\Delta\rho_s/\rho_s)_x$ is

$$\left(\frac{\Delta\rho_s}{\rho_s}\right)_x=-\left[\frac{\rho\tilde{\kappa}I_l\alpha_V}{4\rho_n c_2^2\beta_V}\right], \tag{21}$$

i.e., one half of the longitudinal value.

III. SOME EXPERIMENTAL CONSEQUENCES

In this section we discuss some experimental consequences of the results obtained above. Although the suppression of superfluid density is pretty large, of the order of

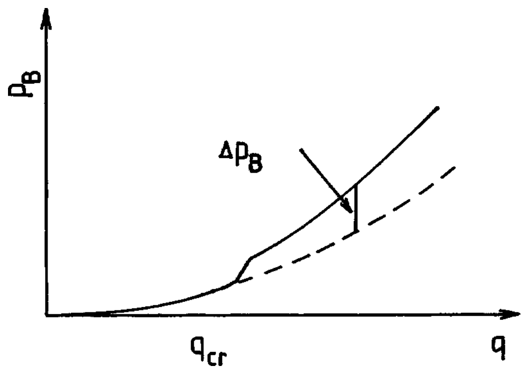

FIG. 4. Jump of Bernoulli pressure appearing in the counterflow after exceeding the critical value of the heat flux (schematically). A dashed line corresponds to pure Bernoulli pressure without suppression of the superfluid density.

5–10 % of the ''bare'' superfluid density, we do not know how it can be measured directly. Instead we propose two hydrodynamic effects which are connected with the suppression of the superfluid density in supercritical counterflow.

A. Jump of the Bernoulli pressure

One possibility to detect the suppression of superfluid density is a measurement of the pressure difference Δp_B between points inside and outside a channel where the counterflow of He II is realized. It is known (see, e.g., Ref. 15) that under some suppositions the analog of the Bernoulli law takes place in superfluid hydrodynamics. Namely, in steady flows the combination

$$p+\left(\frac{1}{2}\rho_s\mathbf{V}_s^2+\frac{1}{2}\rho_n\mathbf{V}_n^2\right) \tag{22}$$

should be constant. Following Ref. 15 we refer to the term inside the parentheses as the Bernoulli pressure p_B. In the counterflow the quantity p_B is expressed via applied heat flux q as

$$p_B=\frac{\rho_n\rho}{2\rho_s}(q/ST)^2, \tag{23}$$

where S is the entropy density. The circumstance that the expression for the Bernoulli pressure (23) includes the quantities ρ_s and ρ_n gives an opportunity to detect and to measure the effect of the suppression of superfluid density described above. Really, after exceeding the critical velocity, the vortex tangle develops in the channel which results in a change of the superfluid density. According to relation (23), the change of the superfluid density leads to the jump Δp_B of the Bernoulli pressure as is schematically shown in Fig. 4. As for numerical values, the evaluation with the help of relations (12),(23) gives for the jump of the Bernoulli pressure Δp_B the value of order 10 Dyn/cm^2 for the temperature $T=1.62$ K and for heat flux of order 1W/cm^2. The excessive pressure drop Δp in superfluid turbulent helium has actually been observed in a number of works (see, e.g. Ref. 16), but usually it was attributed to nonuniformity of the counterflow (in transverse direction) and to the ''eddy'' viscosity.

B. Transverse second sound experiment

The second hydrodynamic effect which we would like to discuss, is related to the propagation of second sound in the supercritical counterflow. The second sound testing is one of the most popular experimental methods of studying superfluid turbulence. The expression for the second sound velocity includes superfluid density ρ_s. Thus the measurement of the second sound velocity in the supercritical counterflow would enable us to study the effect of the suppression of superfluid density. The longitudinal second sound modulates the value of $\mathcal{L}_v$, therefore the effect of the suppression of ρ_s may be confused with other phenomena arising from the vortex line density dynamics. For this reason we discuss here only the case of transverse second sound.

