Gaussian model of vortex tangle in He II

Sergey K. Nemirovskii*
Institute of Thermophysics, 630090 Novosibirsk, Russia
(Received 20 November 1996)

A description of the chaotic vortex tangle in superfluid turbulent He II is developed. Unlike current phenomenological theory dealing with only the macroscopic variable, the vortex line density $\mathcal{L}_v(t)$ and describing thereby only the macroscopic hydrodynamic phenomena, our approach allows us to describe effects due to the arrangement of the vortex tangle and the interaction of lines. To develop this approach we introduce a trial distribution function in the space of vortex loop configurations which absorbs all properties of superfluid turbulence known both from experiment and from numerical simulations. This trial distribution function is built in terms of the path integral. A number of allowed configurations is obtained evaluating the path integral with constraints connected with the established properties of the vortex tangle. Using the trial distribution function we also build the characteristic (generating) functional which allows us to evaluate any average over the vortex loop configuration. On the basis of the developed approach we briefly discuss some simple statistical characteristics of the vortex tangle. A more extended example of the developed approach studying superfluid mass current induced by vortex tangle is reported in a subsequent paper. [S0163-1829(98)04709-2]

I. INTRODUCTION AND SCIENTIFIC BACKGROUND

It is widely accepted (see, e.g., Ref. 1 and Refs. 2 and 3) that after exceeding some critical (fairly small) value of velocity (or of the relative velocity $\mathbf{V}_{ns}=\mathbf{V}_n-\mathbf{V}_n$ if one considers the case of counterflow) the entangled mass of the chaotic vortex filament or the vortex tangle appears in the superfluid component of He II. The wide class of hydrodynamic phenomena associated with the presence of the vortex tangle is called superfluid turbulence. The most standard scheme to study superfluid turbulence is depicted in Fig. 1. The counterflow is created by the application of a heat load q to the end of the channel filled by He II. When the heat load is small, the counterflow is supported by an extremely small drop of the temperature ($\Delta T \propto q$) along the channel needed to overcome the viscous flow of the normal component. After exceeding some critical value of the heat flux (of order of 10^{-3} W/cm^2) the temperature drop increases rapidly ($\Delta T \propto q^3$), which indicates that an additional strong dissipative mechanism appears. Feynman[4] proposed that this mechanism was the friction between the normal component and a set of chaotically distributed filaments of quantized vortices. He also proposed a qualitative scenario describing the evolution of the vortex tangle. In accordance with this scenario the friction force between the normal component and vortices causes *in average* a growth of the total length of the vortex filaments. When the vortex tangle becomes dense enough, the collisions of lines come into play. In the processes of collision or self-collision the lines reconnect. Subsequent self-reconnections of the rings result in the appearance of a cascadelike breaking down of the vortex loops which leads to an eventual reduction of total length (see Fig. 2). The competition of these two mechanisms results in the ''equilibrium'' state,[5] when the total length of lines per unit of volume, or vortex line density $\mathcal{L}_v$ is established in the system. The quantity $\mathcal{L}_v$ is a function of the counterflow velocity V_{ns} and of parameters of the system, such as the bath temperature T and pressure p.

Feynman's qualitative model was further developed in the classical works of Vinen[6,7] who brought these ideas into quantitative relations. In particular Vinen obtained the equation bearing his name which governs macroscopic dynamics of the vortex tangle, i.e., evolution of the vortex line density $\mathcal{L}_v(t)$. This equation reads

$$\frac{d\mathcal{L}_v}{dt}=\alpha_v|V_{ns}|\mathcal{L}_v^{3/2}-\beta_v\mathcal{L}_v^2, \qquad (1)$$

where α_v and β_v are the parameters (dependent on the bath temperature T and pressure p) specified by Vinen in the experiment. The first term on the right-hand side of Eq. (1) corresponds to the growth of the vortex line density due to mutual friction, the second one is connected to a decay due to the breaking down of the vortex rings. To find the form of the two components Vinen used dimensional considerations

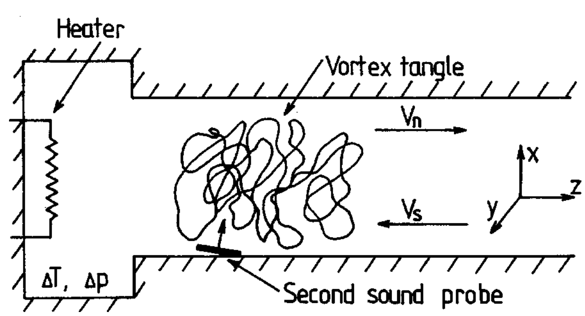

FIG. 1. Turbulent counterflow in He II. The normal component flows from the heater carrying the heat flux $q=ST\mathbf{V}_n$; the superfluid component flows toward the heater. Total mass current $\mathbf{j}=\rho_n\mathbf{V}_n+\rho_s\mathbf{V}_s=0$. The usual measured quantities are the drop of temperature ΔT or/and pressure Δp, attenuation and velocity of the second sound propagating at different angles through the counterflow, the shape of heat pulses, etc. Here are also depicted the axes used in the present paper, the z axis is directed along the relative velocity $\mathbf{V}_{ns}$, axes x and y are arbitrary, however, symmetry between x and y is assumed.

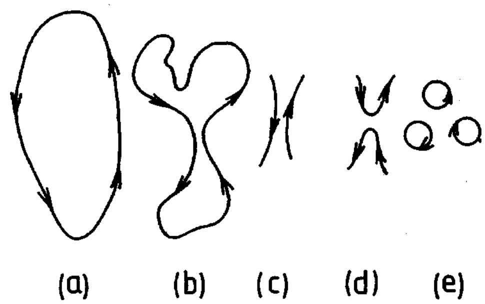

FIG. 2. Cascadelike process of the break down of the vortex ring due to reconnection (Feynman, 1955): (a) initial stage; (b),(c) stage of approaching of the line elements; (d) stage of collapse and reconnection; (e) stage of cascadelike degeneration of the vortex loops into thermal excitations.

as well as the results of the dynamics of single vortex rings and the experimental data. In stationary cases Vinen's equation yields the relation

$$\mathcal{L}_v = \frac{\alpha_v^2}{\beta_v^2} |V_{ns}|^2. \tag{2}$$

Relation (2) supplemented by some assumptions such as the assumptions of isotropic and uniform distribution of the vortex lines in space served as a basis to evaluate the various quantities, e.g., the sound attenuation, mutual friction force, temperature gradient, etc. A large number of works exist where relations similar to Eqs. (1) and (2) were used to explain the various physical effects and further modifications and corrections to the classical Feynman-Vinen theory were made (see, e.g., Ref. 1).

The Vinen equation can be incorporated into the classical hydrodynamics of He II (Refs. 8–10). This unified hydrodynamics of superfluid turbulence describes a huge variety of nonstationary processes in the superfluid turbulent helium such as propagation of strong thermal pulses, evolution of the temperature and the velocities fields, decay of the vortex tangle, and other phenomena (see, e.g., Ref. 3). The approach developed by Feynman and Vinen is frequently referred to as the phenomenological theory of superfluid turbulence, for the main constituents of this theory were Feynman's acute conjecture as well as Vinen's experimental data.

Further progress in the understanding of the nature of superfluid turbulence and its relation to the vortex line dynamics can be found in a series of works by Schwarz (the key papers are Refs. 11–13). In a striking paper of this series[13] Schwarz reported on the results of a direct numerical simulation of the vortex filament dynamics. Starting from the equation of motion of the vortex line elements in He II and assuming the vortex lines to reconnect while approaching each other, Schwarz showed that initially smooth vortex rings developed into a chaotic vortex tangle. He calculated some of the characteristics of this vortex tangle, which he called the structure parameters of the vortex tangle. It has to be said that performing numerical simulations Schwarz primarily concentrated on the phenomena and effects studied before in phenomenological theory. In particular he calculated the force exerted by vortices on the normal component which turned out to have a more complicated structure than the one obtained with the help of Eq. (2), the attenuation of longitudinal and transverse second sounds, averaged curvature of the filaments, and a number of other quantities. Schwarz also managed to express the rate of change of the vortex line density $d\mathcal{L}_v/dt$ via the structure parameters and demonstrated that it was equivalent to the Vinen equation (1), although with a different interpretation.

Although the phenomenological theory successfully explains many phenomena, it has a number of serious problems and open questions (see, e.g., Ref. 3). However, the following aspect seems to be more important and topical. Being formulated in terms of averaged macroscopical variable $\mathcal{L}_v(t)$ the phenomenological theory ignores discreteness of the vortex tangle and, correspondingly, it fails to describe effects connected to distribution of the filaments, their interaction, etc., unless one draws some additional suppositions. Although Schwarz's numerical modeling expands considerably the limits of the phenomenological theory it also has restricted possibilities to study effects connected to the fine structure of the vortex tangle. Indeed, as is often the case in numerical simulations, the structure parameters calculated by Schwarz can hardly be used to evaluate other quantities (e.g., the various correlation functions), than the ones he had calculated. It is understood that the above mentioned statement does not concern the quantities which are directly expressed via vortex line density and via the structure parameters. Meanwhile there exist many other physical quantities related with other physical phenomena which cannot be expressed in terms of Schwarz theory. We will give some examples of such quantities (and associated physical effects) later in the second section. Thus the question of developing the appropriate stochastic theory of chaotic vortex filaments to calculate various averaged quantities arises.

Of course the most honest way to develop such a theory is to study stochastic dynamics of vortex filaments on the basis of equations of motions with some source of chaos, for instance, introducing the Langevin force. However, because of extremely involved dynamics of vortex lines this way seems almost hopeless. Indeed the deterministic dynamics of the vortex line elements in He II is governed by an essentially nonlinear equation with nonpolynomial and even nonanalytical nonlinearities (see, e.g., Ref. 13 or 3). This equation also includes nonlocal terms due to Biot-Savart law. Because of the mutual friction between vortices and normal component the usual conservation laws (e.g., the conservation of energy) are violated. In addition the reconnection processes permanently change the topology of the system. Probably the most serious obstacle is that the stochastic behavior is expected to be essentially one of nonequilibrium. For instance, analytical and numerical investigations of the far simpler model problem devoted to the stochastic behavior of the vortex ring in a local approach, without friction and reconnection, showed that the strongly nonequilibrium state, characterized by the flux of the local curvature in Fourier space, is established (see Ref. 14). For this reason we think that an advanced theory of chaotic vortex filaments will not be developed in the near future and the question of the proper calculation of various properties due to the distribution and interaction of discrete vortex filaments on the basis of rigorous theory remains open as before.

In this paper another, far more modest approach is developed. The main idea and strategy are the following. Although the phenomenological theory of the superfluid turbulence deals with macroscopical characteristics of the vortex tangle, it conveys rich information concerning the *instantaneous* structure of the vortex tangle. The main goal of the present work is to construct a trial distribution function in the space of vortex loop configurations of the most general form which satisfies all the established properties of the vortex tangle. It is assumed that this trial distribution function will enable us to calculate any averaged quantities due to the vortex tangle (see, however, remarks made in the Conclusion). In particular we will discuss some stochastic properties of the vortex lines at the end of this paper. A more extended example of the developed approach, which concerns a superfluid mass current induced by the vortex tangle, will be exposed in the subsequent paper.[15]

The structure of the paper is the following. In Sec. II we discuss the properties of the *instantaneous* vortex tangle structure known from experiment and numerical simulations. We also give several examples of quantities due to the vortex lines arrangement which are of definite physical interest and which cannot be obtained within the framework of phenomenological theory. In Sec. III we construct the trial distribution function of a general form satisfying all the known properties of the vortex tangle. Performing this procedure we widely use the ideas and methods of the theory of polymer chains. Section IV is devoted to the calculation of the characteristic functional. As it will be shown the use of a characteristic functional not only significantly simplifies the evaluation of various averages, but also plays a key role in the construction of the trial distribution function. In Sec. V we discuss some statistical properties of the vortex tangle.

