# Theory of a spherical-quantum-rotors model: Low-temperature regime and finite-size scaling

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The quantum-rotors model can be regarded as an effective model for the low-temperature behavior of the quantum Heisenberg antiferromagnets. Here, we consider a $d$-dimensional model in the spherical approximation confined to a general geometry of the form $L^{d-d'}\times\infty^{d'}\times L_\tau^z$ ($L$-linear space size and $L_\tau$-temporal size) and subjected to periodic boundary conditions. Due to the remarkable opportunity it offers for rigorous study of finite-size effects at arbitrary dimensionality this model may play the same role in quantum critical phenomena as the popular Berlin-Kac spherical model in classical critical phenomena. Close to the zero-temperature quantum critical point, the ideas of finite-size scaling are utilized to the fullest extent for studying the critical behavior of the model. For different dimensions $1<d<3$ and $0\leqslant d'\leqslant d$ a detailed analysis, in terms of the special functions of classical mathematics, for the susceptibility and the equation of state is given. Particular attention is paid to the two-dimensional case. [S0163-1829(98)03209-3]

## I. INTRODUCTION

In recent years there has been a renewed interest[1–3] in the theory of zero-temperature quantum phase transitions initiated in 1976 by Hertz's quantum dynamic renormalization group[4] for itinerant ferromagnets. Distinctively from temperature-driven critical phenomena, these phase transitions occur at zero temperature as a function of some nonthermal control parameter (or a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature.

It is well known from the theory of critical phenomena that for the temperature-driven phase transitions quantum effects are unimportant near critical points with $T_c>0$. It could be expected, however, that at rather small (as compared to characteristic excitation in the system) temperature, the leading $T$ dependence of all observables is specified by the properties of the zero-temperature critical points, which take place in quantum systems. The dimensional crossover rule asserts that the critical singularities of such a quantum system at $T=0$ with dimensionality $d$ are formally equivalent to those of a classical system with dimensionality $d+z$ ($z$ is the dynamical critical exponent) and critical temperature $T_c>0$. This makes it possible to investigate low-temperature effects (considering an effective system with $d$ infinite space and $z$ finite time dimensions) in the framework of the theory of finite-size scaling (FSS). The idea of this theory has been applied to explore the low-temperature regime in quantum systems (see Refs. 5–7), when the properties of the thermodynamic observables in the *finite-temperature quantum critical region* have been the main focus of interest. The very *quantum critical region* was introduced and studied by Chakravarty *et al.*[5] using the renormalization-group methods. The most famous model for discussing these properties is the quantum nonlinear $\mathcal{O}(n)$ sigma model (QNL$\sigma$M).[5–10]

Recently an equivalence between the QNL$\sigma$M in the limit $n\to\infty$ and a quantum version of the spherical model or more precisely the ''spherical-quantum-rotors'' model (SQRM) was announced.[11] The SQRM is an interesting model in its own. Due to the remarkable opportunity it offers for rigorous study of finite-size effects at arbitrary dimensionality SQRM may play the same role in quantum critical phenomena as the popular Berlin-Kac spherical model in classical critical phenomena. The last one became a touchstone for various scaling hypotheses and source of new ideas in the general theory of finite-size scaling (see, for example, Refs. 12–19, and references therein). Let us note that an increasing interest related with the spherical approximation (or large-$n$ limit) generating tractable models in quantum critical phenomena has been observed in the last few years.[11,20–25]

In Ref. 11, the critical exponents for the zero-temperature quantum fixed point and the finite-temperature classical one as a function of dimensionality was obtained. What remains beyond the scope of Ref. 11 is to study in an exact manner the scaling properties of the model in different regions of the phase diagram including the *quantum critical region* as a function of the dimensionality of the system. In the context of the finite-size scaling theory both cases: (i) The infinite $d$-dimensional quantum system at low temperatures $\infty^d\times L_\tau^z$ [$L_\tau\sim(\hbar/k_BT)^{1/z}$ is the finite size in the imaginary time direction] and (ii) the finite system confined to the geometry $L^{d-d'}\times\infty^{d'}\times L_\tau^z$ ($L$-linear space size), are of crucial interest.

Earlier a class of exactly solvable lattice models intended to study the displacive structural phase transition have been intensively considered in both finite-size and bulk geometry.[26–29] The main feature of these models is that the real anharmonic interaction is substituted with its quantum mean spherical approximation reducing the problem to an exactly solvable one. We expect that the analytical technique proposed below will apply to these models too.