Formula (21) for the transverse change of ρ_s was derived for a static case. To use relation (21) for dynamical phenomena we have to develop our approach a bit further. The general theory (see, e.g., Ref. 17) asserts that for a nonstationary process, when $\delta\mathbf{V}_{ns}^x$ is a harmonic function of time,

$$\delta\mathbf{V}_{ns}^x\propto\exp(i\omega t), \tag{24}$$

formula (21) should be improved. Namely, the suppression of superfluid density in this situation becomes a function of frequency ω in the following form:

$$\Delta\rho_s^x(\omega)=(\Delta\rho_s^x)_{\text{static}}\frac{1}{1+i\omega\tau_J}. \tag{25}$$

Here $(\Delta\rho_s^x)_{\text{static}}$ is the transverse change of superfluid density in the static transverse experiment introduced by Eq. (21), τ_J is the time of the relaxation of the superfluid mass current $\mathbf{J}_V$, which is supposed to satisfy the condition $\omega\tau_J\ll 1$. Thus the shift of $\Delta\rho_s^x(\omega)$ is the complex quantity, which implies that besides the change of the second sound velocity there will also be an additional dissipation and dispersion. Taking the real part of relation (25) and employing condition $\omega\tau_J\ll 1$ let us rewrite $\Delta\rho_s^x(\omega)$ in the form

$$\Delta\rho_s^x(\omega)=(\Delta\rho_s^x)_{\text{static}}\left(\frac{1}{\omega\tau_J}\right)^2. \tag{26}$$

The relaxation time of the superfluid mass current $\mathbf{J}_V^x$ can be found from a dynamical consideration. Let us consider first the dynamics of the full current $\mathbf{J}_V$. Using the prescription made in Sec. II B of the previous paper[2] we have to revert a time dependence for the line element positions $\mathbf{s}_j(\xi_j)\to\mathbf{s}_j(\xi_j,t)$ and to differentiate (with respect to variable t) the expression for the Lamb impulse (4) with the subsequent use of the chain rule. Accomplishing such a procedure we arrive at the following expression for the rate of change of quantity $\mathbf{J}_V$:

$$\begin{aligned}\frac{d\mathbf{J}_V}{dt}&=\rho_s\tilde{\kappa}\sum_j\int\langle\dot{\mathbf{s}}_j(\xi_j,t)\times\mathbf{s}_j'(\xi_j)\rangle d\xi_j\\&=-\rho_s\tilde{\kappa}\sum_j\int\langle\mathbf{s}_j'(\xi_j)\times[\mathbf{V}_s+\mathbf{V}_i+\alpha\mathbf{s}_j'\\&\quad\times(\mathbf{V}_{ns}-\mathbf{V}_i)]\rangle\ d\xi_j.\end{aligned} \tag{27}$$

Here we used for $\dot{\mathbf{s}}_j(\xi_j,t)$ the right-hand side of the equation of motion of the vortex line, discussed in detail, e.g., in Ref.

1, the quantity α is the friction coefficient, $\mathbf{V}_i$ is the velocity of the line elements induced by the vortex filament configuration (including the contribution from image vortices) and expressed by the Biot-Savart law. With good accuracy (see, e.g., Ref. 8) the quantity $\mathbf{V}_i$ can be taken from the so-called local approximation $\mathbf{V}_i=\beta\mathbf{s}_j'(\xi_j)\times\mathbf{s}_j''(\xi_j)$, where, we recall, $\beta=(\tilde{\kappa}/4\pi)\ln(1/\mathcal{L}_V^{1/2}a_0)$ (a_0 is the core radius). It is easy to see that if one took a local approximation for $\mathbf{V}_i$ the contribution in the integral from the second term on the right-hand side would vanish due to symmetry. Furthermore, the usual supposition is that the external superfluid velocity $\mathbf{V}_s$ is uniform. Therefore the contribution in the integral from the first term in the right-hand side should also vanish. The rest of the right-hand side of Eq. (27) is nothing but the force $\mathbf{F}_{sn}$ exerted by the normal component on the vortex tangle so we arrive at the following result:

$$\frac{d\mathbf{J}_V}{dt}=\mathbf{F}_{sn}. \tag{28}$$

This result expresses the obvious fact that the rate of change of the Lamb impulse is equal to the applied external force. Nevertheless Eq. (28) needs some comments. The most unpleasant thing about this equation is that the quantity $\mathbf{J}_V$ is not conserved, but, on the contrary, it will either vanish or grow up to infinity depending on the initial conditions. The situation is identical to the behavior of the vortex rings moving in He II which either grow up to infinity or vanish depending on their radius and polarization. However, in the steady case $\mathbf{J}_V$ should be constant and the question of what equilibrates the action of the friction force to sustain the stationary situation arises.