II. ARRANGEMENT OF THE VORTEX TANGLE

This section is devoted to summarizing our knowledge on the arrangement of the vortex tangle obtained from the investigations of superfluid turbulence. Primarily this knowledge was accumulated from the experimental works, however, while interpretating of one or other experiments, investigators used the conception of superfluid turbulence as a set of vortex filaments chaotically distributed in space. Fetching various semi-quantitative speculations investigators drew a number of conclusions concerning vortex tangle structure. As it was said in the Introduction these pure phenomenological results were confirmed in numerical simulations of the vortex lines dynamics made by Schwarz.[13] The numerical modeling not only established numerical values for a number of characteristics of the vortex tangle, but also allowed us to determine them in temperature regions where the experimental data were absent. For this reason we widely use the structure parameters calculated by Schwarz as a basis, although in principle we could appeal only to the experimental results. Therefore in addition to the simple introduction of the quantities characterizing the vortex tangle structure, we will briefly discuss what experimental results led to them.

A. What we know about the vortex tangle structure

The current view on the vortex tangle arrangement can be summarized as follows. The vortex tangle developed in counterflowing He II (Fig. 1) consists of a set of closed lines labeled by index j. They can be described as a set of functions $\mathbf{s}_j(\xi_j)$, where $\mathbf{s}_j(\xi_j)$ is the radius vector of the points resting on the j loop. Variable ξ_j labels the points of the j loop. It is convenient to choose variable ξ_j to be equal to the arc length ξ_j $(0 \le \xi_j \le L_j)$ (see Fig. 3). We remind the reader that we are interested in the instantaneous picture of the vortex tangle, therefore dependence on time is omitted. The whole configuration of the vortex tangle $\{\mathbf{s}_j(\xi_j)\}$ is the unification of all of the curves $\{\mathbf{s}_j(\xi_j)\} = \cup_j \mathbf{s}_j(\xi_j)$. Due to frequent reconnections, both the number of loops and their lengths L_j are arbitrary quantities. In addition each of the loops can take any arbitrary shape $\mathbf{s}_j(\xi_j)$. It should be understood, however, that in spite of the arbitrariness of these quantities the whole configuration should meet a number of requirements. For instance, the total length of the loops per unit of volume-vortex line density $\mathcal{L}_v$ is the well determined quantity satisfying the relation

$$\left\langle \frac{1}{\mathcal{V}} \sum_j \int_0^{L_j} |\mathbf{s}_j'(\xi_j)| d\xi_j \right\rangle = \mathcal{L}_v . \tag{3}$$

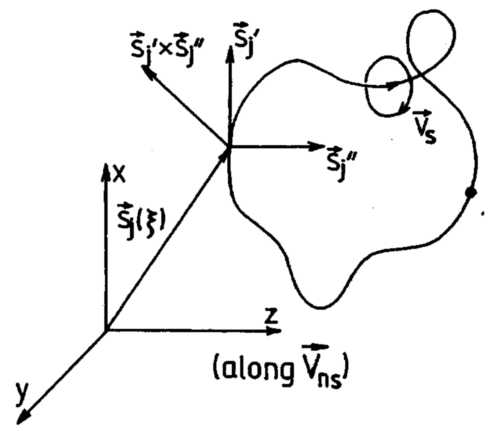

FIG. 3. Space curve representing a j-vortex loop. The position of the vortex line element is described by curve $\mathbf{s}_j(\xi_j)$, where ξ_j is the arc length, $\mathbf{s}_j'(\xi_j) = d\mathbf{s}_j(\xi_j)/d\xi_j$ is a tangent vector, the unit vector along the vortex line; $\mathbf{s}_j''(\xi_j) = d^2\mathbf{s}_j(\xi_j)/d\xi_j^2$ is the local curvature vector. Vector production $\mathbf{s}_j'(\xi_j) \times \mathbf{s}_j''(\xi_j)$ is responsible for the mutual orientation of the tangent vector and the vector of curvature. The initial point of the curve denoted by a dot is chosen as arbitrary.

Here $\mathcal{V}$ is the volume, the prime denotes derivative with respect to the arc length ξ_j. The angle brackets denote overall averaging over vortex loop configurations $\{\mathbf{s}_j(\xi_j)\}$. Since the variable ξ_j is chosen to be the arc length, the absolute value of the tangent vector is the unit

$$|\mathbf{s}_j'(\xi_j)| = 1, \tag{4}$$

which leads to the relation

$$\left\langle \sum_j \int_0^{L_j} d\xi_j \right\rangle = \mathcal{V} L_v . \tag{5}$$

The filaments comprising the vortex tangle are distributed in space in an anisotropic manner. There are two kinds of anisotropy. The first one is connected to orientations of the line elements and has been discovered in experiments on the attenuation of transverse (with respect to the counterflow, see Fig. 1) and of longitudinal second sound.[16] The measure of this anisotropy can be defined with the use of the structure parameters $I_\|$, $I_\perp$ introduced by Schwarz (see Ref. 13) who confirmed the anisotropy of the vortex tangle in the numerical modeling. For our purposes it is more convenient to use other parameters $I_{\alpha\alpha}$ $(\alpha=x,y,z)$ simply connected with the ones introduced by Schwarz. The fraction of the vortex line elements orientated along the z axis is

$$\left\langle \frac{1}{\mathcal{V}\mathcal{L}_v}\sum_j \int_0^{L_j} \mathbf{s}'_{jz}(\xi_j)\mathbf{s}'_{jz}(\xi_j)d\xi_j \right\rangle = 1-I_\| = I_{zz}.. \tag{6}$$

To move further we have to discuss what we mean by the phrase ''overall averaging over vortex loop configuration.'' Based on what has been said above, we state that the overall averaging includes averaging over (i) the shape of each of the loops, (ii) the number of the loops and their lengths, and (iii) the initial points of each of the loops. As far as the second and third items are concerned, unfortunately neither experiment nor theory give any clue to the according distributions. To overcome this problem we accept the supposition of full uniformity of the vortex tangle made by many investigators and also confirmed in numerical simulations. Because of uniformity overall *local* averages such as $\langle \mathbf{s}_{j\alpha}(\xi_j)\rangle$, $\langle \mathbf{s}_{j\alpha}(\xi_j)\mathbf{s}'_{j\beta}(\xi_j)\rangle$, etc., should not depend on both ξ_j and j . Therefore relation (6) is factorized as

$$\langle \mathbf{s}'_{jz}(\xi_j)\mathbf{s}'_{jz}(\xi_j)\rangle \left\langle \frac{1}{\mathcal{V}L_v}\sum_j \int_0^{L_j} d\xi_j \right\rangle = I_{zz}$$

and, in combination with Eq. (5), can be rewritten in the local form

$$\langle \mathbf{s}'_{jz}(\xi_j)\mathbf{s}'_{jz}(\xi_j)\rangle = I_{zz}. \tag{7}$$

The essence of the performed procedure is the separation of an overall averaging into the one over the shapes of the loops and other averaging. Therefore accomplishing preaveraging over loop lengths, over the number of loops and their initial points, a trivial matter due to full uniformity, we are left with an average over the shape of some ''averaged'' loop having the same structure parameters as the whole vortex tangle. Thus the problem is reduced to constructing a distribution function in the space of the configuration, where the ''averaged'' curve takes various shapes. We, however, retain the index j for consistency of presentation.

Relations analogous to Eq. (7) can be written down for x and y components:

$$\langle \mathbf{s}'_{jx}(\xi_j)\mathbf{s}'_{jx}(\xi_j)\rangle = 1-I_\perp = I_{xx}, \tag{8}$$

$$\langle \mathbf{s}'_{jy}(\xi_j)\mathbf{s}'_{jy}(\xi_j)\rangle = 1-I_\perp = I_{yy}. \tag{9}$$

Due to $|\mathbf{s}'_j(\xi_j)|=1$ [see Eq. (4)] parameters I_{xx}, I_{yy}, and I_{zz} obey the obvious identity

$$I_{xx}+I_{yy}+I_{zz}=1. \tag{10}$$

The second kind of anisotropy, the so-called polarization, is connected with a mutual orientation of the tangent vector $\mathbf{s}'_j(\xi_j)$ of the filament segments and the vector of curvature $\mathbf{s}''_j(\xi_j)$. The measure of polarization I_l has quantitatively been introduced by Schwarz by the relation

$$\left\langle \frac{1}{\mathcal{V}\mathcal{L}_v^{3/2}}\sum_j \int_0^{L_j} \mathbf{s}'_j(\xi_j)\times\mathbf{s}''_j(\xi_j)d\xi_j \right\rangle = I_l\mathbf{e}_z, \tag{11}$$

where $\mathbf{e}_z$ stands for the unit vector in the z direction (along the counterflow $\mathbf{V}_{ns}$). On the strength of the arguments discussed above, condition (11) can also be brought into the local form

$$\langle \mathbf{s}'_j(\xi_j)\times\mathbf{s}''_j(\xi_j)\rangle = I_l\mathcal{L}_v^{1/2}\mathbf{e}_z. \tag{12}$$

Though relation (11) was taken from numerical simulation it can be readily obtained from experimental data. Indeed, the combination $\langle \mathbf{s}'_j(\xi_j)\times\mathbf{s}''_j(\xi_j)\rangle$ appears to be a positive term in the equation for the rate of change of length of the line element. Therefore it can be extracted by comparing this equation with the first term on the right-hand side of the Vinen equation (1). This procedure has been carried out in Refs. 10 and 13.

The next property of the vortex tangle concerns the mean curvature of the lines. The idea that the mean curvature of the vortex tangle should be of the order of interline space goes back to Hall's work on superfluid turbulence.[17] Later this view was discussed by many authors and was rigorously confirmed in Schwarz's numerical simulation. He calculated the coefficient c_2^2 connecting the averaged squared curvature with the quantity $\mathcal{L}_v$. In our notation this property reads

$$\left\langle \frac{1}{\mathcal{V}\mathcal{L}_v}\sum_j \int_0^{L_j} \mathbf{s}''_j(\xi_j)\mathbf{s}''_j(\xi_j)d\xi_j \right\rangle = c_2^2\mathcal{L}_v \tag{13}$$

or, in the local form,

$$\langle \mathbf{s}''_j(\xi_j)\mathbf{s}''_j(\xi_j)\rangle = c_2^2\mathcal{L}_v. \tag{14}$$

The uniform superfluid turbulence which we are interested in is realized in wide channels. More rigorously it implies that interline space $\delta=\mathcal{L}_v^{-1/2}$ should be much smaller than the size of channel. In this case we can disregard lines ending on surfaces and consider all lines to be closed loops. Condition of the closeness of the lines can be written as

$$\int_0^{L_j} \mathbf{s}'_j d\xi = 0. \tag{15}$$

In addition we suppose that the length of each of the loops is greater than the mean radius of the curvature, as observed by Schwarz,[13]

$$L_j \gg \langle |\mathbf{s}''_j(\xi_j)|\rangle^{-1}. \tag{16}$$

Apart from the direct evidence following from Schwarz's work we can bring forward the following argument in favor of condition (16). The length of the vortex filament is first changed due to the deterministic process of ballooning or of shrinking of the loops and, secondly, due to reconnection processes. The reconnection can decay the loops into two smaller loops in the case of self-collision or can fuse them into a larger loop in the case of collision with other loops. Clearly the loops collide more frequently with other loops than are subjected to the self-collision. Therefore it seems plausible that long loops prevail due to the reconnection processes.

The properties expressed by relations (3)–(16) are about the extent of our knowledge concerning the vortex tangle arrangement in superfluid turbulent helium II. Following Schwarz we will call the quantities $I_\perp$,$I_\parallel$ (and I_{xx} , I_{yy} , I_{zz}), I_l , c_2 , as the structure parameters of the vortex tangle. They depend on the temperature T and the pressure p and do not depend on the applied counterflow velocity $\mathbf{V}_{ns}$. Numerical values of the structure parameters as a function of temperature are given in the original work by Schwarz[13] and in Refs. 1 and 3.

B. Other quantities of interest

The structure parameters introduced in the previous subsection convey valuable information on the vortex tangle arrangement. In addition they can serve as a basis for the evaluation of various quantities connected to the observed physical phenomena, such as the Gorter-Mellink constant, coefficients of the extraattenuation of the second sound (both transverse and longitudinal), drift velocity of the vortex tangle, etc. Details of the corresponding procedure can be found in Ref. 13. However, there are many other important characteristics of the vortex tangle which are connected with other physical phenomena and which cannot be directly expressed via structure parameters or a combination of them. Let us give several examples of these quantities.