In this paper a detailed theory of the scaling properties of the SQRM with nearest-neighbor interaction is presented. The plan of the paper is as follows: we start with a brief review of the model and the basic equation for the quantum spherical field in the case of periodic boundary conditions (Sec. II). Since we would like to exploit the ideas of the FSS theory, the bulk system in the low-temperature region is considered like an effective $(d+1)$-dimensional classical system with one finite (temporal) dimension. This is done to enable contact to be made with other results based on the spherical-type approximation, e.g., in the framework of the spherical model and the QNL$\sigma$M in the limit $n\to\infty$ (Sec. III). In Sec. IV we consider the FSS form of the spherical field equation for the system confined to the general geometry $L^{d-d'}\times\infty^{d'}\times L_\tau^z$. This equation turns out to allow for analytic studies of the finite-size and low-temperature asymptotes for different $d$ and $d'$. Special attention is laid on the two-dimensional system. The remainder of the paper contains the details of the calculations: Appendixes A, B.

## II. THE MODEL

The model we will consider here describes a magnetic ordering due to the interaction of quantum spins. This has the following form:[11]

$$\mathcal{H}=\frac{1}{2}g\sum_{\ell}\mathcal{P}_\ell^2-\frac{1}{2}\sum_{\ell\ell'}\boldsymbol{J}_{\ell\ell'}\mathcal{S}_\ell\mathcal{S}_{\ell'}+\frac{\mu}{2}\sum_{\ell}\mathcal{S}_\ell^2-H\sum_{\ell}\mathcal{S}_\ell, \tag{2.1}$$

where $\mathcal{S}_\ell$ are spin operators at site $\ell$, the operators $\mathcal{P}_\ell$ are ''conjugated'' momenta (i.e., $[\mathcal{S}_\ell,\mathcal{S}_{\ell'}]=0$, $[\mathcal{P}_\ell,\mathcal{P}_{\ell'}]=0$, and $[\mathcal{P}_\ell,\mathcal{S}_{\ell'}]=i\delta_{\ell\ell'}$, with $\hbar=1$), the coupling constants $\boldsymbol{J}_{\ell,\ell'}=\boldsymbol{J}$ are between nearest neighbors only,[30] the coupling constant $g$ is introduced so as to measure the strength of the quantum fluctuations (below it will be called quantum parameter), $H$ is an ordering magnetic field, and finally the spherical field $\mu$ is introduced so as to ensure the constraint

$$\sum_{\ell}\langle\mathcal{S}_\ell^2\rangle=N. \tag{2.2}$$

Here $N$ is the total number of the quantum spins located at sites ''$\ell$'' of a hypercubical lattice of size $L_1\times L_2\times\cdots\times L_d=N$ and $\langle\cdots\rangle$ denotes the standard thermodynamic average taken with $\mathcal{H}$.

Many aspects of the physics of SQRM and QNL$\sigma$M in the limit $n\to\infty$ are similar, but there is an important difference: while the last has a continuous $\mathcal{O}(n)$ symmetry, the Hamiltonian of SQRM possesses a global $\mathbb{Z}_2$ symmetry. As in the Ising model in a transverse field (the other popular model in the theory of quantum phase transitions) Hamiltonian (2.1) is invariant under the unitary transformation $\mathcal{S}_\ell\to-\mathcal{S}_\ell$. An external field coupling to $\mathcal{S}_\ell$ would break the $\mathbb{Z}_2$ symmetry.

Let us note that the commutation relations for the operators $\mathcal{S}_\ell$ and $\mathcal{P}_\ell$ together with the quadratic kinetic term in the Hamiltonian (2.1) do not describe quantum Heisenberg-Dirac spins but quantum rotors as it was pointed out in Ref. 11.

Under periodic boundary conditions, Eq. (2.2) takes the form

$$1=\frac{\lambda}{2N}\sum_{\boldsymbol{q}}\frac{1}{\sqrt{\phi+2\sum_{i=1}^{d}(1-\cos q_i)}}\times\coth\left(\frac{\lambda}{2t}\sqrt{\phi+2\sum_{i=1}^{d}(1-\cos q_i)}\right)+\frac{h^2}{\phi^2}, \tag{2.3}$$

where we have introduced the following notations: $\lambda=\sqrt{g/\boldsymbol{J}}$ is the normalized quantum parameter, $t=T/\boldsymbol{J}$ is the normalized temperature, $h=H/\boldsymbol{J}$ is the normalized magnetic field, $b=2\pi t/\lambda$, and $\phi=\mu/\boldsymbol{J}-2d$ is the shifted spherical field.

In Eq. (2.3) the vector $\boldsymbol{q}$ is a collective symbol, which has for $L_j$ odd integers the components:

$$\left\{\frac{2\pi n_1}{L_1},\ldots,\frac{2\pi n_d}{L_d}\right\},\quad n_j\in\left\{-\frac{L_j-1}{2},\ldots,\frac{L_j-1}{2}\right\}.$$

A previous direct analysis[11] of Eq. (2.3) in the thermodynamic limit shows that there can be no long-range order at finite temperature, for $d\leqslant2$ (in accordance with the Mermin-Wagner theorem). For $d>2$ one can find long-range order at finite temperature up to a critical temperature $t_c(\lambda)$. Here we shall consider the low-temperature region for $1<d<3$.