One of the possible ways to cure this situation is to abandon the assumptions made while deriving the equation, namely, the use of the local approximation and the supposition of the uniformity of $\mathbf{V}_s$ [Eq. (28)]. Then the right-hand side of Eq. (28) should include the additional terms

$$-\rho_s\tilde{\kappa}\sum_j\int\langle\mathbf{s}_j'(\xi_j)\times\mathbf{V}_s^{\text{total}}\rangle d\xi_j, \tag{29}$$

where $\mathbf{V}_s^{\text{total}}$ is the sum of the external and full self-induced velocities, including a nonlocal contribution as well as the one that appears from image vorticity near the boundaries. Expression (29) is called the vortex force (see, e.g., Ref. 10). It can be rigorously proved that a nonlocal contribution as well as the local one does not change the value of the total Lamb impulse. As for the contribution from the image vorticity, it does change the dynamics of the vortex lines especially near the walls (this question is of independent interest). However, the according effects are too weak to resolve the paradox. One more omitted effect during the derivation of Eq. (28) is related with the reconnection processes. However, reconnection processes do not change the Lamb impulse (see Ref. 18) therefore the consideration of them does not resolve the problem either (Ref. 19).

We think that the resolution of the discussed problem is deeper and simpler simultaneously and lies in the fact that the deterministic equation of motion of the vortex line cannot be applied to describe the evolution of quantities having statistical origin. Moreover, it cannot be applied to describe the stochastic dynamics of the line itself. Indeed because of the nonlinearity the dynamics of vortex filaments should be highly unstable producing a huge amount of uncontrolled perturbations in the motion of the line. These chaotic perturbations are usually introduced into equations of motion as random (or Langevin) forces supplemented by some supposition concerning their stochastic properties. The steady case is reached as a result of the competition of the deterministic terms and of the random forces. The description of the chaotic vortex filament dynamics appears to apply the same procedure. The introduction of random force should also change the equations for the evolution of macroscopic quantities, in particular, the equation for the Lamb impulse [in comparison with relation (28)] which would resolve a problem of steady states. We would like to remind the reader that a similar conclusion was made by Schwarz[8] who also faced the problem of the degeneration of the vortex tangle in his numerical simulations and was obliged to use a procedure of artificial mixing. A random forcing of the filaments may solve this degeneration problem in a more physical manner.

In spite of their undoubted importance, the problems exposed above are obviously beyond the scope of our paper. Therefore to proceed and to determine the time of relaxation τ_J of quantity $\mathbf{J}_V^x$ entering Eqs. (25) and (26) we use a rough estimate, casting Eq. (28) into standard relaxation form

$$\frac{d\mathbf{J}_V^x}{dt}=\frac{\delta\mathbf{F}_{sn}^x/\delta\mathbf{V}_{ns}^x}{\delta\mathbf{J}_V^x/\delta\mathbf{V}_{ns}^x}\mathbf{J}_V^x. \tag{30}$$

The fraction on the right-hand side of relation (30) can be considered as an estimate (upper limit) of the inverse relaxation time $1/\tau_j$. Denominator $\delta\mathbf{J}_V^x/\delta\mathbf{V}_{ns}^x$ should be extracted from relation (20). As far as the numerator is concerned, it can be evaluated by the use of Eq. (27), where we have to substitute $\mathbf{V}_{ns}\rightarrow\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x$, $\mathbf{V}_s\rightarrow\mathbf{V}_s+\delta\mathbf{V}_s^x$, average them over the total flow $\mathbf{V}_{ns}+\delta\mathbf{V}_{ns}^x$, and, finally, to take the x component. Accomplishing the corresponding algebraic manipulations we finally obtain[20]

$$\frac{1}{\tau_J}\approx\frac{8\alpha(1-I_{xx})}{\beta}\mathbf{V}_{ns}^2. \tag{31}$$