For the first example we take the average vorticity of the superfluid velocity $\mathbf{\Omega}(\mathbf{r})$ and its Fourier transform $\mathbf{\Omega}_\mathbf{k}$. These quantities may have nonzero values and should be taken into account when more complicated flows than a one-dimensional counterflow are considered. They have to be introduced as the following averages:

$$\mathbf{\Omega}(\mathbf{r})=\left\langle \tilde{\kappa}\sum_j \oint d\mathbf{s}_j\,\delta[\mathbf{r}-\mathbf{s}_j(\xi_j)]\right\rangle \tag{17}$$

and

$$\mathbf{\Omega}_\mathbf{k}=\left\langle \frac{\tilde{\kappa}}{(2\pi)^{3/2}}\int d^3\mathbf{r}e^{-i\mathbf{kr}}\sum_j \oint d\mathbf{s}_j\,\delta[\mathbf{r}-\mathbf{s}_j(\xi_j)]\right\rangle$$
$$=\left\langle \frac{\tilde{\kappa}}{(2\pi)^{3/2}}\sum_j \oint d\mathbf{s}_j e^{-i\mathbf{k}\mathbf{s}_j(\xi_j)}\right\rangle, \tag{18}$$

where $\tilde{\kappa}$ is the quantum of circulation. The closest example of the flow of the superfluid component with a nonzero averaged vorticity created by a set of vortex lines is the case of rotating He II. In this case the superfluid component simulates (in average) the solid-body rotation, and the averaged vorticity is just the areal density of the vortex lines multiplied by $\tilde{\kappa}$ as follows from relation (17).

Another quantity which influences the hydrodynamic properties of the superfluid turbulence is mean superfluid velocity created by vortex tangle $\mathbf{v}_s^V(\mathbf{r})$. It is obtained by averaging the Biot-Savart law

$$\mathbf{v}_s^V(\mathbf{r})=\left\langle \frac{\tilde{\kappa}}{4\pi}\sum_j \int_0^{L_j}\frac{\mathbf{s}_j'(\xi_j)\times[\mathbf{r}-\mathbf{s}_j(\xi_j)]}{|\mathbf{r}-\mathbf{s}_j(\xi_j)|^3}d\xi_j\right\rangle. \tag{19}$$

Accordingly, the full momentum $\mathbf{P}_V$ of the additional superfluid motion connected to the presence of the vortex tangle is

$$\mathbf{P}_V=\rho_s\int \mathbf{v}_s^V(\mathbf{r})d^3\mathbf{r}. \tag{20}$$

With the direct use of relations (19),(20) a problem typical for vortex flows is encountered. The integral in relation (20) diverges for both small and large $|\mathbf{r}-\mathbf{s}_j(\xi_j)|$ (see, e.g., Refs. 18, 19). Therefore the question of the averaged velocity or of the full momentum generated by vortices cannot be resolved in a straightforward way. On the other hand, it is known that in many respects the so-called Lamb impulse plays the role of momentum. In general the Lamb impulse density is defined as

$$\mathbf{J}_V=\frac{\rho_s}{2\mathcal{V}}\int \mathbf{r}\times\omega(\mathbf{r})d^3\mathbf{r}, \tag{21}$$

where $\omega(\mathbf{r})$ is the distribution [microscopical, not averaged $\mathbf{\Omega}(\mathbf{r})$] of the vorticity. For the singular distribution of the vorticity, viz., for chaotic vortex filaments, Eq. (21) can be rewritten as

$$\mathbf{J}_V=\left\langle \frac{\rho_s\tilde{\kappa}}{2\mathcal{V}}\sum_j \int_0^{L_j}\mathbf{s}_j(\xi_j)\times\mathbf{s}_j'(\xi_j)d\xi_j\right\rangle. \tag{22}$$

The following paper[15] is devoted the study of the quantity $\mathbf{J}_V$.

Due to the interaction between vortices, the vortex tangle should display some kind of elasticity. As a result it is expected that the long-range interaction between different vortex lines element will lead to macroscopic modes such as waves of the vortex line density i.e., three-dimensional (3D) analogs of the Tkachenko waves.[1] The measure of elasticity is determined by the energy of the interaction. The energy of the vortex tangle is defined by the average

$$E=\left\langle \frac{1}{2}\int \rho_s\mathbf{v}_s^2 d^3\mathbf{r}\right\rangle$$
$$=\left\langle \frac{\rho_s\kappa^2}{8\pi}\sum_{j,i}\int_0^{L_i}\int_0^{L_j}\frac{\mathbf{s}_i'(\xi_i)\mathbf{s}_j'(\xi_j)}{|\mathbf{s}_i(\xi_i)-\mathbf{s}_j(\xi_j)|}d\xi_i d\xi_j\right\rangle. \tag{23}$$

By use of the well-known formula

$$\frac{1}{|\mathbf{r}|}=\int_\mathbf{k}\frac{4\pi d^3\mathbf{k}}{\mathbf{k}^2}e^{i\mathbf{kr}}, \tag{24}$$

the average energy E [Eq. (23)] can be rewritten as

$$E=\left\langle \frac{\rho_s\kappa^2}{2}\sum_{i,j}\int_\mathbf{k}\frac{d^3\mathbf{k}}{\mathbf{k}^2}\right.$$
$$\left.\times\int_0^{L_i}\int_0^{L_j}\mathbf{s}_j'(\xi_i)\mathbf{s}_j'(\xi_j)d\xi_i d\xi_j e^{i\mathbf{k}(\mathbf{s}_i(\xi_i)-\mathbf{s}_j(\xi_j))}\right\rangle. \tag{25}$$

For the next example, we would like to point out that the proper entropy of the vortex tangle is

$$S^V=k_B\langle \ln\Gamma(\{\mathbf{s}_j(\xi_j)\})\rangle, \tag{26}$$

where $\Gamma(\{\mathbf{s}_j(\xi_j)\})$ is the number of the vortex loop configurations (see next section). The quantity S^V enters the equations of hydrodynamics of superfluid turbulence (see Refs. 3 and 8) and the knowledge of it is necessary for the correct study of unsteady hydrodynamic processes.

Let us consider the average

$$\mathcal{A}=\left\langle \sum_j \int_0^{L_j} (\dot{\mathbf{s}}_j(\xi_j)\times \mathbf{s}_j'(\xi_j))d\xi_i \right\rangle, \tag{27}$$

where $\dot{\mathbf{s}}_j(\xi_j)$ is the velocity of the line element. The quantity $\dot{\mathbf{s}}_j(\xi_j)$ is expressed (generally as a functional) via the instantaneous configuration of the vortex tangle $\{\mathbf{s}_j(\xi_j)\}$ with the help of the equation of motion (see, e.g., Ref. 13). The right-hand side of Eq. (27) is an averaged net area swept out by the motion of the lines. The quantity $\mathcal{A}$ bears a manyfold physical interest. For example, the equation of motion of the vortex line can be derived from the variational principle, and the contribution of the lines into the action is proportional to the area swept out by the moving lines (see Ref. 20). Furthermore, the z component of $\mathcal{A}$ is just the rate of the phase slippage caused by the motion of vortex lines transverse to the counterflow $\mathbf{V}_{ns}$ (see Ref. 21). Finally, the integrand in (Eq. 27) is the discrete variant of quantity $\mathbf{v}_s\times(\nabla\times\mathbf{v}_s)$, which is called the vortex force and plays a significant role in the vortex dynamics (see Ref. 19). Of course all of the properties of $\mathcal{A}$ discussed above are not independent and we listed them just to stress an importance of $\mathcal{A}$ for applications.

Using the distribution function it is even possible to describe dynamics of various quantities $A(\{\mathbf{s}_j(\xi_j)\})$ averaged over loop configurations $\langle A(\{\mathbf{s}_j(\xi_j)\})\rangle$. Indeed, reverting the time dependence for the line elements positions $\mathbf{s}_j(\xi_j)\to\mathbf{s}_j(\xi_j,t)$ and using a chain rule, we can write down the rate of change of the quantity $\langle A(\{\mathbf{s}_j(\xi_j,t)\})\rangle$ in the form

$$\frac{\partial\langle A(\{\mathbf{s}_j(\xi_j,t)\})\rangle}{\partial t}=\left\langle \sum_i \int_0^{L_j} \frac{\delta A(\{\mathbf{s}_j(\xi,t)\})}{\delta \mathbf{s}_i(\xi_i',t)}\frac{\partial \mathbf{s}_i(\xi_i',t)}{\partial t}d\xi_i' \right\rangle. \tag{28}$$

Expressing the velocity of the line element $\dot{\mathbf{s}}_j(\xi_j)$ with the help of the equation of motion[13] we find that the right-hand side of Eq. (28) is an average of some functional of the vortex line configuration. Thus we have obtained a rule for the calculation of the evolution of quantity $\langle A(\{\mathbf{s}_j(\xi_j,t)\})\rangle$. A word of caution should be given. The future distribution function will correspond to the instantaneous picture of the vortex tangle, i.e., to the ''equilibrium'' state of the vortex tangle. Therefore, by saying dynamics, we have in mind only small deviations from the ''equilibrium.'' Nevertheless the possibility to introduce macroscopic dynamics of the vortex tangle in a regular way seems to be very important. For example, while derivating of the governing equation (1) for the evolution of the vortex line density, Vinen made a very important assumption that the rate of change of the vortex line density $d\mathcal{L}_v/dt$ is a function only of the instantaneous value of $\mathcal{L}_v$. This property was called the self-preserving assumption. As was discussed in Ref. 3 this assumption is justified if and only if the other characteristics of the vortex tangle other than the quantity $\mathcal{L}_v(t)$ relax to the ''equilibrium'' state much faster than $\mathcal{L}_v(t)$ itself. The self-preserving-assumption can be confirmed or refuted by inspecting relations similar to Eq. (28).

We gave several examples of the averages responsible for interesting phenomena in the turbulent counterflowing He II. Apart from independent interest, the study of these phenomena can supply important information concerning the fine structure of the vortex tangle. It is easy to see that none of the quantities introduced can be expressed via structure parameters of the phenomenological theory of superfluid turbulence. To evaluate them one has to have some rules to accomplish averaging. We propose to do it with the help of a trial distribution function construction which is the main purpose of this work. Before we proceed to this procedure we have to introduce and discuss one more averaged quantity — the characteristic (or generating) functional.

C. Introduction of characteristic functional

Let us consider the following averaged quantity, the so-called characteristic functional:

$$W(\{\mathbf{P}_j(\xi_j)\})=\left\langle \exp\left(i\sum_j \int_0^{L_j}\mathbf{P}_j(\xi_j)\mathbf{s}_j'(\xi_j)d\xi_j\right)\right\rangle. \tag{29}$$

The characteristic functional is of special interest. The point is that one is able to calculate any average depending on the vortex line configuration by using simple functional differentiation. For instance the average tangent vector $\langle \mathbf{s}_{j\alpha}'(\xi_j)\rangle$, the average vector of curvature $\langle \mathbf{s}_{j\alpha}''(\xi_j)\rangle$, or the correlation function between the orientation of different elements of the vortex filaments $\langle \mathbf{s}_{j\alpha}'(\xi_j)\mathbf{s}_{j\beta}'(\xi_j)\rangle$ are readily expressed via a characteristic functional according to the following rules:

$$\langle \mathbf{s}_{j\alpha}'(\xi_j)\rangle=\left.\frac{\delta W}{i\,\delta\mathbf{P}_j^{\alpha}(\xi_j)}\right|_{\text{all }\mathbf{P}=0}, \tag{30}$$

$$\langle \mathbf{s}_{j\alpha}''(\xi_j)\rangle=\left.\frac{\partial}{\partial\xi_j}\frac{\delta W}{i\,\delta\mathbf{P}_j(\xi_j)}\right|_{\text{all }\mathbf{P}=0}, \tag{31}$$

$$\langle \mathbf{s}_{j\alpha}'(\xi_j)\mathbf{s}_{j\beta}'(\xi_j)\rangle=\left.\frac{\delta^2 W}{i\,\delta\mathbf{P}_j^{\alpha}(\xi_{j1})\,i\,\delta\mathbf{P}_j^{\beta}(\xi_{j2})}\right|_{\text{all }\mathbf{P}=0}. \tag{32}$$

The other quantities are expressed via a characteristic functional in a bit more sophisticated way. For instance the Fourier transform of the averaged vorticity $\mathbf{\Omega}_{\mathbf{k}}$ [Eq. (18)] can be evaluated with the help of the characteristic functional by the use of the following procedure:

$$\mathbf{\Omega}_{\mathbf{k}}=\sum_j \int_0^{L_j} d\xi_j e^{-i\mathbf{k}\mathbf{s}_j(0)}\left.\frac{\delta W}{i\,\delta\mathbf{P}_j(\xi_j)}\right|_{\{\mathbf{P}_j(\xi_j')\}=-\mathbf{k}\theta(\xi_j')\theta(\xi_j-\xi_j')}, \tag{33}$$

where $\theta(\xi_j')$ is the unit stepwise function. The production $\theta(\xi_j')\theta(\xi_j-\xi_j')$ selected out points lying in range $0\leq\xi_j'\leq\xi_j$ on the j curve. This choice assures the appearance of the correct quantity $e^{i\mathbf{k}\mathbf{s}_j(\xi_j)}$ after integration of the exponent in relation (29). The quantity $\mathbf{s}_j(0)$ is the initial point of the j curve which is chosen to be arbitrary.

