Eliashberg equations of electrons on a square lattice: Interplay of strong electron-phonon interactions and electron correlations

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We discuss vertex corrections to the Eliashberg equations of correlated electrons due to strong electron-phonon interactions. Local electron correlations have been taken into account in the mean-field approximation for slave-boson fields in the $U\rightarrow\infty$ limit of the two-dimensional Hubbard model. The modification of the wave-function renormalization factor originating from vertex corrections has been evaluated in a self-consistent way. We have considered both ionic and covalent parts of the electron-phonon interaction as possible pairing mechanisms. For the ionic contribution at low doping (strong-coupling limit), inclusion of vertex corrections leads to a decrease of the superconducting transition temperature T_c for d-wave symmetry. This decrease agrees with the experimentally observed dependence of T_c on the concentration of holes and originates from the enhancement of the density of states near the van Hove singularities due to correlations. Another important effect originating from inclusion of vertex corrections is a pronounced reduction of the isotope shift exponent α at optimal doping. This supports the view that experimentally observed small values of α still allow for phonon-induced superconductivity in strongly correlated systems. We discuss the controversial question with respect to the sign with which the vertex corrections contribute to the pairing kernels of the Eliashberg equations of strongly correlated electron-phonon systems. [S0163-1829(98)07101-X]

I. INTRODUCTION

It is generally accepted that the symmetry of the order parameter of high-temperature superconductors is related to the underlying pairing mechanism. There is evidence that strong electron-phonon interaction[1–4] will lead to a modification of phononic properties below the superconducting transition temperature.[5–8] This supports the view that coupling of electrons to phonons is involved in the formation of the superconducting state. This possibility has also been considered in Refs. 2 and 9–21. On the other hand, the proximity of the antiferromagnetic and superconducting phase indicates that Coulomb correlations can lead to a nonphononic pairing mechanism originating from the exchange of antiferromagnetic spin fluctuations.[22–29] The resulting gap function with d-wave symmetry is in agreement with increasing experimental data.[30–33] However, the data do not exclude admixture of components with s-like symmetry[34,35] nor do the data allow one to decompose the gap function into contributions due to phonon or correlation induced pairing.

High-temperature superconductors can be characterized as systems, where correlations are responsible for narrow quasiparticle bands, lifetime effects of one-particle states close to the Fermi energy, and Fermi energies of the order of the Debye energy which are smaller than in conventional superconductors.[36,37] To some extent superconducting fullerenes have similar properties.[38,39] Therefore, one has to be careful when constructing a strong-coupling theory of high-temperature superconductivity, since both electron correlations as well as electron-phonon interactions will be responsible for strong-coupling features. Also there is no guarantee that Migdal's theorem[40] is fulfilled *a priori*. This means that we need an extension of Eliashberg's formalism[41] which takes care of the interplay of electron-phonon interactions of strongly correlated electrons and of the hitherto neglected vertex corrections to the Eliashberg equations. Related investigations can be found in a few recent papers.[42–51] In particular the role of van Hove singularities in nonadiabatic effects has been discussed.[51] There are also some contradictory results with respect to the question, whether vertex corrections play a dominant role in the weak-coupling[42] or in the strong coupling limit[46] (see discussion in Ref. 49). The aim of the present paper is to clarify certain aspects connected with vertex corrections for the case of phonon-mediated anisotropic superconductivity in a strongly correlated system described by the two-dimensional Hubbard model. Local correlations will be considered in the mean-field approximation for slave-boson fields.[52] Although this approximation neglects dynamic features of correlated electrons, it allows us to compare the pairing originating from the ionic part of electron-phonon interaction (which remains finite for vanishing concentration of holes) with that from the covalent part (which is strongly reduced close to half filling). Using this scheme we find that d-wave symmetry dominates for small doping. We have found that vertex corrections of the ionic part of the electron-phonon interaction lead to a remarkable decrease of T_c at low doping. This is due to the fact that contributions to the pairing kernel and wave-function renormalization factor originating from the vertex corrections are of the same order as the contributions without vertex corrections. This effect is connected with the enhancement of the quasiparticle density of states near the van Hove singularities due to correlations at half-filling. The inclusion of vertex corrections modifies the concentration dependence

of the superconducting transition temperature making it similar to experimental data. This is not the case when we only consider contributions originating from the covalent part of electron-phonon interaction.