## III. THE INFINITE SYSTEM

In the thermodynamic limit the $d$-dimensional sum over the momentum vector $\boldsymbol{q}$ in Eq. (2.3) changes in $d$ integrals over the $q_i$'s in the first Brillouin zone and the equation for the shifted spherical field $\phi$ reads

$$1=\frac{t}{(2\pi)^d}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}dq_1\cdots\times\int_{-\pi}^{\pi}dq_d\frac{1}{\phi+2\sum_{i=1}^{d}(1-\cos q_i)+b^2m^2}+\frac{h^2}{\phi^2}. \tag{3.1}$$

After some algebra (see Appendix A), Eq. (3.1) takes the form $(1<d<3)$

$$\frac{1}{\lambda}-\frac{1}{\lambda_c}=-\frac{1}{(4\pi)^{(d+1)/2}}\left|\Gamma\left(\frac{1-d}{2}\right)\right|\phi^{(d-1)/2}+\frac{2}{(4\pi)^{(d+1)/2}}\phi^{(d-1)/2}\mathcal{K}\left(\frac{d-1}{2},\frac{\lambda}{2t}\phi^{1/2}\right)+\frac{h^2}{\phi^2}, \tag{3.2}$$

where $\lambda_c$ is the quantum critical point and

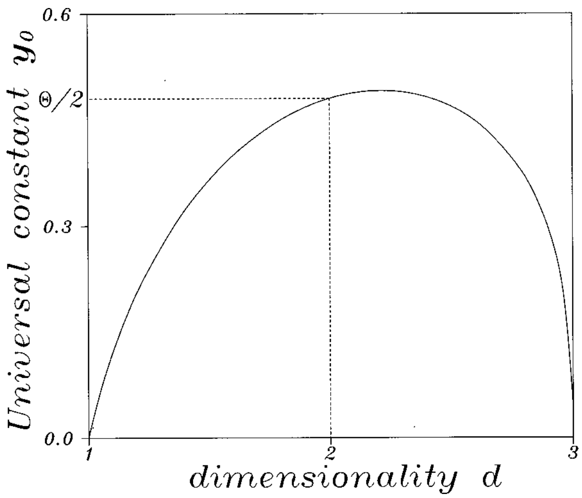


FIG. 1. The dependence of the universal constant $y_0$ upon the dimensionality $d$. The constant $\Theta = 0.962\,424\ldots$ is obtained for the two-dimensional system [see Eq. (3.13c)].

$$\mathcal{K}(\nu,y) \equiv \mathcal{K}_1\left(\frac{d-1}{2}\middle|1,y\right) = 2\sum_{m=1}^{\infty}(ym)^{-\nu}K_\nu(2my). \tag{3.3}$$

Here $K_\nu(x)$ is the MacDonald function (second modified Bessel function). The asymptotic forms of the functions $\mathcal{K}_1[(d-1)/2|1,y]$ are studied in Appendix B. It is easy to show that Eq. (3.2) may be written in a scaling form and consequently the correlation length $\xi = \phi^{-1/2}$, as a solution of that equation has the following scaling form:

$$\xi = \frac{\lambda}{t} f_\xi\left\{\delta\lambda\left(\frac{t}{\lambda}\right)^{-1/\nu},\ h\left(\frac{t}{\lambda}\right)^{-\Delta/\nu}\right\}. \tag{3.4}$$

In the remainder of this section we will study the effect of the temperature on the susceptibility and the equation of state near the quantum critical fixed point.

### A. Zero-field susceptibility

After making vanished the field $h$, from Eq. (3.2) we find that the normalized zero-field susceptibility $\chi = \phi^{-1}$ on the line $\lambda = \lambda_c$ $(t \to 0^+)$ is given by

$$\chi = \frac{\lambda_c^2}{4y_0^2}t^{-2}, \tag{3.5}$$

where $y_0$ is the universal solution of

$$\left|\Gamma\left(\frac{1-d}{2}\right)\right| = 2\mathcal{K}\left(\frac{d-1}{2},y\right). \tag{3.6}$$

The behavior of the universal constant $y_0$ as a function of the dimensionality $d$ of the system is shown in Fig. 1.

One can see that the low-temperature behavior of the susceptibility increases as the inverse of the square of the temperature above the quantum critical point.