In a similar way the mean energy $\langle E\rangle$ [Eq. (25)] can be calculated as

$$\langle E\rangle=\frac{\rho_s\kappa^2}{2}\sum_{i,j}\int_{\mathbf{k}}\frac{d^3\mathbf{k}}{\mathbf{k}^2}\int_0^{L_i}\int_0^{L_j}\times d\xi_i d\xi_j e^{-i\mathbf{k}[\mathbf{s}_i(0)-\mathbf{s}_j(0)]}\times\frac{\delta^2 W}{i\,\delta\mathbf{P}_i^\alpha(\xi_i)\,i\,\delta\mathbf{P}_j^\alpha(\xi_j)}. \quad (34)$$

Here a set of $\mathbf{P}_n(\xi_n')$ in the characteristic functional $W(\{\mathbf{P}_n(\xi_n')\})$ is again determined with the help of the θ functions

$$\mathbf{P}_i(\xi_i')=-\mathbf{k}\theta(\xi_i')\theta(\xi_i-\xi_i'),$$

$$\mathbf{P}_j(\xi_j')=-\mathbf{k}\theta(\xi_j')\theta(\xi_j-\xi_j'), \quad (35)$$

$$\mathbf{P}_n(\xi_n)=0, n\neq i,j.$$

Relation (35) implies that we choose an integrand of the characteristic functional (29) from only points lying in the interval from 0 to ξ_i on the i curve and from 0 to ξ_j on the j curve. When evaluating the self-energy of the same loop $i=j$, one has to distinguish points ξ_i, and put them to be, e.g., ξ_i' and ξ_i''.

Although the characteristic functional is defined via the distribution function as some auxiliary quantity, it plays a significant independent role in stochastic theory. For instance, in many body problems the use of a characteristic functional (in this theory it is usually called a generating functional) allows one to get the shortened description of statistical properties in terms of the Green function and equations for them. Another example would be the case of classical turbulence, where one derives the master equation for the characteristic functional directly from the equation of motion of the fluid avoiding the use of a distribution function (see, e.g., Ref. 22). Similarly in our work the characteristic functional is used not only for the calculation of different averages but it also plays a key role in the derivation of the trial distribution function.

III. CONSTRUCTING A TRIAL DISTRIBUTION FUNCTION OF GENERAL FORM

A. Main definitions

The averages introduced in the previous section can be calculated using a distribution function $\mathcal{P}(\{\mathbf{s}_j(\xi_j)\})$ in the space of vortex loop configurations. According to the general prescriptions the average of any quantity $\langle\mathcal{B}(\{\mathbf{s}_j(\xi_j)\})\rangle$ depending on the vortex loop configurations is

$$\langle\mathcal{B}(\{\mathbf{s}_j(\xi_j)\})\rangle=\sum_{\{\mathbf{s}_j(\xi_j)\}}\mathcal{B}(\{\mathbf{s}_j(\xi_j)\})\mathcal{P}(\{\mathbf{s}_j(\xi_j)\}). \quad (36)$$

Here $\mathcal{P}(\{\mathbf{s}_j(\xi_j)\})$ is the probability of the vortex tangle to have a particular configuration $\{\mathbf{s}_j(\xi_j)\}$. The meaning of the summation over all vortex loop configurations $\Sigma_{\{\mathbf{s}_j(\xi_j)\}}$ in formula (36) will be clear from the further presentation.

We use the usual supposition in statistical physics that all configurations corresponding to the same macroscopic state have equal probabilities. Thus the probability $\mathcal{P}(\{\mathbf{s}_j(\xi_j)\})$ for the vortex tangle to have a particular configuration $\{\mathbf{s}_j(\xi_j)\}$ should be proportional to $1/N_{\mathrm{allowed}}$, where N_{allowed} is the number of allowed configurations which is, of course, infinite:

$$\mathcal{P}(\{\mathbf{s}_j(\xi_j)\})\propto\frac{1}{N_{\mathrm{allowed}}}. \quad (37)$$

By the term ''allowed configurations'' N_{allowed} we mean only the configurations that will lead to the correct values for all average quantities known from experiment and given by relations (3)–(16). The following subsection will be devoted to a detailed elaboration of the ideas given above.

B. Introduction of the constraints and the effective Lagrangian

The number of allowed configurations, or number of curves, is expressed by the path integral in space of 3D curves supplemented by some constraints which follow from conditions (3)–(16), or, to be precise, which will lead to these conditions:

$$N_{\mathrm{allowed}}\propto\int\mathcal{D}\{\mathbf{s}_j(\xi_j)\}\times\mathrm{constraints}\{\mathbf{s}_j(\xi_j)\}. \quad (38)$$

In this subsection we will introduce the constraints dictated by conditions (3)–(16), and modify expression (38) into a standard and tractable form. Let us begin with condition (3) concerning the total length of vortex lines. This condition implies that among the possible curves, labeled by ξ_j, we have to choose only curves whose lengths are fixed and equal to L_j. Taking the local form of this condition (4) we impose the corresponding constraint into the integrand of Eq. (38) as a δ function:

$$N_{\mathrm{allowed}}^{(j)}\propto\int\mathcal{D}\mathbf{s}_j(\xi_j)\times\delta(|\mathbf{s}_j'(\xi_j)|-1). \quad (39)$$

Because of the absolute value of $\mathbf{s}_j'(\xi_j)$, which is a nonanalytical function, this expression will lead to a theory which is not tractable. We will use here a trick known from the theory of polymer chains, where a similar problem appears. Let us divide the j vortex line into a set of discrete points and change the path integral by production of the usual ones

$$\int\mathcal{D}\mathbf{s}_j(\xi_j)=\int J\mathcal{D}\mathbf{s}_j'(\xi_j)\to\prod_n\int Jd(\mathbf{s}_{j(n+1)}-\mathbf{s}_{jn}). \quad (40)$$

Here J is a Jacobian corresponding to the change from the variable $\mathbf{s}_j(\xi_j)$ to the derivative $\mathbf{s}_j'(\xi_j)$. The explicit shape of the Jacobian J is not significant since it is usually cancelled against the one in the normalizing factor for the probability $\mathcal{P}[\mathbf{s}_j(\xi_j)]$ in Eq. (37). Then the integration for each of the links $\mathbf{s}_{j(n+1)}-\mathbf{s}_{jn}=\boldsymbol{L}_n$ in relation (40) should be accomplished with the delta function constraints $\delta(|\boldsymbol{L}_n|-1)$ corresponding to the fixed length of the link. In the theory of polymer chains they offer to relax rigorous condition $|\boldsymbol{l}_n|$

$=l_n$ and to replace it by using the smeared-out (Gaussian) distribution of the link length with the same value of the integral (see, e.g., Ref. 23)

$$\int d^3\boldsymbol{l}_n \delta(|\boldsymbol{l}_n|-l_n) \Rightarrow \int d^3\mathbf{l}_n\left(\frac{4}{\sqrt{\pi} l_n}\right) e^{-\mathbf{l}_n^2/l_n^2}. \tag{41}$$

Parameter l_n here, the so-called effective bond length, is usually established in experiment. Gathering contributions from all links and coming back to the continuous case

$$\prod_n e^{-\mathbf{l}_n^2/a^2} = e^{-\Sigma_n(\mathbf{s}_{jn+1}-\mathbf{s}_{jn})^2/a^2} \Rightarrow e^{-\lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j} \tag{42}$$

one obtains that the number of configurations $N_{\text{allowed}}^{(j)}$ of the j vortex loop supplemented with constraint $\delta(|\mathbf{s}_j'(\xi_j)|-1)$ can be written

$$N_{\text{allowed}}^{(j)} \propto \int J\mathcal{D}\mathbf{s}_j(\xi_j) \times e^{-\lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j}. \tag{43}$$

Here the Jacobian J differs from the one used in Eq. (40), however, this has no effect because of the remark made after Eq. (40). The quantity λ_1 is a parameter of our theory which will be determined later. Before we move further it is worth discussing the sense of the fulfilled procedure once more. As for the theory of a polymer, the introduction of the Gaussian chains instead of real polymers does not significantly influence most of the important physical applications (see Ref. 23). One can say that a loss of some rigorousness is an acceptable sacrifice for the obvious simplicity. This conclusion concerns our model to a greater extent than polymer theory. Indeed, the choice of the arc length ξ_j as a label for the vortex filament was a question of convenience. The same treatment might be applied to another label, say, the arc length $\widetilde{\xi_j}$ at some moment of time. While the lines are in motion, some parts of the curve shrink whereas other parts stretch. Therefore the real value of $|\mathbf{s}_j{}'(\xi_j)|$ does not have to be equal to unity exactly, but should be smeared-out.

Returning to relation (43) we conclude that the use of the constraint (4) leads to the familiar form for the number of allowed configurations of the j loop as a path integral over all configurations with weight proportional to $\exp(-\int\mathcal{L}_j)$ where the effective Lagrangian is determined by

$$\int \mathcal{L}_j = \lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j. \tag{44}$$

Using formula (37) we conclude that the probability $\mathcal{P}_j[\mathbf{s}_j(\xi_j)]$ for the j loop to have configuration $\mathbf{s}_j(\xi_j)$ is the Wiener distribution (see Ref. 23)

$$\mathcal{P}_j[\mathbf{s}_j(\xi_j)] = \mathcal{N}_j \exp\left(-\lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j\right), \tag{45}$$

where $\mathcal{N}_j$ is the corresponding normalizing factor.

In a similar way we can account for other loops so the full Lagrangian corresponding to the constraint $|\mathbf{s}_j'(\xi_j)|=1$ is determined by

$$\int \mathcal{L} = \lambda_1\sum_j \int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j. \tag{46}$$

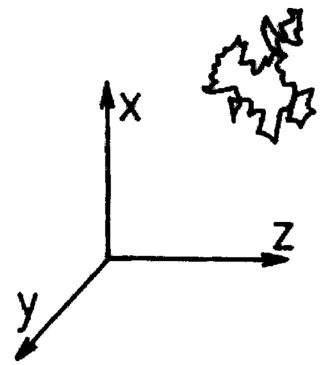

FIG. 4. Schematic picture of the curve averaged with weight $\exp(-\lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j)$. Although the averaging was performed over smooth lines, the resulting curve is a fractal object with a kink at each point.

The constant λ_1 was factored out of the sum, otherwise the average would depend on index j which would contradict the supposition of full uniformity.