Vertex corrections play an important role when considering the isotope effect. We have found that the isotope shift exponent takes on vanishingly small values at optimal doping. This feature remains in agreement with experimental data for high-temperature superconductors.

II. ELECTRON-PHONON VERTEX CORRECTIONS FOR THE TWO-DIMENSIONAL HUBBARD MODEL

We describe correlated electrons by the two-dimensional Hubbard model in the limit $U\rightarrow\infty$ including electron-phonon interactions. In the mean-field slave-boson approximation[52] the Hamiltonian takes the form[11,16,18,19]

$$H=H_0+H_I, \tag{1}$$

where

$$H_0=\sum_{\mathbf{k}}\ \widetilde{\varepsilon}_{\mathbf{k}}\Psi^+_{\mathbf{k}}\tau_3\Psi_{\mathbf{k}}+\sum_{\mathbf{q}}\ \omega_{\mathbf{q}}B^+_{\mathbf{q}}B_{\mathbf{q}}, \tag{2}$$

and

$$H_I=\sum_{\mathbf{k},\mathbf{q}}\ \widetilde{g}_{\mathbf{k}\mathbf{k}+\mathbf{q}}\Psi^+_{\mathbf{k}+\mathbf{q}}\tau_3\Psi_{\mathbf{k}}(B_{\mathbf{q}}+B^+_{-\mathbf{q}}) \tag{3}$$

with the Nambu notation $\Psi^+_{\mathbf{k}}=(f^+_{\mathbf{k}\uparrow}f_{-\mathbf{k}\downarrow})$.[41] The renormalized band energy is $\widetilde{\varepsilon}_{\mathbf{k}}=r^2\varepsilon_{\mathbf{k}}-\widetilde{\mu}$ with $\varepsilon_{\mathbf{k}}=-t\gamma(\mathbf{k})$ and $\gamma(\mathbf{k})=2(\cos k_xa+\cos k_ya)$ for nearest-neighbor hopping t; $\widetilde{\mu}$ is the chemical potential which includes a Lagrange multiplier introduced to guarantee the exclusion of double occupancy of lattice sites;[52] $r^2=1-n$ is the bandwidth narrowing factor with the average number of electrons per site given by $n=(1/N)\Sigma_{\mathbf{k}\sigma}\langle f^+_{\mathbf{k}\sigma}f_{\mathbf{k}\sigma}\rangle\leqslant 1$; $\widetilde{g}_{\mathbf{k}\mathbf{k}+\mathbf{q}}$ is the effective electron-phonon interaction. We will separately consider two contributions, (i) the ionic part for which $\widetilde{g}_{\mathbf{k}\mathbf{k}+\mathbf{q}}=g_{\mathbf{q}}$ holds and which is thought to arise from the volume dependence of the on-site energy $-t_0$ of the correlated electrons, and (ii) the covalent part due to the volume dependence of the two-center integral $-t$, for which we obtain $\widetilde{g}_{\mathbf{k}\mathbf{k}+\mathbf{q}}=r^2g_{\mathbf{k}\mathbf{k}+\mathbf{q}}$ in the mean-field slave-boson approximation.[11,16,18,19] In what follows phonons will be modeled by an Einstein oscillator of frequency ω_0.

We start the calculation by writing the Green's function in Wannier representation as

$$G_{ij}(i\omega_l)=\langle\langle\Psi_i|\Psi^+_j\rangle\rangle=\begin{pmatrix}\langle\langle f_{i\uparrow}|f^+_{j\uparrow}\rangle\rangle & \langle\langle f_{i\uparrow}|f_{j\downarrow}\rangle\rangle\\ \langle\langle f^+_{i\downarrow}|f^+_{j\uparrow}\rangle\rangle & \langle\langle f^+_{i\downarrow}|f_{j\downarrow}\rangle\rangle\end{pmatrix}$$
$$=G_{ij}(i\omega_l)_d+G_{ij}(i\omega_l)_{nd}, \tag{4}$$