In what follows we will try to investigate Eq. (3.2) for different dimensions $(1<d<3)$ of the system and in different regions of the $(t,\lambda)$ phase diagram. Introducing the ''shifted'' critical value of the quantum parameter due to the temperature by

$$\frac{1}{\lambda_c^{\mp}(t)} \approx \frac{1}{\lambda_c} \mp \frac{1}{2\pi^{(d+1)/2}}\left(\frac{t}{\lambda_c}\right)^{d-1}\Gamma\left(\frac{d-1}{2}\right)|\zeta(d-1)|, \tag{3.7}$$

[where $\zeta(x)$ is the Rieman zeta function] one has to make a difference between the two cases $d<2$ ''sign $-$'' and $d>2$ ''sign $+$.'' In the first case $(1<d<2)$, it is possible to define the *quantum critical region* by the inequality

$$\left|\frac{1}{\lambda} - \frac{1}{\lambda_c}\right| \leqslant \frac{1}{2\pi^{(d+1)/2}}\left(\frac{t}{\lambda_c(t)}\right)^{d-1}\Gamma\left(\frac{d-1}{2}\right)|\zeta(d-1)|. \tag{3.8}$$

For $1<d<2$ the function $\mathcal{K}(\nu,y) \sim y^{-1}$ and by substitution in Eq. (3.2) we obtain for $\lambda < \lambda_c$ (outside of the *quantum critical region*)

$$\chi \approx \left[\frac{|\Gamma(1-d/2)|}{(4\pi)^{d/2}\delta\lambda}\right]^{2/(d-2)} t^{2/(d-2)}, \tag{3.9}$$

where

$$\delta\lambda = \frac{1}{\lambda_c} - \frac{1}{\lambda}.$$

In Eq. (3.9) we see that the susceptibility is going to infinity with power-law degree when the quantum fluctuations become important $(t \to 0^+)$ and there is no phase transition driven by $\lambda$ in the system for dimensions between 1 and 2.

In the second case $(2<d<3)$, one has

$$\chi \approx \left[\frac{|\Gamma(1-d/2)|}{(4\pi)^{d/2}}\frac{\lambda\lambda_c(t)}{\lambda - \lambda_c(t)}\right]^{2/(d-2)} t^{2/(d-2)}, \tag{3.10}$$

as a solution for $\lambda$ less than $\lambda_c$ and greater than the critical value $\lambda_c(t)$ of the quantum parameter. Here for finite temperatures there is a phase transition driven by the quantum parameter $\lambda$ with critical exponent of the $d$-dimensional classical spherical model $\gamma = 2/(d-2)$. This however is valid only for very close values of $\lambda$ to $\lambda(t)$. For $\lambda < \lambda_c(t)$ the susceptibility is infinite.

In the region where $\lambda > \lambda_c$ the zero-field susceptibility is given by

$$\chi \approx \left[\frac{(4\pi)^{(d+1)/2}}{\Gamma[(1-d)/2]}\delta\lambda\right]^{2/(1-d)}. \tag{3.11}$$

This result is valid for every $d$ between the lower and the upper quantum critical dimensions, i.e., $1<d<3$.

The important case $d=2$ can be solved easily and one gets

$$\phi^{1/2} = \frac{2t}{\lambda}\text{arcsinh}\left\{\frac{1}{2}\exp\left[\frac{2\pi\lambda}{t}\delta\lambda\right]\right\}. \tag{3.12}$$

For the susceptibility, Eq. (3.12) yields

$$\chi \approx \frac{\lambda^2}{t^2} \exp\left( -\frac{4\pi\lambda_c}{t} \delta\lambda \right) \tag{3.13a}$$

for $(2\pi/t)|\lambda/\lambda_c - 1| \gg 1$ and $\lambda < \lambda_c$, i.e., in the renormalized classical region. For $\lambda = \lambda_c = 3.1114\ldots$

$$\chi = \frac{1}{\Theta^2}\left(\frac{\lambda_c}{t}\right)^2, \tag{3.13b}$$

where the universal constant

$$\Theta = 2y_0 = 2\ln\left(\frac{\sqrt{5}+1}{2}\right) = -2\ln\left(\frac{\sqrt{5}-1}{2}\right) = 0.962\,424\ldots \tag{3.13c}$$

was obtained in the framework of the three-dimensional classical mean spherical model with one finite dimension.[12] Finally for $(2\pi/t)|\lambda/\lambda_c - 1| \gg 1$ and $\lambda > \lambda_c$, i.e., in the quantum disordered region:

$$\chi \approx [4\pi\delta\lambda]^{-2}\left\{ 1 + \frac{2t}{\pi\lambda_c\delta\lambda} \exp\left[ \frac{4\pi\lambda_c}{t}\delta\lambda \right]\right\}. \tag{3.13d}$$

The first term of Eq. (3.13d) is a particular case of Eq. (3.11) for $d=2$.

From Eqs. (3.13) one can transparently see the different behaviors of $\chi(T)$ in three regions: (a) renormalized classical region with exponentially divergence as $T \to 0$, (b) *quantum critical region* with $\chi(T) \sim T^{-2}$ and crossover lines $T \sim |\lambda - \lambda_c|$, and (c) quantum disordered region with temperature-independent susceptibility (up to exponentially small corrections) as $T \to 0$. The above results (3.12) and (3.13) coincide in form with those obtained in Refs. 6,8 for the two-dimensional QNL$\sigma$M in the $n \to \infty$ limit. The only differences are that (i) in Eq. (3.12) the temperature is scaled by $\lambda$, and (ii) the critical value $\lambda_c$ is given by Eq. (A13), while for the QNL$\sigma$M it depends upon the regularization scheme.