The next constraint to be discussed is connected to the curvature of the lines. It is known that though the resulting curve averaged according to Eq. (45) has a correct length L_j, it is not smooth, but wiggly at each point as schematically depicted in Fig. 4. Indeed the distance between points $\mathbf{s}_j(\xi_{j1})$ and $\mathbf{s}_j(\xi_{j2})$ of the curve is expressed by the relation (see, e.g., Ref. 23)

$$\langle|\mathbf{s}_j(\xi_{j1})-\mathbf{s}_j(\xi_{j2})|\rangle = \sqrt{(\xi_{j1}-\xi_{j2})/\lambda_1} \tag{47}$$

nonanalytical when $\xi_{j1}\to\xi_{j2}$. In other words, the average curve $\langle\mathbf{s}_j(\xi_j)\rangle$ does not even have a first derivative. To make it smooth we have to introduce into the effective Lagrangian $\mathcal{L}_j$ a term with a second derivative and, further, in order to make it have finite curvature we have to introduce a term with a third derivative, etc.,

$$\int \mathcal{L}_j = \lambda_1\int_0^{L_j}|\mathbf{s}_j{}'|^2 d\xi_j + \lambda_2\int_0^{L_j}|\mathbf{s}_j{}''|^2 d\xi_j + \lambda_3\int_0^{L_j}|\mathbf{s}_j{}'''|^2 d\xi_j + \cdots . \tag{48}$$

The effective Lagrangian (48) is isotropic, therefore it will give an isotropic distribution of the line which is wrong. To improve the situation we have to impose that the coefficients λ be matrices (so far diagonal). For instance, relation (44) should be changed by the following expression:

$$\lambda_{1x}\int_0^{L_j}\mathbf{s}'_{jx}\mathbf{s}'_{jx}d\xi_j + \lambda_{1y}\int_0^{L_j}\mathbf{s}'_{jy}\mathbf{s}'_{jy}d\xi_j + \lambda_{1z}\int_0^{L_j}\mathbf{s}'_{jz}\mathbf{s}'_{jz}d\xi_j = \int_0^{L_j}\mathbf{s}'_{j\alpha}\Lambda^{\alpha\beta}\mathbf{s}'_{j\beta}d\xi_j. \tag{49}$$

The next step is to take into account the polarization of the vortex tangle. To have a nonzero value for the averaged polarity we have to add nondiagonal terms into the matrix $\Lambda^{\alpha\beta}$. The according correction to the Lagrangian is

$$\int \delta\mathcal{L}_j = \lambda_p \int_0^{L_j} (\mathbf{s}_j' \times \mathbf{s}_j'')_z d\xi_j . \tag{50}$$

The index z implies that we have to take only the z coordinate in the vector production $\mathbf{s}_j' \times \mathbf{s}_j''$.

Finally, since the effective Lagrangian includes derivatives of different orders ($\mathbf{s}_j'$, $\mathbf{s}_j''$, and $\mathbf{s}_j'''$) it is convenient to perform the one-dimensional Fourier transform along ξ_j:

$$\mathbf{s}_j(\xi_j) = \sum_\kappa \mathbf{s}_j(\kappa) e^{i\kappa\xi_j}, \kappa = 2\pi n / L_j . \tag{51}$$

Because of the closure condition (15)

$$\frac{1}{L_j}\int_0^{L_j} \mathbf{s}_j' d\xi_j = \mathbf{s}_j'(\kappa)\big|_{\kappa=0} = 0 \tag{52}$$

the zero harmonic of derivative $\mathbf{s}_j'(\xi_j)$ is zero, therefore we will further exclude harmonic $\kappa=0$ in the summation in 1D Fourier space. Correspondingly, evaluation of the path integral in the κ representation should be accomplished according to the following rule:

$$\int \mathcal{D}\{\mathbf{s}_j(\kappa)\} = \prod_j \prod_{\kappa\neq 0} \int d\mathbf{s}_j(\kappa). \tag{53}$$

Further we will use both sides of this relation interchangeably.

Summarizing everything concerning the effective Lagrangian and using relation (37) it can be inferred that probability $\mathcal{P}(\{\mathbf{s}_j(\kappa)\})$ that the vortex tangle has the particular configuration $\{\mathbf{s}_j(\kappa)\}$ is

$$\mathcal{P}(\{\mathbf{s}_j(\kappa)\}) = \mathcal{N} \exp\left(-\sum_{\kappa\neq 0} \mathcal{L}(\{\mathbf{s}_j(\kappa)\}) \right), \tag{54}$$

where $\mathcal{N}$ is overall normalization.

The density of Lagrangian $\mathcal{L}\{\mathbf{s}_j(\kappa)\}$ in κ space is[24]

$$\mathcal{L}\{\mathbf{s}_j(\kappa)\} = \sum_j \begin{pmatrix} \mathbf{s}_{jx}(\kappa) \\ \mathbf{s}_{jy}(\kappa) \\ \mathbf{s}_{jz}(\kappa) \end{pmatrix} \begin{pmatrix} \Lambda^{xx}(\kappa) & \Lambda^{xy}(\kappa) & \Lambda^{xz}(\kappa) \\ \Lambda^{yx}(\kappa) & \Lambda^{yy}(\kappa) & \Lambda^{yz}(\kappa) \\ \Lambda^{zx}(\kappa) & \Lambda^{zy}(\kappa) & \Lambda^{zz}(\kappa) \end{pmatrix} \times \begin{pmatrix} \mathbf{s}_{jx}(-\kappa) \\ \mathbf{s}_{jy}(-\kappa) \\ \mathbf{s}_{jz}(-\kappa) \end{pmatrix}. \tag{55}$$

The diagonal terms of the matrix $\Lambda^{\alpha\beta}$ have the following structure:

$$\Lambda^{\alpha\alpha} = \lambda_{1\alpha}\kappa^2 + \lambda_{2\alpha}\kappa^4 + \lambda_{3\alpha}\kappa^6 + \cdots, \alpha = x,y,z. \tag{56}$$

From the definition of the nondiagonal part of the Lagrangian (50) and from the assumed symmetry in the plain x,y it follows that

$$\Lambda^{xy} = (i\kappa)^3 \lambda_p , \Lambda^{yx} = -(i\kappa)^3 \lambda_p . \tag{57}$$

Expressions (54)–(57) determine the probability of the allowed vortex loop configuration of the most general form satisfying all the known properties of the vortex tangle. Thus they can be considered to be the trial distribution function which we were looking for. Of course to use this function one has to specify all of the parameters λ and cancel the uncertainty in the expansion (56). The most convenient way to do this is to calculate the characteristic functional and study its properties to specify an explicit form of the parameters entering the trial distribution function. The following section is devoted to this procedure.

IV. CONSTRUCTION OF THE CHARACTERISTIC FUNCTIONAL

A. Calculation of the characteristic functional in κ space

Our first step is to calculate the characteristic functional defined by relation (29). It is convenient to do so in κ space. In κ space the characteristic functional can be obtained by accomplishing a 1D Fourier transform (51) in Eq. (29):

$$W(\{\mathbf{P}_j(\kappa)\}) = \left\langle \exp\left(i\sum_j \sum_{\kappa\neq 0} L_j \mathbf{P}_j(\kappa)\mathbf{s}_j'(-\kappa) \right) \right\rangle. \tag{58}$$

The various averages are readily obtained using this definition. For instance, the averaged values of the tangent vector (30) and of the vector of curvature (31) can be evaluated with the use of the following rules:

$$\langle \mathbf{s}_{j\alpha}'(\xi_j)\rangle = \sum_{\kappa\neq 0} e^{-i\kappa\xi_j} \frac{\delta W}{iL_j \delta \mathbf{P}_j^\alpha(\kappa)}\bigg|_{\text{all } \mathbf{P}(\kappa)=0} , \tag{59}$$

$$\langle \mathbf{s}_{j\alpha}''(\xi_j)\rangle = \sum_{\kappa\neq 0} (-i\kappa) e^{-i\kappa\xi_j} \frac{\delta W}{iL_j \delta \mathbf{P}_j^\alpha(\kappa)}\bigg|_{\text{all } \mathbf{P}(\kappa)=0} . \tag{60}$$

Likewise the two point correlation function [relation (32)] is expressed via the characteristic functional in κ space as follows:

$$\langle \mathbf{s}_{j\alpha}'(\xi_j)\mathbf{s}_{j\beta}'(\xi_j)\rangle = \sum_{\kappa_1,\ \kappa_2\neq 0} e^{-i\kappa_1\xi_{j1}} e^{-i\kappa_2\xi_{j2}} \times \frac{\delta^2 W}{iL_j \delta\mathbf{P}_j^\alpha(\kappa_1) iL_j \delta\mathbf{P}_j^\beta(\kappa_2)}\bigg|_{\text{all } \mathbf{P}(\kappa)=0} . \tag{61}$$

To calculate the characteristic functional in the κ space [relation (58)] we employ the trial distribution function introduced in the previous section. Using relations (54)–(57) one can rewrite expression (58) in the following form:

$$W(\{\mathbf{P}_j(\kappa)\}) = \mathcal{N}\int J\mathcal{D}\{\mathbf{s}_j(\kappa)\} \times \exp\left[-\sum_j \sum_{\kappa\neq 0} \mathbf{s}_{j\alpha}(\kappa)\Lambda^{\alpha\beta}(\kappa)\mathbf{s}_{j\beta}(-\kappa) \right] \times \exp\left(i\sum_j \sum_{\kappa\neq 0} \left\{ L_j \frac{1}{2}[\mathbf{P}_j(\kappa)\mathbf{s}_j'(-\kappa) + \mathbf{P}_j(-\kappa)\mathbf{s}_j'(\kappa)] \right\} \right), \tag{62}$$

where

$$\mathcal{N}=\left\{\int J\mathcal{D}\mathbf{s}_j(\kappa)\exp\left(-\sum_j\sum_{\kappa\neq 0}\mathbf{s}_j^{\alpha}(\kappa)\Lambda^{\alpha\beta}(\kappa)\mathbf{s}_j^{\beta}(-\kappa)\right)\right\}^{-1}$$

is the overall normalization.

The right-hand side of relation (62) is evaluated by the standard ''full square procedure'' expressed by identity

$$\int\prod_k dz_k\exp\left[-\sum_{nm}z_nA_{nm}z_m^*+\sum_n(z_nu_n^*+z_n^*u_n)\right]$$
$$=\exp\left(-\sum_{nm}u_nA_{nm}^{-1}u_n^*\right)*\int\prod_k dz_k$$
$$\times\exp\left(-\sum_{nm}z_nA_{nm}z_m^*\right). \tag{63}$$

The integral in Eq. (63) is taken over a set of complex variables z_k, $\int dz_k=\int d\mathrm{Re}z_k d\mathrm{Im}z_k$. The matrix A_{nm} is supposed to be the Hermitian, and the matrix A_{nm}^{-1} is inverse to the matrix A_{nm}. Using this rule for each of the Fourier harmonics in Eq. (62) we get

$$W(\{\mathbf{P}_j(\kappa)\})=\exp\left(-\sum_j\sum_{\kappa\neq 0}L_j^2\mathbf{P}_j^{\alpha}(\kappa)N^{\alpha\beta}(\kappa)\mathbf{P}_j^{\beta}(-\kappa)\right), \tag{64}$$

where the matrix $N^{\alpha\beta}(\kappa)$ is equal to $1/4\kappa^2[\Lambda^{\alpha\beta}(\kappa)]^{-1}$. Elements of both the matrix $N^{\alpha\beta}(\kappa)$ and the matrix $\Lambda^{\alpha\beta}(\kappa)$ do not depend on index j, otherwise the local averages would depend on j which contradicts the full uniformity supposition.

The second step in realizing the scheme outlined at the end of Sec. III B is to study general properties of the characteristic functional (64) issuing from method it has been built. Before doing it let us connect the matrix $N^{\alpha\beta}(\kappa)$ (so far not determined explicitly) with the characteristics of the vortex tangle expressed by formulas (3)–(16). The functional derivatives entering relations (59)–(61) as applied to the characteristic functional (64) are evaluated according to the following rules:

$$\frac{\delta W}{iL_j\delta\mathbf{P}_j^{\alpha}(\kappa)}=\frac{L}{i}2N^{\alpha\nu}(\kappa_1)\mathbf{P}_j^{\nu}(-\kappa_1)$$
$$\times\exp\left(-\sum_j\sum_{\kappa\neq 0}L_j^2\mathbf{P}_j^{\mu}(\kappa)N^{\mu\nu}(\kappa)\mathbf{P}_j^{\nu}(-\kappa)\right). \tag{65}$$

Here it has been taken into account that $N^{\mu\nu}(\kappa)$ is the Hermitian matrix $N^{\alpha\nu}(\kappa)=N^{\nu\alpha}(-\kappa)$. Likewise the second derivative is

$$\frac{\delta^2 W}{iL_j\delta\mathbf{P}_j^{\alpha}(\kappa_1)iL_j\delta\mathbf{P}_j^{\beta}(\kappa_2)}$$
$$=\{2N^{\alpha\beta}(\kappa_1)\delta_{-\kappa_1,\kappa_2}-4N^{\alpha\nu}(\kappa_1)\mathbf{P}_j^{\nu}(-\kappa_1)N^{\beta\gamma}(\kappa_2)$$
$$\times\mathbf{P}_j^{\gamma}(-\kappa_2)\}$$
$$\times\exp\left[-\sum_j\sum_{\kappa\neq 0}L_j^2\mathbf{P}_j^{\mu}(\kappa)N^{\mu\nu}(\kappa)\mathbf{P}_j^{\nu}(-\kappa)\right]. \tag{66}$$