where ω_n is the Matsubara frequency $\omega_n=\pi/\beta(2n+1)$; $\beta=(k_BT)^{-1}$. The superconducting order parameter can be evaluated from the nondiagonal part of $\langle\langle\Psi_i|\Psi^+_j\rangle\rangle(G_{nd})$, whereas the diagonal part (G_d) determines the normal-state properties for vanishing gap function. In the strongly correlated case ($U\rightarrow\infty$) only intersite Cooper pairs will survive. This allows us to approximate the momentum dependence of G_{nd} by

$$G_k(i\omega_n)_{nd}=\frac{1}{N}\sum_{i,j}\ e^{i\mathbf{k}(\mathbf{R}_i-\mathbf{R}_j)}G_{ij}(i\omega_n)_{nd}$$
$$\simeq\frac{1}{N}\sum_{\langle i,j\rangle}\ e^{i\mathbf{k}(\mathbf{R}_i-\mathbf{R}_j)}G_{ij}(i\omega_n)_{nd}$$
$$=\frac{1}{N}\sum_{\mathbf{p}}\ \gamma(\mathbf{k}-\mathbf{p})G_{\mathbf{p}}(i\omega_n)_{nd}, \tag{5}$$

where we have neglected second- and higher-order neighbor Cooper pairs. The relation

$$\gamma(\mathbf{k}\pm\mathbf{p})=\frac{1}{4}[\,\gamma(\mathbf{k})\,\gamma(\mathbf{p})+\eta(\mathbf{k})\,\eta(\mathbf{p})\mp\xi(\mathbf{k})\xi(\mathbf{p})$$
$$\mp\zeta(\mathbf{k})\zeta(\mathbf{p})], \tag{6}$$

where

$$\eta(\mathbf{k})=2(\cos\,k_xa-\cos\,k_ya),$$
$$\xi(\mathbf{k})=2(\sin\,k_xa+\sin\,k_ya),$$
$$\zeta(\mathbf{k})=2(\sin\,k_xa-\sin\,k_ya), \tag{7}$$

enables one to distinguish between extended s- [$\gamma(\mathbf{k})$] and d-wave [$\eta(\mathbf{k})$] contributions to singlet superconductivity. Triplet superconductivity given by ξ and ζ, is unlikely to occur in high-temperature superconductors[53,54] and will not be considered here. The identity

$$\frac{1}{N^2}\sum_{\mathbf{p},\mathbf{p}'}\ \gamma(\mathbf{k}-\mathbf{p}')\,\gamma(\mathbf{p}'-\mathbf{p})G_{\mathbf{p}}(i\omega_n)_{nd}$$
$$=\frac{1}{N}\sum_{\mathbf{p}}\ \gamma(\mathbf{k}-\mathbf{p})G_{\mathbf{p}}(i\omega_n)_{nd}, \tag{8}$$

indicates that the approximation in Eq. (5) corresponds to projecting out a given type of symmetry for the gap function.

In order to facilitate calculations we have used Kresin's method of introducing an average phonon frequency $\langle\Omega\rangle$ (Ref. 55)

$$\nu=\frac{\langle\Omega\rangle}{2\pi k_BT_c}, \tag{9}$$

which corresponds to the frequency of an Einstein oscillator ω_0. ($\omega_0=0.1t$ is used throughout this paper.[18,19]) When considering anisotropic superconductivity originating from nearest-neighbor Cooper pairs, one obtains two different electron-phonon coupling functions λ and λ_γ, where the first one determines the magnitude of the renormalization factor Z and the second one is responsible for anisotropic superconductivity.[18,19,56] We do not know the actual momentum dependence of $g_{\mathbf{k}\mathbf{k}+\mathbf{q}}$. Therefore, the electron-phonon coupling functions are parametrized by these two quantities, as defined in Ref. 18. Note that in the $U\rightarrow$ limit, the dominating contribution which determines the symmetry of the order parameter comes from the form factors [Eq. (6)] which