### B. Equation of state

The equation of state of the model Hamiltonian (2.1) near the quantum critical point is obtained after substituting the shifted spherical field $\phi$ by the magnetization $\mathcal{M}$ through the relation

$$\mathcal{M} = \frac{h}{\phi}, \tag{3.14}$$

in Eq. (3.2), which allows us to write the equation of state in a scaling form

$$-\frac{\delta\lambda}{\mathcal{M}^{1/\beta}} + (4\pi)^{-(d+1)/2}\left[\frac{h}{\mathcal{M}^{\delta}}\right]^{1/\gamma}\left\{ \left|\Gamma\left(\frac{1-d}{2}\right)\right| - 2\mathcal{K}\left[\frac{d-1}{2}, \frac{\lambda}{2}\left(\frac{\mathcal{M}^{\nu/\beta}}{t}\right)\left(\frac{h}{\mathcal{M}^{\delta}}\right)^{\beta}\right]\right\} = 1. \tag{3.15}$$

We conclude that near the quantum critical point Eq. (3.15) may be written in general forms as

$$h = \mathcal{M}^{\delta} f_h(\delta\lambda\mathcal{M}^{-1/\beta}, (t/\lambda)^{1/\nu}\mathcal{M}^{-1/\beta}), \tag{3.16a}$$

or

$$\mathcal{M} = \left(\frac{t}{\lambda}\right)^{-\beta/\nu} f_{\mathcal{M}}(\delta\lambda\mathcal{M}^{-1/\beta}, h\mathcal{M}^{-\delta}). \tag{3.16b}$$

In Eqs. (3.16) $f_h(x,y)$ and $f_{\mathcal{M}}(x,y)$ are some scaling functions, furthermore $\gamma = 2/(d-1)$, $\nu = 1/(d-1)$, $\beta = \frac{1}{2}$ and $\delta = (d+3)/(d-1)$ are the familiar bulk critical exponents for the $(d+1)$-dimensional classical spherical model. Equations (3.16) are direct verification of FSS hypothesis in conjunction with classical to quantum critical dimensional crossover. They can be easily transformed into the scaling form [Eq. (21)] obtained in Ref. 11, however here they are verified for $1<d<3$ instead of $2<d<3$ (c.f., Ref. 11), i.e., the noncritical case is included.

Hereafter we will try to give an explicit expression of the scaling function $f_h(x,y)$ [$x \equiv \delta\lambda\mathcal{M}^{-2}$, $y \equiv (t/\lambda)^{d-1}\mathcal{M}^{-2}$] in the neighborhood of the quantum critical fixed point. This may be performed, in the case $(t/\lambda)\sqrt{h/\mathcal{M}} \ll 1$, with the use of the asymptotic form of $\mathcal{K}(\nu,y)$ to get the following result for the scaling function $(d \neq 2)$:

$$f_h(x,y) = \left\{ \frac{(4\pi)^{d/2}}{\Gamma(1-d/2)} y^{-\nu}\left[ 1 + x + \frac{1}{2\pi^{(d+1)/2}}\Gamma\left(\frac{d-1}{2}\right) \times \zeta(d-1)y\right]\right\}^{2/(d-2)}. \tag{3.17}$$

For the special case $d=2$ the scaling function is given by the expression

$$f_h(x,y) = 4y^2\left[\operatorname{arcsinh}\frac{1}{2}\exp\left(2\pi\frac{1+x}{y}\right)\right]^2. \tag{3.18}$$

At $x=0$ and $y \gg 1$ (fixed low temperature and $h \to 0^+$), Eq. (3.18) reduces to

$$f_h(0,y) \approx y^2 \exp\left(\frac{4\pi}{y}\right). \tag{3.19}$$

In the region $x < -1$, and for $y \ll 1$ (fixed weak field and $t \to 0^+$) the corresponding scaling function is

$$f_h(x,y) \approx y^2 \exp\left(4\pi\frac{x+1}{y}\right), \tag{3.20}$$

and in region $x > -1$ and $y \ll 1$ we have

$$f_h(x,y) \approx 16\pi^2(x+1)^2\left[1 + \frac{y}{\pi(1+x)}\exp\left(-4\pi\frac{1+x}{y}\right)\right]. \tag{3.21}$$

This identifies the zero-temperature $(y=0)$ form of the scaling function (3.21) with those of the three-dimensional classical spherical model.