Using Eqs. (65) and (66) in relations (59)–(61) we conclude that

$$\langle\mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}'_{j\beta}(\xi_{j2})\rangle=\sum_{\kappa_1\neq 0}e^{-i\kappa_1(\xi_{j1}-\xi_{j2})}2N^{\alpha\beta}(\kappa_1), \tag{67}$$

hence the averaged squared tangent vector $\langle\mathbf{s}'_{j\alpha}(\xi_j)\mathbf{s}'_{j\alpha}(\xi_j)\rangle$ is just

$$\langle\mathbf{s}'_{j\alpha}(\xi_j)\mathbf{s}'_{j\alpha}(\xi_j)\rangle=\sum_{\kappa\neq 0}2N^{\alpha\alpha}(\kappa). \tag{68}$$

Accordingly, the averaged squared vector of curvature $\langle\mathbf{s}''_{j\alpha}(\xi_j)\mathbf{s}''_{j\alpha}(\xi_j)\rangle$ is

$$\langle\mathbf{s}''_{j\alpha}(\xi_j)\mathbf{s}''_{j\alpha}(\xi_j)\rangle=\sum_{\kappa\neq 0}2\kappa^2N^{\alpha\alpha}(\kappa). \tag{69}$$

Note that formulas (68) and (69) are valid both for each of the components α and for the sum over α. As far as the average polarization of the vortex tangle is concerned it is expressed via the matrix $N^{\alpha\beta}(\kappa)$ as follows:

$$\langle(\mathbf{s}'_{jx}\mathbf{s}''_{jy}-\mathbf{s}'_{jy}\mathbf{s}''_{jx})\rangle$$
$$=\sum_{\kappa_1,\kappa_2\neq 0}e^{-i(\kappa_1+\kappa_2)\xi_j}$$
$$\times[(i\kappa_2)N^{xy}(\kappa_1)\delta_{-\kappa_1,\kappa_2}-(i\kappa_2)N^{yx}(\kappa_2)\delta_{-\kappa_1,\kappa_2}]. \tag{70}$$

By constructing the nondiagonal elements of the Hermitian matrix $\Lambda^{xy}(\kappa)$ and $\Lambda^{yx}(\kappa)$ are odd functions of argument κ. It is obvious that the inverse matrix $N^{\alpha\beta}(\kappa)$ satisfies the same conditions. Therefore

$$N^{yx}(\kappa)=N^{xy}(-\kappa)=-N^{xy}(\kappa). \tag{71}$$

Using this chain of relations we finally arrive at

$$\langle(\mathbf{s}'_{jx}\mathbf{s}''_{jy}-\mathbf{s}'_{jy}\mathbf{s}''_{jx})\rangle=\sum_{\kappa\neq 0}2(i\kappa)N^{xy}(\kappa). \tag{72}$$

B. Trial form of the matrix $N^{\alpha\beta}(\kappa)$

Inspecting the method of constructing a trial distribution function as well as the way of deriving a characteristic functional we can deduce some very general properties of the matrix $N^{\alpha\beta}(\kappa)$: (i) the matrix $N^{\alpha\beta}(\kappa)=N^{\beta\alpha}(-\kappa)$ is the Hermitian one; (ii) the diagonal terms of $N^{\alpha\alpha}(\kappa)$ should be even functions of κ; (iii) the nondiagonal terms $N^{xy}(\kappa),N^{xy}(\kappa)$ should be odd functions of κ; (iv) to guarantee the existence of any $\langle\mathbf{s}_j^{(n)}(\xi_j)\mathbf{s}_j^{(n)}(\xi_j)\rangle$ for any n one has to require that the elements of matrix $N^{\alpha\beta}(\kappa)$ decrease faster than any power function κ^{n+2}.

In addition to these properties, the matrix $N^{\alpha\beta}(\kappa)$ should give the correct values of the mean tangent vector $\langle \mathbf{s}'_{j\alpha}(\xi_j)\mathbf{s}'_{j\alpha}(\xi_j)\rangle$, of the mean squared curvature $\langle \mathbf{s}''_{j\alpha}(\xi_j)\mathbf{s}''_{j\alpha}(\xi_j)\rangle$, and of the polarization $\langle(\mathbf{s}'_{jx}\mathbf{s}''_{jy}-\mathbf{s}'_{jy}\mathbf{s}''_{jx})\rangle$. Furthermore, since only a few characteristics of the vortex tangle are known, the matrix $N^{xy}(\kappa)$ should not include too many parameters. Finally, it should be simple enough and tractable otherwise the whole method would be meaningless.

As a suitable candidate satisfying all the listed properties we propose a matrix $N^{\alpha\beta}(\kappa)$ of the following form:

$$\begin{pmatrix} N^{xx}\exp(-\kappa^2\xi_0^2) & (i\kappa)N^{xy}\exp(-\kappa^2\xi_0^2) & 0 \\ -(i\kappa)N^{yx}\exp(-\kappa^2\xi_0^2) & N^{yy}\exp(-\kappa^2\xi_0^2) & 0 \\ 0 & 0 & N^{zz}\exp(-\kappa^2\xi_0^2) \end{pmatrix}. \tag{73}$$

It will be shown later that quantity ξ_0 is nothing but the correlation length. It will also be shown that ξ_0 is of order of the mean curvature, or of order of the interline space. Thus, besides the above conditions (i)–(iv) one more strong supposition that all the correlation functions $\langle \mathbf{s}^{(n)}_{j\alpha}(\xi_{j1})\mathbf{s}^{(m)}_{j\beta} \times(\xi_{j2})\rangle$ have the same correlation length ξ_0 of the order of the interline space is made. Some semiquantitative proof of that fact based on the consideration of kinematic relations such as $\mathbf{s}'\mathbf{s}''=0$, $\mathbf{s}'\mathbf{s}'''+\mathbf{s}''\mathbf{s}''=0$, etc., has been given by Schwarz.[11]

C. Specifying coefficients $N^{\alpha\beta}$ in κ space

The final step in constructing the characteristic functional is to specify the coefficients $N^{\alpha\beta}$ as well as the quantity ξ_0 . These five quantities can be obtained comparing relations (68)–(72), where matrix $N^{\alpha\beta}(\kappa)$ is taken from Eq. (73) with relations (3)–(16).

Let us start with the calculation of the coefficient N^{xx}. It can be found from a comparison of relation (8) for the mean x fraction of the tangent vector obtained in experiment with relation (68) for the same quantity expressed via the characteristic functional (64) with matrix (73):

$$I_{xx}=2N^{xx}\left\{\sum_{n\ \mathrm{all}}\exp\left[-n^2\left(\frac{2\pi\xi_0}{L_j}\right)^2\right]-1\right\}. \tag{74}$$

Employing condition $\xi_0\ll L_j$ and changing $\Sigma_{n\mathrm{all}}\to\int dn$ we arrive at

$$N^{xx}=I_{xx}\frac{\xi_0\sqrt{\pi}}{L_j}\frac{1}{(1-2\xi_0\sqrt{\pi}/L_j)}. \tag{75}$$

Of course this result is valid for each of the components with corresponding I_{yy},I_{zz}. Note that we retained the small term $2\xi_0\sqrt{\pi}/L_j$ in the denominator on the right-hand side of Eq. (75). Its origin is from the closeness of the vortex lines and it plays a significant role in questions where the closeness of the vortex lines is relevant (see below).

To specify the quantity ξ_0 one has to use the relations for the average squared vector of the curvature. Comparing Eqs. (13)and (69) and using the expression for $N^{\alpha\alpha}$ obtained above one concludes that

$$c_2^2\mathcal{L}_v=\frac{\xi_0\sqrt{\pi}}{L_j}\sum_{n\ \mathrm{all}}2n^2\left(\frac{2\pi}{L_j}\right)^2\exp\left[-n^2\left(\frac{2\pi\xi_0}{L_j}\right)^2\right]. \tag{76}$$

Changing again $\Sigma_{n\ \mathrm{all}}\to\int dn$ we obtain

$$\xi_0^2=\frac{1}{2c_2^2\mathcal{L}_v}. \tag{77}$$

Since c_2^2 is of the order of unity (see Ref. 13), the quantity ξ_0 is of the order of the interline space $\mathcal{L}_v^{-1/2}$.

Analogous calculations for polarization of the vortex tangle (11) allow us to determine the coefficient N^{xy} in nondiagonal terms of matrix $N^{\alpha\beta}(\kappa)$:

$$N^{xy}=2\sqrt{\pi}I_l\frac{\xi_0^3\mathcal{L}_v^{1/2}}{L_j}=\sqrt{\frac{\pi}{2}}\frac{I_l}{L_jc_2^3\mathcal{L}_v}. \tag{78}$$

An evaluation of the pre-exponent factors $N^{\alpha\beta}$ (75)–(78) and of the quantity ξ_0 in the matrix $N^{\alpha\beta}(\kappa)$ [Eq. (73)] completes the calculation of the characteristic functional and consequently of the trial distribution function.

D. The characteristic functional in ξ space

For many purposes it is more convenient to deal with a ξ representation of the characteristic functional, i.e., with $W(\{\mathbf{P}_j(\xi_j)\})$ [Eq. (29)]. To obtain it we have to perform an inverse Fourier transformation in the expression $\Sigma_{\kappa\neq 0}L_j^2\mathbf{P}_j^{\mu}(\kappa)N^{\mu\nu}(\kappa)\mathbf{P}_j^{\nu}(-\kappa)$ entering the exponent in the characteristic functional (64). The according calculations lead to the following result:

$$\sum_{\kappa\neq 0}L_j^2\mathbf{P}_j^{\mu}(\kappa)N^{\mu\nu}(\kappa)\mathbf{P}_j^{\nu}(-\kappa)$$
$$=\int_0^{L_j}\int_0^{L_j}d\xi'\,d\xi''\mathbf{P}_j^{\mu}(\xi')N^{\mu\nu}(\xi'-\xi'')\mathbf{P}_j^{\nu}(\xi_j''), \tag{79}$$

where $N_j^{\mu\nu}(\xi_j'-\xi_j'')$ is the Fourier pretransform of $N_j^{\mu\nu}(\kappa)$ minus zero harmonic

$$N^{\mu\nu}(\xi_j'-\xi_j'')=\sum_{\kappa\ \mathrm{all}}e^{i\kappa(\xi_j'-\xi_j'')}N^{\mu\nu}(\kappa)-N^{\mu\nu}(\kappa=0). \tag{80}$$

The matrix $N^{\mu\nu}(\xi_j'-\xi_j'')$ will be evaluated separately for diagonal and for nondiagonal terms of the matrix $N^{\alpha\beta}(\kappa)$ [Eq. (73)]. Let us start with the diagonal terms which are to be obtained from

$$N^{\alpha\alpha}(\xi_j'-\xi_j'')=N^{\alpha\alpha}\left\{\sum_{\kappa \text{ all}} e^{i\kappa(\xi_j'-\xi_j'')}\exp\left[-\left(\frac{2\pi\xi_0}{L_j}n\right)^2\right]-1\right\}. \tag{81}$$

Using again $\Sigma_{n\text{all}}\to\int dn$ one obtains

$$N^{\alpha\alpha}(\xi_j'-\xi_j'')=\frac{I_{\alpha\alpha}}{2(1-2\xi_0\sqrt{\pi}/L)}\left\{\exp\left[-\frac{(\xi_j'-\xi_j'')^2}{4\xi_0^2}\right]-\frac{2\xi_0\sqrt{\pi}}{L_j}\right\}. \tag{82}$$

As for the nondiagonal terms, similar calculations lead to the result

$$N^{xy}(\xi_j'-\xi_j'')=\frac{I_l\mathcal{L}_v^{1/2}}{2}(\xi_j'-\xi_j'')\exp\left[-\frac{(\xi_j'-\xi_j'')^2}{4\xi_0^2}\right],$$

$$N^{xy}=-N^{yx}. \tag{83}$$

Because of the change $\Sigma_{n\text{ all}}\to\int dn$ formulas (82) and (83) do not satisfy the condition of periodicity and the closeness of the loops:

$$N^{\mu\nu}(\xi_j'-\xi_j'')=N^{\mu\nu}(L_j+\xi_j'-\xi_j'').$$

This disagreement can be remedied by the substitution of

$$N^{\mu\mu}(\xi_j'-\xi_j'')\to\{N^{\mu\mu}(\xi_j'-\xi_j'')+N^{\mu\mu}[L_j-(\xi_j'-\xi_j'')]\}, \tag{84}$$

for diagonal (even) elements and by

$$N^{xy}(\xi_j'-\xi_j'')\to\{N^{xy}(\xi_j'-\xi_j'')-N^{xy}[L_j-(\xi_j'-\xi_j'')]\}, \tag{85}$$

for nondiagonal (odd) elements. Since $N^{\mu\nu}(\xi_j'-\xi_j'')$ is a sharply decreasing function [for $(\xi_j'-\xi_j'')\geqslant\xi_0$] this substitution does not significantly change the behavior of $N^{\mu\nu}(\xi_j'-\xi_j'')$ for small $(\xi_j'-\xi_j'')$ but improves the situation for $(\xi_j'-\xi_j'')\simeq L_j$. We will retain hereafter the previous notation for redefined matrix elements for it will not lead to confusion.