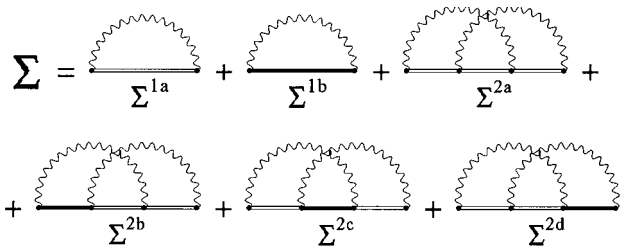

FIG. 1. Diagrammatic representation of diagonal and nondiagonal contributions to the self-energy at $T=T_c$ including vertex corrections denoted by Σ^{2a}, Σ^{2b}, Σ^{2c}, and Σ^{2d}. The double (thick) line represents the normal (anomalous) propagator and the wiggle line stands for the phonon Green's function.

originate from the intersite pairing. One should bear in mind that in the uncorrelated case the ratio λ_γ/λ depends on the explicit form of $g_{\mathbf{k}\mathbf{k}+\mathbf{q}}$ and determines the relative role of d- and s-wave symmetries. In general λ and λ_γ will depend on the occupation number,[56] however, in order to simplify the numerical work, we will consider them as parameters and use $\lambda_{(\gamma)}(n)=\lambda_{(\gamma)}=\text{const}$ for the ionic and $\lambda_{(\gamma)}(n)=(1-n)^2\lambda_{(\gamma)}$ $(n=0)$ for the covalent part of electron-phonon interaction.[11,18,19] This simplification does not influence the qualitative aspects of our numerical results.[56]

The matrix self-energy $\Sigma_{\mathbf{k}}(i\omega_n)$ is defined by Dyson's equation

$$G_{\mathbf{k}}(i\omega_n)=[i\omega_n\tau_0-\tilde{\varepsilon}_{\mathbf{k}}\tau_3-\Sigma_{\mathbf{k}}(i\omega_n)]^{-1}. \tag{10}$$

The lowest-order vertex corrections to $\Sigma_{\mathbf{k}}(T=T_c)$ (due to multiphonon contributions) which have been taken into account in this work, are sketched in Fig. 1. According to Migdal's theorem, multiphonon contributions of this type are of the order of $\sqrt{m/M}$ and usually are neglected. However, for the case of strong electron-phonon interaction and large effective mass renormalization due to strong local Coulomb correlations, these contributions will play a non-negligible role. The different contributions to the matrix self-energy are given by

$$\Sigma^{1a}(i\omega_l)=\frac{1}{\beta}\sum_n\frac{1}{N}\sum_{\mathbf{p}}\lambda\frac{\nu^2}{(l-n)^2+\nu^2}\tau_3 G_{\mathbf{p}}(i\omega_n)_d\tau_3, \tag{11}$$

$$\Sigma^{1b}_{\mathbf{k}}(i\omega_l)=\frac{1}{\beta}\sum_n\frac{1}{N}\sum_{\mathbf{p}}\gamma(\mathbf{k}-\mathbf{p})\lambda_\gamma\frac{\nu^2}{(l-n)^2+\nu^2}\times\tau_3 G_{\mathbf{p}}(i\omega_n)_{nd}\tau_3, \tag{12}$$

$$\Sigma^{2a}_{\mathbf{k}}(i\omega_l)=\frac{1}{\beta^2}\sum_{n,m}\frac{1}{N^2}\sum_{\mathbf{p},\mathbf{p}'}\lambda^2\frac{\nu^2}{(l-n)^2+\nu^2}\frac{\nu^2}{(n-m)^2+\nu^2}\times\tau_3 G_{\mathbf{p}}(i\omega_n)_d\tau_3 G_{\mathbf{p}'}(i\omega_m)_d\times\tau_3 G_{\mathbf{k}+\mathbf{p}'-\mathbf{p}}(i\omega_l+i\omega_m-i\omega_n)_d\tau_3, \tag{13}$$