## IV. SYSTEM CONFINED TO A FINITE GEOMETRY

When the model Hamiltonian (2.1) is confined to the general geometry $L^{d-d'} \times \infty^{d'} \times L_\tau$, with $0 \leq d' \leq d$, Eq. (2.3) of the spherical field $\phi$ takes the form (for derivational details see Appendix A)

$$\frac{1}{\lambda}=\frac{1}{\lambda_c}-(4\pi)^{-(d+1)/2}\left|\Gamma\left(\frac{1-d}{2}\right)\right|\phi^{(d-1)/2}+\frac{\phi^{(d-1)/2}}{(2\pi)^{(d+1)/2}}\times\sum_{m,\boldsymbol{l}(d-d')}{}' \frac{K_{(d-1)/2}[\phi^{1/2}\{(\lambda m/t)^2+(L|\boldsymbol{l}|)^2\}^{1/2}]}{[\phi^{1/2}\{(\lambda m/t)^2+(L|\boldsymbol{l}|)^2\}^{1/2}]^{(d-1)/2}}+\frac{h^2}{\phi^2}, \tag{4.1}$$

where

$$|\boldsymbol{l}|=(l_1^2+l_2^2+\cdots+l_{d-d'}^2)^{1/2}$$

and the primed summation indicates that the vector with components $m=l_1=l_2=\cdots=l_{d-d'}=0$ is excluded.

### A. Shift of the critical quantum parameter

The FSS theory (for a review, see Ref. 31) asserts, for the temperature-driven phase transition, that the phase transition occurring in the system at the thermodynamic limit persists, if the dimension $d'$ of infinite sizes is greater than the lower critical dimension of the system. In this case the value of the critical temperature $T_c(\infty)$ at which some thermodynamic functions exhibit a singularity is shifted to $T_c(L)$ critical temperature for a system confined to the general geometry $L^{d-d'}\times\infty^{d'}$, when the system is infinite in $d'$ dimensions and finite in $(d-d')$-dimensions. In the case when the number of infinite dimensions is less than the lower critical dimension, there is no phase transition in the system and the singularities of the thermodynamic functions are altered. The critical temperature $T_c(\infty)$ in this case is shifted to a pseudocritical temperature, corresponding to the center of the rounding of the singularities of the thermodynamic functions, holding in the thermodynamic limit.

In our quantum case, having in mind that we have considered the low-temperature behavior of model (2.1) in the context of the FSS theory, it is convenient to choose the quantum parameter $\lambda$ as a critical instead of the temperature $t$ and to consider our system confined to the geometry $L^{d-d'}\times\infty^{d'}\times L_\tau$. So the shifted critical quantum parameter $\lambda_c(t,L)\equiv\lambda_{tL}$ is obtained by setting $\phi=0$ in Eq. (4.1). This gives

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}=\frac{\Gamma[(d-1)/2]}{4\pi^{(d+1)/2}}\sum_{m,\boldsymbol{l}(d-d')}{}' [(\lambda_{tL}m/t)^2+(L|\boldsymbol{l}|)^2]^{(1-d)/2}. \tag{4.2}$$

The sum in the right-hand side (rhs) of Eq. (4.2) is convergent for $d'>2$, however it can be expressed in terms of the Epstein zeta function

$$\mathcal{Z}\begin{vmatrix}0\\0\end{vmatrix}\left[L^2\boldsymbol{l}^2+\left(\frac{\lambda}{t}\right)^2m^2;d-1\right]=\sum_{m,\boldsymbol{l}(d-d')}{}' \left[L^2\boldsymbol{l}^2+\left(\frac{\lambda}{t}\right)^2m^2\right]^{(1-d)/2}, \tag{4.3}$$

which can be regarded as the generalized $(d-d'+1)$-dimensional analog of the Riemann zeta function $\zeta[(d-1)/2])$ (see Ref. 32). In the case under consideration the Epstein zeta function has only a simple pole at $d'=2$ and may be analytically continued for $0\leqslant d'<2$ to give a meaning to Eq. (4.2) for $d'<2$ as well. It is hard to investigate the sum appearing in Eq. (4.3). The anisotropy of the sum $L^2l_1^2+\cdots+L^2l_{d-d'}^2+(\lambda/t)^2m^2$ is an additional problem. That is why we will try to solve it asymptotically, considering different regimes of the temperature, depending on whether $L\ll\lambda_{tL}/t$ or $L\gg\lambda_{tL}/t$, which will be called, respectively, the very low-temperature regime and the low-temperature regime.