Finally the characteristic functional in ξ space has the following form:

$$W(\{\mathbf{P}_j(\xi_j)\})=\exp\left(-\sum_j\int_0^{L_j}\int_0^{L_j}\mathbf{P}_j^{\mu}(\xi_j')\times N^{\mu\nu}(\xi_j'-\xi_j'')\mathbf{P}_j^{\nu}(\xi_j'')\right), \tag{86}$$

where the diagonal terms of matrix $N^{\mu\nu}(\xi_j'-\xi_j'')$ are

$$N^{\alpha\alpha}(\xi_j'-\xi_j'')=\frac{I_{\alpha\alpha}}{2(1-2\xi_0\sqrt{\pi}/L)}\left(\exp\left[-\frac{(\xi_j'-\xi_j'')^2}{4\xi_0^2}\right]+\exp\left[-\frac{[L_j-(\xi_j'-\xi_j'')]^2}{4\xi_0^2}\right]-\frac{2\xi_0\sqrt{\pi}}{L_j}\right), \tag{87}$$

and the nondiagonal ones are

$$N^{xy}(\xi_j'-\xi_j'')=\frac{I_l\mathcal{L}_v^{1/2}}{2}\left((\xi_j'-\xi_j'')\exp\left[-\frac{(\xi_j'-\xi_j'')^2}{4\xi_0^2}\right]-[L_j-(\xi_j'-\xi_j'')]\exp\left[-\frac{(L_j-\xi_j'-\xi_j'')^2}{4\xi_0^2}\right]\right). \tag{88}$$

The element $N^{yx}(\xi_j'-\xi_j'')=-N^{xy}(\xi_j'-\xi_j'')$; the other terms are zero.

It is easy to check that the use of the matrix $N^{\mu\nu}(\xi_j'-\xi_j'')$ leads to correct (with accuracy up to ξ_0/L_j) values for quantities $\langle\mathbf{s}'\mathbf{s}'\rangle,\langle\mathbf{s}''\mathbf{s}''\rangle,\langle(\mathbf{s}'^x\mathbf{s}''^y-\mathbf{s}'^y\mathbf{s}''^x)\rangle$, etc. Indeed using the rules for working with the characteristic functional $W(\{\mathbf{P}_j(\xi_j)\})$ in ξ space described in Sec. II C one obtains

$$\langle\mathbf{s}_{j\alpha}'(\xi_j)\mathbf{s}_{j\alpha}'(\xi_j)\rangle=\left.\frac{\delta^2W}{i\,\delta\mathbf{P}_j^{\alpha}(\xi_j)i\,\delta\mathbf{P}_j^{\alpha}(\xi_j)}\right|_{\text{all }\mathbf{P}_j=0}=I_{\alpha\alpha}, \tag{89}$$

$$\langle(\mathbf{s}_{jx}'\mathbf{s}_{jy}''-\mathbf{s}_{jy}'\mathbf{s}_{jx}'')\rangle=2\frac{\partial}{\partial\xi_{j2}}\left.\frac{\delta^2W}{i\,\delta\mathbf{P}_j^{x}(\xi_j)i\,\delta\mathbf{P}_j^{y}(\xi_{j2})}\right|_{\text{all }\mathbf{P}_j=0,\ \xi_1=\xi_2}=I_l\mathcal{L}_v^{1/2}, \tag{90}$$

$$\langle\mathbf{s}_{j\alpha}''(\xi_j)\mathbf{s}_{j\alpha}''(\xi_j)\rangle=\frac{\partial^2}{\partial\xi_{j1}\partial\xi_{j2}}\times\left.\frac{\delta^2W}{i\,\delta\mathbf{P}_j^{\alpha}(\xi_j)i\,\delta\mathbf{P}_j^{\alpha}(\xi_j)}\right|_{\text{all }\mathbf{P}_j=0,\ \xi_1=\xi_2}=\frac{\partial^2}{\partial\xi_{j1}\partial\xi_{j2}}\exp\left[-\frac{(\xi'-\xi'')^2}{4\xi_0^2}\right]=\frac{1}{2\xi_0^2}=c_2^2\mathcal{L}_v. \tag{91}$$

These expressions are in full agreement with relations (3)–(16) making our scheme self-consistent.

V. SOME STATISTICAL PROPERTIES OF THE VORTEX TANGLE

In this section we describe some statistical properties of the vortex tangle which emerged from the formalism developed above. We restrict ourselves to the calculation of the simplest characteristics to see what the possible arrangement of the vortex tangle stemming from the trial distribution function is. In particular we calculate the correlation function between orientations of different elements of the lines, between the tangent vector and the vector of curvature, etc. We also calculate the average distance between different parts of the loops and, correspondingly, their sizes. Using these calculations we discuss the distribution of different loops over their lengths.

A. Correlation functions

Let us begin with an evaluation of the correlation function between orientations of different elements of the line. It is immediately obtained taking the functional derivative from characteristic functional $W(\{\mathbf{P}_j(\xi_j)\})$ in ξ_j representation (86) and from relation (32):

$$\langle \mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}'_{j\alpha}(\xi_{j2})\rangle = \frac{I_{\alpha\alpha}}{(1-2\xi_0\sqrt{\pi}/L)}\left(\exp\left[-\frac{(\xi_{j1}-\xi_{j2})^2}{4\xi_0^2}\right] + \exp\left[-\frac{[L_j-(\xi_{j1}-\xi_{j2})]^2}{4\xi_0^2}\right] - \frac{2\xi_0\sqrt{\pi}}{L_j}\right). \tag{92}$$

Inspecting relation (92) one concludes that close points $\xi_{j1}-\xi_{j2}\leqslant\xi_0$ and points satisfying $[L_j-(\xi_{j1}-\xi_{j2})]\leqslant\xi_0$ are strongly correlated (the latter condition appears because of the closeness of the loops). Then this correlation weakens as $\exp[-(\xi_{j1}-\xi_{j2})^2/4\xi_0^2]$ turning into a δ-correlated structure:

$$\exp[-(\xi_{j1}-\xi_{j2})^2/4\xi_0^2]\sim 2\sqrt{\pi}\xi_0\delta(\xi_{j1}-\xi_{j2}). \tag{93}$$

Thus we arrive at a very important conclusion. The correlation length of orientations of different parts of the curve is of the order of the mean radius of curvature or, in accordance with Eq. (77), of the order of the interline space. This view corresponds to current notions of the vortex tangle and discussed previously by Schwarz.[11,13] It is worth noting that there is a small negative correlation between distant points due to the term $-2\xi_0\sqrt{\pi}/L_j$ on the right-hand side of relation (92). The origin of this term is connected with the closeness of the line because each of the elements of the line ''remembers'' that the whole line should return to the initial point. Discarding this effect, the correlations between remote (along the curve, $\xi_0\ll\xi_{j1}-\xi_{j2}$) points vanish and the line takes on a random walk structure.

The correlation between different vectors of curvature $\langle \mathbf{s}''_{j\alpha}(\xi_{j1})\mathbf{s}''_{j\alpha}(\xi_{j2})\rangle$ behaves in a similar manner. The only exception is that a small negative correlation disappears because of the differentiation over ξ_j.

In a similar way the correlation between derivatives of different orders can be evaluated. Let us consider, e.g., the correlation between the tangent vector and the vector of curvature $\langle \mathbf{s}'^{\alpha}_j(\xi_{j1})\mathbf{s}''^{\alpha}_j(\xi_{j2})\rangle$. It has to be evaluated as

$$\langle \mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}''_{j\alpha}(\xi_{j2})\rangle = \frac{\partial}{\partial\xi_{j2}}\langle \mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}'_{j\alpha}(\xi_{j2})\rangle.$$

The interesting feature of this quantity is that it is zero when $\xi_{j1}=\xi_{j2}$, then it grows reaching its maximum value at point $\xi_{j1}-\xi_{j2}\sim\xi_0/2$, then this growth is changed with the usual exponential decay.

As far as the correlations between different components of vector $\mathbf{s}_j(\xi_j)$ and its derivatives are concerned, it follows from a similar consideration that xz and yz correlations are absent. The strong correlation between x and y components appears due to nonzero polarization and depends on the order of derivatives entering the expression. If the difference of orders is an odd number then the correlation function behaves in a usual way having a maximum value in point $\xi_{j1}-\xi_{j2}=0$ with a subsequent exponential decay. If this difference is an even number the correlation function behaves similar to the correlation function $\langle \mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}''_{j\alpha}(\xi_{j2})\rangle$.

B. Average size of loops

Let us now calculate the quantity $\langle(\mathbf{s}^{\alpha}(\xi_j)-\mathbf{s}^{\alpha}(0))^2\rangle$ (here is assumed a summation over α) which is the average squared distance between the initial point of the curve $\mathbf{s}(0)$ and the points $\mathbf{s}(\xi_j)$. Note that we deal with real distance in the usual space (not along the curve), therefore this consideration concerns the real size of the vortex loop embedded in 3D space. Note also that if one did not accomplish the summation over α, this quantity would describe the size of the loop along the α axes. Using the ξ presentation of the characteristic functional (86), the quantity $\langle(\mathbf{s}^{\alpha}(\xi_j)-\mathbf{s}^{\alpha}(0))^2\rangle$ is rewritten as follows:

$$\begin{aligned}\langle(\mathbf{s}_{\alpha}(\xi_j)-\mathbf{s}_{\alpha}(0))^2\rangle &= \int_0^{\xi_j}\int_0^{\xi_j} d\xi_{j1}d\xi_{j2}\langle \mathbf{s}'_{j\alpha}(\xi_{j1})\mathbf{s}'_{j\alpha}(\xi_{j2})\rangle \\ &= \int_0^{\xi_j}\int_0^{\xi_j} d\xi_{j1}d\xi_{j2}\frac{I_{\alpha\alpha}}{(1-2\xi_0\sqrt{\pi}/L)} \\ &\times\left(\exp\left[-\frac{(\xi_{j1}-\xi_{j2})^2}{4\xi_0^2}\right] + \exp\left[-\frac{[L_j-(\xi_{j1}-\xi_{j2})]^2}{4\xi_0^2}\right] - \frac{2\xi_0\sqrt{\pi}}{L_j}\right).\end{aligned} \tag{94}$$

For $\xi_j\leqslant\xi_0$ the exponent is close to the unit and with accuracy $2\xi_0\sqrt{\pi}/L_j$ we conclude that the average squared distance in the α direction is

$$\langle[\mathbf{s}_{\alpha}(\xi_j)-\mathbf{s}_{\alpha}(0)]^2\rangle = \xi_j^2 I_{\alpha\alpha}, \tag{95}$$

or the full distance is

$$\langle(\mathbf{s}(\xi_j)-\mathbf{s}(0))^2\rangle = \xi_j^2. \tag{96}$$

In the intermediate region of argument $\xi,\xi_0\leqslant\xi_j\leqslant L_j-\xi_0$, the exponent can be approximately replaced by a δ function [see relation (93)], which together with Eq. (94) gives the following result (with accuracy $2\xi_0\sqrt{\pi}/L_j$):

$$\langle[\mathbf{s}_{\alpha}(\xi_j)-\mathbf{s}_{\alpha}(0)]^2\rangle \sim 2\xi_0 I_{\alpha\alpha}\sqrt{\pi}(\xi_j-\xi_j^2/L). \tag{97}$$

Note that the quantity $-2\xi_0\sqrt{\pi}/L_j$ was disregarded only in the denominator of relation (94) whereas it was retained in the numerator, where its contribution is comparable with the