$$\Sigma^{2b}_{\mathbf{k}}(i\omega_l)=\frac{1}{\beta^2}\sum_{n,m}\frac{1}{N^3}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\gamma(\mathbf{k}-\mathbf{p})\lambda\lambda_\gamma\frac{\nu^2}{(l-n)^2+\nu^2}\times\frac{\nu^2}{(n-m)^2+\nu^2}\tau_3 G_{\mathbf{p}}(i\omega_n)_{nd}\tau_3 G_{\mathbf{q}}(i\omega_m)_d\times\tau_3 G_{\mathbf{p}'}(i\omega_l+i\omega_m-i\omega_n)_d\tau_3, \tag{14}$$

$$\Sigma^{2c}_{\mathbf{k}}(i\omega_l)=\frac{1}{\beta^2}\sum_{n,m}\frac{1}{N^3}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\gamma(\mathbf{k}-\mathbf{p})\lambda\lambda_\gamma\frac{\nu^2}{(l-m)^2+\nu^2}\times\frac{\nu^2}{(m-n)^2+\nu^2}\tau_3 G_{\mathbf{p}'}(i\omega_m)_d\tau_3 G_{\mathbf{p}}(i\omega_n)_{nd}\times\tau_3 G_{\mathbf{q}}(i\omega_l+i\omega_n-i\omega_m)_d\tau_3, \tag{15}$$

$$\Sigma^{2d}_{\mathbf{k}}(i\omega_l)=\frac{1}{\beta^2}\sum_{n,m}\frac{1}{N^3}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\gamma(\mathbf{k}-\mathbf{p})\lambda\lambda_\gamma\frac{\nu^2}{(l-n)^2+\nu^2}\times\frac{\nu^2}{(n-m)^2+\nu^2}\tau_3 G_{\mathbf{p}'}(i\omega_l+i\omega_m-i\omega_n)_d\times\tau_3 G_{\mathbf{q}}(i\omega_m)_d\tau_3 G_{\mathbf{p}}(i\omega_n)_{nd}\tau_3. \tag{16}$$

The difference with respect to the usual form of the Eliashberg equations[49] originates from the fact that we have parametrized the actual momentum dependence of $g_{\mathbf{k}\mathbf{k}+\mathbf{q}}$ by λ and λ_γ. Here, we are dealing with a strongly correlated system where only anisotropic superconductivity (as being brought about by the nonlocal pairing) is possible. This shows up in the presence of the form factor $\gamma(\mathbf{k}-\mathbf{p})$ which comes from Eq. (5). The usual ansatz for the matrix self-energy is of the form[41,18,19]

$$\Sigma_{\mathbf{k}}(i\omega_n)=[1-Z_{\mathbf{k}}(i\omega_n)]i\omega_n\tau_0+\phi_{\mathbf{k}}(i\omega_n)\tau_1+\chi_{\mathbf{k}}(i\omega_n)\tau_3. \tag{17}$$

Here $\phi_{\mathbf{k}}(i\omega_n)$ denotes the momentum-dependent gap function determined by Eqs. (12) and (14)–(16):

$$\phi_{\mathbf{k}}(i\omega_n)=\gamma(\mathbf{k})\phi_\gamma(i\omega_n)+\eta(\mathbf{k})\phi_\eta(i\omega_n), \tag{18}$$

$Z_{\mathbf{k}}(i\omega_n)$ is the wave-function renormalization factor [Eqs. (11) and (13)] which can be considered as a momentum-independent quantity $Z_{\mathbf{k}}(i\omega_n)=Z(i\omega_n)$.[41,18,19] In what follows we will neglect the energy shift $\chi_{\mathbf{k}}(i\omega_n)$ which has been proved to be a small quantity in the electron-phonon problem.[41,19] This may not be the case when considering fluctuations of the slave-boson fields over the mean-field value.[56] For simplicity we assume that this shift can be absorbed in the chemical potential without changing the physical nature of the Cooper pairs. Substituting the above form of $\Sigma_{\mathbf{k}}(i\omega_n)$ into Eqs. (10)–(16), one can easily obtain a system of self-consistent equations for the gap function $\phi_{\gamma(\eta)}$ and the renormalization factor Z. This system of equations constitutes the Eliashberg equations which have been extended to account for the lowest-order vertex corrections. Since extended s- and d-wave symmetries separate at $T=T_c$, both cases can be investigated independently. The two-dimensional band structure has explicitly been taken into account when integrating Eqs. (11)–(16).