#### 1. Low-temperature regime $\lambda_{tL}/t\ll L$

In this case after some algebra the resulting expression is

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}=\frac{1}{2\pi^{(d+1)/2}}\left(\frac{t}{\lambda_{tL}}\right)^{d-1}\Gamma\left(\frac{d-1}{2}\right)\zeta(d-1)+\frac{t}{\lambda_{tL}}\frac{L^{2-d}}{4\pi^{d/2}}\Gamma\left(\frac{d}{2}-1\right)\sum_{\boldsymbol{l}(d-d')}{}' |\boldsymbol{l}|^{2-d}+\left(\frac{t}{\lambda_{tL}}\right)^{d/2}\frac{L^{1-d/2}}{\pi}\sum_{\boldsymbol{l}(d-d')}{}' \sum_{m=1}^{\infty}\left(\frac{m}{|\boldsymbol{l}|}\right)^{d/2-1}\times K_{d/2-1}\left(2\pi\frac{t}{\lambda_{tL}}Lm|\boldsymbol{l}|\right). \tag{4.4}$$

The first term of the rhs of Eq. (4.4) is the shift of the critical quantum parameter [see Eq. (3.7)] due to the presence of the quantum effects in the system. The second term is a correction resulting from the finite sizes. It is just the shift due the finite-size effects in the $d$-dimensional spherical model[19] multiplied by the temperature scaled to the quantum parameter. Here the $(d-d')$-fold sum may be continued analytically beyond its domain of convergence with respect to $d$ and $d'$ (which is $2<d'<d$). The last term is exponentially small in the considered limit, i.e., $\lambda_{tL}/t\ll L$.

In the borderline case $d=2$, Eq. (4.4) reduces to

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}=\frac{t}{2\pi\lambda_{tL}}\left\{B_0+\gamma_E+\ln\frac{tL}{2\lambda_{tL}}\right\}, \tag{4.5}$$

where $\gamma_E=0.577\ldots$ is the Euler constant and $B_0$ is a constant depending on the dimensionality $d-d'$: in the case of strip geometry $(d'=1)$ $B_0=-\ln 2\pi$, and in the fully finite geometry case $(d'=0)$ $B_0=-\ln[\Gamma(1/4)]^2/2\sqrt{\pi}$. Let us note that in the rhs of Eq. (4.5) exponentially small corrections are omitted.

#### 2. Very-low-temperature regime $\lambda/t\gg L$

The final result in this case is given by the expression

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}=\frac{L^{1-d}}{4\pi^{(d+1)/2}}\Gamma\left(\frac{d-1}{2}\right)\sum_{\boldsymbol{l}(d-d')}{}' |\boldsymbol{l}|^{1-d}+\frac{L^{d'-d}}{2\pi^{(d'+1)/2}}\Gamma\left(\frac{d'-1}{2}\right)\left(\frac{t}{\lambda_{tL}}\right)^{d'-1}\zeta(d'-1)$$

$$+\frac{L^{1/2+d'/2-d}}{\pi}\left(\frac{t}{\lambda_{tL}}\right)^{(d'-1)/2}\times\sum_{\mathbf{l}(d-d')}{}'\sum_{m=1}^{\infty}\left(\frac{|\mathbf{l}|}{m}\right)^{(d'-1)/2}K_{(d'-1)/2}\times\left(2\pi\frac{\lambda_{tL}}{tL}m|\mathbf{l}|\right). \tag{4.6}$$

Here, in the rhs, the first term is the expression of the shift of the critical quantum parameter, at zero temperature,[28] due to the finite sizes of the system. This is equivalent to the shift of a $(d+1)$-dimensional spherical model confined to the geometry $L^{d+1-d'}\times\infty^{d'}$. The second term gives a correction due to the quantum effects. This is the shift of critical quantum parameter of a $d'$-dimensional infinite system multiplied by the volume of a $(d-d')$-dimensional hypercube. The third term is exponentially small in the limit of very low temperatures. For $d'=1$ Eq. (4.6) yields

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}=\frac{L^{1-d}}{2\pi}\left\{\ln\frac{\lambda_{tL}}{2\sqrt{\pi}tL}+\frac{\gamma_E}{2}+[2\pi^{(d-1)/2}]^{-1}\Gamma\left(\frac{d-1}{2}\right)C_0\right\}. \tag{4.7}$$

Here the expressions for the constants $C_0$ are quite complicated expect for some special cases: see Refs. 33, e.g., for $d=2$, $d'=1$, one has $C_0=\gamma_E-\ln 4\pi$ [c.f., Ref. 34, Eq. (30.104)].

For the case $d=2$, $d'=1$, comparing between Eqs. (4.5) and (4.7) one can see the crucial role (in symmetric form) of $L$ or $\lambda_{tL}/t$ in the low-temperature regime and the very low-temperature one, respectively.

In the other important case of a two-dimensional bloc geometry $d'=0$ and $d=2$, from Eq. (4.6) one gets (again up to exponentially small corrections)

$$\frac{1}{\lambda_{tL}}-\frac{1}{\lambda_c}\approx\frac{L^{-1}}{\pi}\zeta\left(\frac{1}{2}\right)\beta\left(\frac{1}{2}\right)-\frac{\lambda_{tL}}{tL^2}\zeta(-1), \tag{4.8}$$

where

$$\beta(s)=\sum_{l=1}^{\infty}\frac{(-1)^l}{(2l+1)^s}.$$

Instead of the previous case of the low-temperature regime, here the lower quantum critical dimension $d'=1$ is responsible for the logarithmic dependence in Eq. (4.7). This is the reason for the significant difference between Eqs. (4.7) and (4.8).