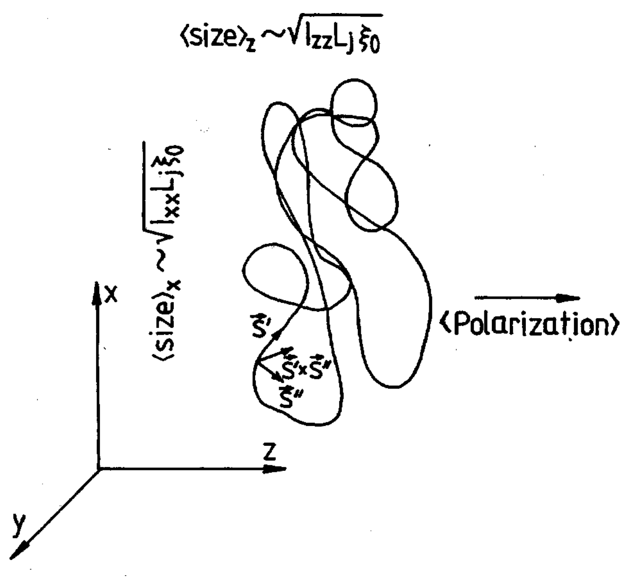

FIG. 5. Snapshot of the ''average'' vortex loop obtained from the analysis of the statistical properties. Close ($\Delta\xi \ll R$) parts of the line are separated in 3D space by distance $\Delta\xi$. The distant parts ($R \ll \Delta\xi$) are separated in 3D space by the distance $\sqrt{2\pi R\Delta\xi}$ (with a correction due to the closeness, see text). As a whole the loop is not isotropic having a ''pancake'' shape with different sizes in longitudinal and transverse directions. In addition the loop has a total average polarization $\langle \int \mathbf{s}_j'(\xi_j)\times\mathbf{s}_j''(\xi_j)d\xi_j\rangle$ forcing the loop to drift along the vector $\mathbf{V}_n$.

one from the exponential terms. The reason for this is that for ξ_j larger than ξ_0 (but smaller than L_j) the vortex line has a random walk structure and the distance between initial point $\mathbf{s}^\alpha(0)$ and point $\mathbf{s}^\alpha(\xi_j)$ increases as $\sqrt{\xi_j}$. The role of the term $-2\xi_0\sqrt{\pi}/L_j$ is to force the line back to assure that $[\mathbf{s}_\alpha(\xi_j)-\mathbf{s}_\alpha(0)]\to 0$, when $\xi_j \to L_j$.

Relation (97) should, however, be corrected in the region near the end of the line $L_j-\xi_0 \leqslant \xi_j$. In this region the main contribution will appear from the second exponent in the right-hand side of relation (94). This contribution will prevail the quantity $(\xi_j-\xi_j^2/L)$ and the final result is

$$\langle[\mathbf{s}_\alpha(\xi_j)-\mathbf{s}_\alpha(0)]^2\rangle=(L_j-\xi_j)^2,$$

which is obvious due to the periodicity.

Summarizing the results obtained in this subsection we conclude that the vortex loop behaves as a flexible polymer (see, e.g., Ref. 23). The small parts of the line behave as rodlike polymers whose lengths are exactly equal to distance $\xi'-\xi''$ along the curve (see Fig. 5). At larger distances the filament has a random walk structure with the effective bond length of the order of the correlation length ξ_0 or of the order of the mean radius of the curvature, or of the order of interline space. Because of the closeness condition the pure random walk structure $|\mathbf{s}_\alpha(\xi_j)-\mathbf{s}_\alpha(0)| \propto \sqrt{\xi_j}$ is violated and changes by dependence (97). In addition, due to anisotropy the whole average loop has a ''pancake'' form in the z direction. In addition, since the vortex filaments are orientated, unlike polymer chains, there is an anisotropy related with the mutual orientation of vectors $\mathbf{s}'$ and $\mathbf{s}''$. Thus the vortex loop as a whole has nonzero polarization $\langle\int_0^{L_j}\mathbf{s}_j'(\xi_j)\times\mathbf{s}_j''(\xi_j)d\xi_j\rangle$, and as a result it should have some drift velocity as well as inducing a nonzero mean superfluid velocity.

The analysis carried out above shows that the loop of length L_j has a size $D_j\sim\sqrt{\xi_0 L_j}$, therefore the volume occupying by the j loop is $V_j\sim\xi_0^{3/2}L_j^{3/2}$. The corresponding vortex line density $\mathcal{L}_j$ is $\mathcal{L}_j\sim L_j/\xi_0^{3/2}L_j^{3/2}=(1/\xi_0^2)(\xi_0^{1/2}/L_j^{1/2})$ $\sim\mathcal{L}_v(\xi_0^{1/2}/L_j^{1/2})$. Because of the condition $\xi_0\ll L_j$, the vortex line density $\mathcal{L}_j$ of the single loop is smaller than total vortex line density $\mathcal{L}_v$. This implies that the real vortex tangle has to consist of many loops. Due to the lack of information about the distribution of the loop lengths over j (see Sec. II A) we are not able to ascertain the fine structure of the many-loop vortex tangle. This obstacle is not serious since many physical effects are determined only by the orientation of the line element and their polarization. In other words, the corresponding quantities are additive over the number of loops (see Sec. II B). However, for other problems the question of the distribution of loops over their lengths can be relevant. In this case it is possible to introduce some averaged length L_j which should be considered as a parameter of the developed approach.

VI. CONCLUSION

We now summarize the obtained results and revise the main steps of the developed approach. The main result can be formulated as follows. Based on the well-established experimental data on the vortex tangle structure in He II we constructed a trial distribution function in the space of the vortex loop configurations of the most general form compatible with these data. We assume further that a trial distribution function obtained in this way will enable us to calculate various averages over vortex loop configurations. The use of the characteristic functional simplifies the calculation of these quantities.

Let us discuss once more the assumptions made while developing the whole procedure and outline the class of the problems which can be resolved with the method of the trial distribution function. The main premise of our approach was relation (37) expressing that the allowed configurations corresponding to the same macroscopic state have equal probabilities. This assumption is widely used to solve problems of equilibrium states and it seems quite reasonable for our problem. We can refer to the work of Polyakov[25] on classical turbulence where it was noted that, ''One can say that while Gibbs' distributions are uniform on surfaces of fixed values of conserved quantities, the turbulent distributions are located on surfaces of constant fluxes of the corresponding quantities.''

The next question which we would like to discuss concerns the constraints imposed by relations (3)–(16). Of course these few properties are by no means the full description of the vortex tangle structure and the question of whether the trial distribution function, satisfying only a few selected conditions, is adequate enough to evaluate correctly other quantities. One more question is what the possible restrictions on the class of these quantities would be. We can give the following answer to these questions. Regardless of the fact that there were not many input conditions (3)–(16), they include almost all the requested information concerning the orientation of the vortex line elements and their curvatures. In other words these input conditions involve almost all information concerning the first and second derivatives of functions $\mathbf{s}_j(\xi_j)$, $\mathbf{s}_j'(\xi_j)$, and $\mathbf{s}_j''(\xi_j)$, respectively. This in turn implies that the quantities of interest containing derivatives of not too high order can be evaluated correctly by the use of the trial distribution function. But it also implies that expressions containing derivatives of higher orders can hardly be calculated correctly in this way. In any case the reliability of the according calculations will not be too high, although they can be taken as a rough estimation. However, we do not know an example of the quantities expressed via high order derivatives and bearing any physical interest. On the contrary, the quantities of physical interest are expressed via derivatives of first and second order. There is a wide class of such quantities and of the corresponding effects. A number of examples were given in Sec. II.

The trial distribution function was derived based on an instantaneous picture of the vortex tangle and, as a consequence, the dynamical properties (except those which deal with the small deviation from the ''equilibrium'' state) drop out of consideration. In particular we are not able to answer the question of how the structure of the vortex tangle develops. We are also not able to answer how it appeared. We suppose that this structure is the result of very subtle and very involved dynamical nonequilibrium processes which unfortunately cannot be described analytically because of the complexity of the problem. In this connection we would like to note that attempts to describe stochastic properties of classical vortex filaments based on principles of the equilibrium Boltzman's statistics seem incorrect. In contrast, our model is based mainly on the experimental data, therefore it is a rather phenomenological one. It does not explain how certain arrangements of the vortex tangle appear, instead it assigned to calculate various averages over the vortex loop configurations. Put another way, the developed approach can be considered as a convenient and simple ''tool'' for the evaluation or estimation of various effects due to the presence of the vortex tangle in turbulent superfluid helium.

ACKNOWLEDGMENTS

This work was partly supported by Grant No. 96-02-19414 from the Russian Foundation of Basic Research. Different parts of the work were presented at the University of Tsukuba (Japan, 1996), at workshops *Microgravity in Low Temperature Physics*, Pasadena, 1995, 1996, at the *Conference on Low Temperature Physics LT21*, Prague, 1996, and at the *Symposium on Quantum Fluids and Crystals QFS97*, Paris, 1997. I would like to thank the participants of these meetings for very fruitful discussions. I am particularly grateful to Professor G. Ahlers, Professor M. Tsubota, Professor W. Glaberson, Professor C. Barenghi, Professor D. Samuels, Professor G. Williams, and Professor M. Murakami for their stimulating questions and discussions.

*Electronic address: nemirov@otani.thermo.nsk.su
Internet address: nemir@itp.nsc.ru

[1] R.J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, England, 1991)

[2] J.T. Tough, in *Progress in Low Temperature Physics*, edited by D.F. Brewer (North-Holland, Amsterdam, 1982), Vol. 8, p. 133.

[3] S.K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. **67**, 37 (1995).

[4] R.P. Feynman, in *Progress in Low Temperature Physics*, edited by C.J. Gorter (North-Holland, Amsterdam, 1955), Vol. I, p. 17.

[5] We place the term equilibrium inside quotation marks since this state is not one of equilibrium in the thermodynamic sense. Indeed, the presence of the friction force and of the cascadelike processes of the break down of the vortex loops imply that the steady state is reached as a result of essentially nonequilibrium processes (see discussion below). At the same time we do not use the term steady state since in general the vortex line density can change in time following the change of the counterflow velocity.

[6] W.F. Vinen, Proc. R. Soc. London, Ser. A **240**, 114 (1957); **240**, 128 (1957); **242**, 493 (1957).

[7] W.F. Vinen, Proc. R. Soc. London, Ser. A **243**, 400 (1958).

[8] S.K. Nemirovskii and V.V. Lebedev, Sov. Phys. JETP **57**, 1009 (1983).

[9] J.A. Geurst, Physica B **154**, 327 (1989).

[10] K. Yamada, S. Kashiwamura, and K. Miyake, Physica B **154**, 318 (1989).

[11] K.W. Schwarz, Phys. Rev. B **18**, 245 (1978).

[12] K.W. Schwarz, Phys. Rev. B **31**, 5782 (1985).

[13] K.W. Schwarz, Phys. Rev. B **38**, 2398 (1988).

[14] S.K. Nemirovskii, J. Pakleza, and W. Poppe, Rus. J. Eng. Thermophysics **3**, 369 (1993).

[15] S.K. Nemirovskii, following paper, Phys. Rev. B **57**, 5987 (1998).

[16] R.T. Wang, C.E. Swanson, and R.J. Donnelly, Phys. Rev. B **36**, 5240 (1987).

[17] H.E. Hall, Adv. Phys. **9**, 89 (1960).

[18] G.K. Batchelor, *An Introduction to Fluid Mechanics* (Cambridge University Press, Cambridge, England, 1967).

[19] P.G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, England, 1992).

[20] M. Rasetti and T. Regge, Physica A **80**, 217 (1975).

[21] C.E. Swanson and R.J. Donnelly, J. Low Temp. Phys. **61**, 363 (1985).

[22] A.S. Monin and A.M. Yaglom, *Statistical Fluid Dynamics* (MIT, Cambridge, MA, 1975), Vol. II.

[23] M. Doi and S.F. Edwards, *The Theory of Polymer Dynamics* (Clarendon, Oxford, 1986).

[24] The effective Lagrangian has a bilinear form over the ''field'' variables $\mathbf{s}_j(\kappa)$, therefore, all path integrals entering our approach are Gaussian ones, hence the title of our paper.

[25] A.M. Polyakov, Nucl. Phys. B **396**, 367 (1993).