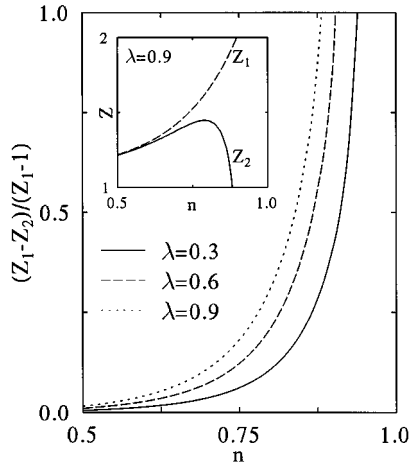

FIG. 2. The impact of vertex corrections to the wave-function renormalization factor as a function of band filling for the ionic part of electron-phonon interaction. Here Z_2 (Z_1) represents the contribution with (without) vertex corrections. The ratio $(Z_1-Z_2)/(Z_1-1)$ is plotted for the lowest Matsubara frequency, $\omega_{n=0}=\pi kT$ and $kT=0.01t$. The inset shows the dependence of Z_1 and Z_2 on band filling.

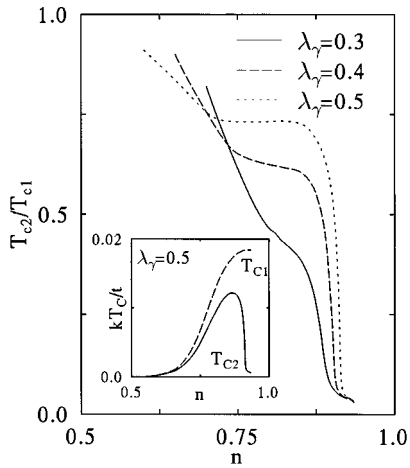

FIG. 4. The modification of the superconducting transition temperature (d wave) due to vertex corrections for the ionic part of electron-phonon interaction as a function of band filling. Here T_{c2} (T_{c1}) denotes T_c with (without) vertex corrections. The inset shows the dependence of T_{c1} and T_{c2} on band filling. In all cases we have taken $\lambda=0.6$.

III. DISCUSSION OF NUMERICAL RESULTS

The superconducting transition temperature and the renormalization factor Z have been evaluated as functions of band filling. Integrations over the momentum automatically take into account effects originating from the enhanced density of states near the van Hove singularities which means that the influence of strong correlations essentially is a band-narrowing effect. In order to get satisfactory convergence 100 Matsubara frequencies have been used in the summations.

The importance of vertex corrections for the wave-function renormalization factor is illustrated in Figs. 2 and 3. Z_2 (Z_1) represents the contribution with and without vertex corrections, respectively. We have plotted the ratio $(Z_1-Z_2)/(Z_1-1)$ for the lowest Matsubara frequency ($\omega_n=\pi kT$ and $kT=0.01t$) as a function of band filling. This ratio measures the convergence of the self-consistent perturbation series when doping the system below half-filling. Figure 2 shows the variation of this quantity as a function of n for the case when the ionic part of electron-phonon interaction is responsible for pairing and Fig. 3 shows the same for the covalent case. One notes that already for moderate values of λ we have corrections for the ionic part at low doping (Fig. 2). This is not the case for the covalent part, as is obvious from Fig. 3. In the latter case there is no significant difference between Z_2 and Z_1. This certainly is related to the prefactor of $(1-n)^2$ which enters the covalent part of electron-phonon coupling function as a result of the mean-field approximation for the slave-boson fields. This prefactor smears out effects connected with the enhancement of the van Hove singularities when approaching half-filling. Therefore, it is expected that superconductivity brought about by the volume dependence of the hopping integral will be less affected by vertex correction than pairing originating from the long-range electron-phonon interaction. In order to see this, we have evaluated the superconducting transition temperature for both cases. Figure 4 shows the ratio of T_{c2}/T_{c1} (with and without vertex corrections, respectively) as a function of the occupation number for the case of the ionic electron-phonon interaction as pairing mechanism. The strong reduction of T_c due to vertex corrections can be observed at low doping. Note, that this result has been obtained for the d-wave channel. For small and moderate doping the extended s-wave component does not play any role as compared to the d-wave contribution. It is remarkable that the inclusion of vertex corrections leads to results which correspond to the experimentally observed dependence of the superconducting transition temperature on the concentration of holes (see inset in Fig. 4). The decrease of T_c (for concentrations of holes exceeding optimal doping) originates from a decrease of $\langle \eta^2(\mathbf{k})\rangle_{\mathrm{FS}}$, whereas the decrease at low doping is caused by vertex corrections. Figure 5 shows the ratio T_{c2}/T_{c1} as a function of the occupation number for the case that now the covalent part of the electron-phonon interaction is responsible for pairing. The decrease of T_c is less pro-