Combining Eqs. (21),(26) with Eq. (31) and inserting the result into the expression for the second sound velocity u_2 one concludes that the transverse second sound propagates slower in counterflowing He II. The relative change $\Delta u_2/u_2$ is

$$\frac{\Delta u_2}{u_2}=-f(T)\frac{\mathbf{V}_{ns}^4}{\omega^2}. \tag{32}$$

Here the function $f(T)$ is composed of structure parameters of the vortex tangle:

$$f(T)=\frac{4\rho\tilde{\kappa}I_l^2\alpha^2(1-I_{xx})^2}{\rho_n c_2^4\beta^3}. \tag{33}$$

Decreasing the second sound velocity in counterflowing He II was really observed about two decades ago by Vidal.[21] To our knowledge there was only one attempt to consider this effect theoretically made by Mehl.[22] He explained the change of the second sound velocity by introducing the imaginary part into the Hall-Vinen constant. However, in addition to some numerical disagreement, Mehl's theory did not explain the strong $\mathbf{V}_{ns}$ dependence in experimental data. Let us compare our result (32) with the Vidal experiment. Using the data on the structure parameters one obtains that, e.g., for temperature 1.44 K, the value of the function $f(T)$ is about 620 s^2/cm^4. Taking the frequency $\omega = 4.3$ rad/s, used in Ref. 21, and $V_{ns} = 2$ cm/s one obtains that $\Delta u_2/u_2 \approx 4 \times 10^{-4}$, which is very close to the observed value.

IV. CONCLUSION

One of the purposes of this paper was to demonstrate the potentiality of the method of the trial distribution function developed in the previous paper.[2] Indeed the exposition of Sec. II convincingly showed that the calculation of the Lamb impulse was reduced to some simple procedure not requiring any additional suppositions.

Of course this purpose was not the only one, moreover it was not even the main goal. We think that the results obtained in this work have independent physical interest. Indeed, though the idea that the chaotic vortex tangle induces nonzero mean flow was discussed earlier, it was not brought into a developed quantitative theory. Meanwhile as was shown in this work the quantitative study of the Lamb impulse leads to a number of interesting physical effects. The major one of them is the suppression of the superfluid density, which, in addition, turned out to be fairly large, of the order of several percent. Unfortunately the uncertainty in the experimental data did not allow us to resolve the very important question concerning the behavior of the quantity $\Delta\rho_s/\rho_s$ near T_λ.

The revealed effect can be detected and measured in hydrodynamic experiments in He II, where the superfluid density is relevant and enters the corresponding formulas in an explicit form. We discussed two hydrodynamic phenomena such as the additional pressure drop and the decrease of the velocity of the second sound propagating in superfluid turbulent He II.

In our opinion one of the most important by-products is the analysis of the dynamics of the Lamb impulse made in Sec. III. The conclusion that the deterministic dynamics of the vortex line cannot be applied to an adequate description of the evolution of the Lamb impulse (and other macroscopic quantities), nor to a description of the vortex line dynamics itself can drastically change our notions of the nature of the superfluid turbulence. This point of view coincides with a similar conclusion made by Schwarz[8] who obtained it in a numerical simulation.

Another important by-product is that the characteristic time of relaxation of the Lamb impulse is close to the one for the vortex line density $\mathcal{L}_v(t)$. This result, implying that macroscopic dynamics of the vortex tangle cannot be reduced to the only Vinen equation, can also force us to revise many of the classical results.

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[19] After the submission of the first of these papers the author visited the Tohoku University (Sendai, Japan). The host scientist Professor M. Tsubota carried out a series of the test numerical simulations, in which it was confirmed that neither nonlocal terms nor reconnection processes change the value of the Lamb impulse. The author is grateful to Professor M. Tsubota for this information and for very useful discussions.

[20] This relation may be the second unpleasant thing about Eq. (28). Really, the time of relaxation of the Lamb impulse is of the order of one for the vortex line density dynamics [Eq. (1) of the

previous paper (Ref. 2)]. But that implies that a macroscopic description of the vortex tangle cannot be restricted by the Vinen equation but should also include the equation for the Lamb impulse, and probably for other quantities. From the point of view of the method elaborated in Ref. 2, the close description of the macroscopic dynamics should consist of the set of equations for all of the structure parameters for they represent a full description of the vortex tangle via the trial distribution function.

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