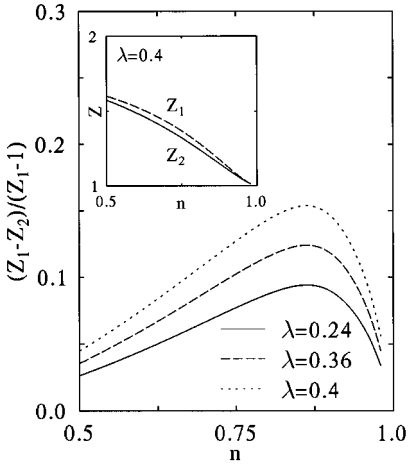

FIG. 3. Same as in Fig. 2 but for the covalent part of electron-phonon interaction. The values of λ refer to $n=0.8$.

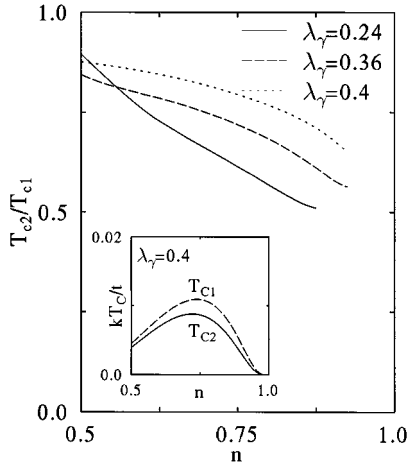

FIG. 5. Same as in Fig. 4 but for the covalent part of electron-phonon interaction. The values of $\lambda_{(\gamma)}$ refer to $n=0.8$. In all cases we have taken $\lambda=0.4$.

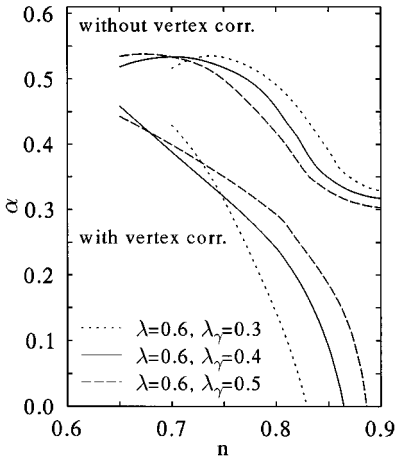

FIG. 6. The isotope shift exponent α originating from the ionic part of electron-phonon interaction as a function of the occupation number. Note the vanishingly small value of α at optimal doping.

nounced than in the previous case. However, in both cases the decrease is more pronounced for lower values of the transition temperature (smaller values of coupling function λ_γ). This feature is in agreement with observations in Ref. 42. Note that for the covalent part of electron-phonon interaction the dependence of T_c on n only is weakly modified when including vertex corrections (see inset in Fig. 4). The maximum around optimal doping is much broader than for the ionic case and, therefore, not comparable to experimental data. For the covalent part of electron-phonon interaction the shape of $T_c=T_c(n)$ predominantly is determined by the competition of $\langle \eta^2(\mathbf{k})\rangle_{\mathrm{FS}}$ (which decreases with the increasing concentration of holes) and $\lambda_{(\gamma)}$ which vanishes at half-filling. Important is that phonon-induced d-wave superconductivity in the presence of correlations can be characterized by moderate or small values of optimal doping for both types (covalent or ionic) of electron-phonon interaction. However, the value of optimal doping and details of the dependence of T_c on n are determined by the relative weight by which these contributions enter the equations.

There are experimental indications that the isotope effect in the copper oxides is inversely correlated with T_c.[57–59] Small values of the isotope shift exponent α may suggest that the electron-phonon interaction plays a secondary role in the formation of the superconducting state. In order to evaluate α we assume that the pairing kernels depends on M only through $\omega_0 \sim M^{-1/2}$. Then

$$\alpha=\frac{\omega_0}{2T_c}\frac{dT_c}{d\omega_0}. \tag{19}$$

Numerical results for α originating from the ionic part of electron-phonon interaction are shown in Fig. 6. The inclusion of vertex corrections leads to pronounced decrease of the isotope shift exponent. In particular, α takes on vanishingly small values at optimal doping, by far lower than the BCS value of 1/2. This is a plausible feature which corresponds to experimental results.[57–59] For lower doping α becomes negative but this may be related to the limited validity of the present theory for $n>0.9$ when vertex corrections are of the same magnitude as second-order contributions.

IV. CONCLUDING REMARKS

We have solved the momentum- and frequency-dependent Eliashberg equations for correlated electrons including vertex corrections caused by strong electron-phonon interactions. The two-dimensional band structure has explicitly been taken into account and strong local correlations have been incorporated in the mean-field approximation for the boson fields. We have made use of a Gauss-Kronrod type of routine for the momentum integration and carried out the summation over 100 Matsubara frequencies. Ionic and covalent parts of the electron-phonon interaction have been considered separately. Considering the symmetry of the superconducting state we have found that for low and moderate doping the extended s-wave component does not play any role in comparison to the d-wave contribution. Renormalization of T_c due to vertex corrections is most important at low doping and is caused by correlations which enhance the quasiparticle density of states at the van Hove singularities. Strong decrease of T_c at low doping occurs for pairing originating from the ionic part of the electron-phonon interaction. Inclusion of vertex corrections causes T_c to peak at optimal doping. This agrees with the experimental observation that T_c depends in a specific way on the concentration of holes. There is no such effect when considering the covalent part of electron-phonon interaction as being responsible for pairing. Vertex corrections give rise to the pronounced reduction of the isotope shift exponent at optimal doping. This result supports the view that small values of α do not eliminate electron-phonon interaction as a possible pairing mechanism in high-temperature superconductors. In order to explain small values of α one has not to assume that the non-phononic mechanism is switched on when approaching optimal doping. This effect arises within the electron-phonon picture due to multiphonon processes contained in vertex corrections.

Our results support the opinion that vertex corrections have a negative influence on superconductivity,[42,44,46] even for the case of anisotropic pairing. Note that in the present calculation the vertex corrections have been incorporated in a self-consistent way. The anomalous and normal parts of the Green's function including the wave-function renormaliza-

tion factor have been taken into account in all diagrams (standard Eliashberg contributions plus vertex corrections). There are also views that vertex corrections as considered here, can have a positive influence on superconductivity in the dynamical limit (small values of the scattering momentum and large values of the exchange frequency).[43,49,51] We have neglected the explicit dependence on the scattering momentum for the sake of a self-consistent evaluation of the wave-function renormalization factor and the superconducting transition temperature. It is the self-consistent wave-function renormalization which leads to the decrease of T_c in the presence of vertex corrections. This renormalization will compete with the possible enhancement of T_c in the dynamical limit and should be taken into account when discussing the sign of the vertex correction function. Therefore, the actual role of vertex corrections in the theory of high-temperature superconductivity needs further investigations. In order to bring this matter to an issue, one must consider the numerically difficult problem of how to account for the dependence on the scattering momentum in the self-consistent evaluation of vertex corrections. This would decide whether a change from negative to positive influence of vertex corrections on superconductivity is possible somewhere between optimal doping and half-filling. In addition one would need to include correlation-mediated contributions to the pairing function as well.

ACKNOWLEDGMENTS

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