The obtained equations for $\lambda_{tL}$ will be exploited later for the study of the two-dimensional case.

### B. Zero-field susceptibility

From Eq. (4.1) one can show [see Eqs. (A14) and (A15)] that the correlation length $\xi=\phi^{-1/2}$ will scale like

$$\xi=Lf_\xi^L\left\{\delta\lambda L^{1/\nu},\frac{tL}{\lambda},hL^{\Delta/\nu}\right\}, \tag{4.9a}$$

or like

$$\xi=\frac{\lambda}{t}f_\xi^t\left\{\delta\lambda\left(\frac{t}{\lambda}\right)^{-1/\nu},\frac{tL}{\lambda},h\left(\frac{t}{\lambda}\right)^{-\Delta/\nu}\right\}, \tag{4.9b}$$

which suggests also that there will be some kind of interplay (competition) between the finite-size and the quantum effects. Equations (4.9) for the finite system are a generalization of Eq. (3.4) for the correlation length for the bulk system.

Hereafter we will try to find the behavior of the susceptibility $\chi=\phi^{-1}$ as a function of the temperature $t$ and the size $L$ of the system. For simplicity, in the remainder of this section, we will investigate the free field case $(h=0)$.

(1) For $(\lambda/t)\,\phi^{1/2}\ll 1$, after using the asymptotic form of the function defined in Eq. (A14b) (see Appendix B) Eq. (4.1) reads $(d'\neq 2,\ 1<d<3)$

$$\delta\lambda+\frac{t}{\lambda}\frac{L^{d'-d}}{(4\pi)^{d'/2}}\Gamma\left(1-\frac{d'}{2}\right)\phi^{(d'-2)/2}+\frac{1}{4\pi^{(d+1)/2}}\Gamma\left(\frac{d-1}{2}\right)\times\sum_{m,\mathbf{l}(d-d)}{}'\left[\left(\frac{\lambda}{t}m\right)^2+(L\mathbf{l})^2\right]^{(1-d)/2}=0. \tag{4.10}$$

Now we will examine Eq. (4.10) in different regimes of $t$ and $L$ and for different geometries of the lattice:

(a) $(\lambda/t)\,\phi^{1/2}\ll 1$ and $tL/\lambda\gg 1$: In this case Eq. (4.10) transforms into [up to an exponentially small correction, cf. Eq. (4.4)]

$$0=\delta\lambda+\frac{t}{\lambda}\frac{L^{d'-d}}{(4\pi)^{d'/2}}\Gamma\left(1-\frac{d'}{2}\right)\phi^{(d'-2)/2}+\frac{1}{2\pi^{(d+1)/2}}\left(\frac{t}{\lambda}\right)^{d-1}\Gamma\left(\frac{d-1}{2}\right)\zeta(d-1)+\frac{t}{\lambda}\frac{L^{2-d}}{4\pi^{d/2}}\Gamma\left(\frac{d}{2}-1\right)\sum_{\mathbf{l}(d-d)}{}'|\mathbf{l}|^{2-d}. \tag{4.11}$$

This equation has different types of solutions depending on whether the dimensionality $d$ is above or below the classical critical dimension 2.

At $\lambda=\lambda_c$ and when $d'<2<d<3$ (i.e., when there is no phase transition in the system) we obtain for the zero-field susceptibility

$$\chi=\left(\frac{t}{\lambda_c}\right)^{-2}\left(\frac{tL}{\lambda_c}\right)^{2(d-d')/(2-d')}\times\left[\frac{2^{d'-1}}{\pi^{(d-d'+1)/2}}\frac{\Gamma[(d-1)/2]}{\Gamma(1-d'/2)}\zeta(d-1)\right]^{2/(2-d')}. \tag{4.12}$$

However for $1<d<2$, Eq. (4.11) has no solution at $\lambda=\lambda_c$ obeying the initial condition $(\lambda_c/t)\,\phi^{1/2}\ll 1$.

Equation (4.12) generalizes the bulk result (3.5) for $d$ close to the upper quantum critical dimension, i.e., $d=3$.

At the shifted critical quantum parameter $\lambda_c(t)$ given by Eq. (3.7) we get

$$\chi = L^2 \left[ \frac{2^{d'-2}}{\pi^{(d-d')/2}} \frac{\Gamma(d/2-1)}{\Gamma(1-d'/2)} \sum_{l(d-d')}' |l|^{2-d} \right]^{2/(2-d')}. \tag{4.13}$$

However this solution is valid only for $3>d>2>d'$, i.e., here again there is no phase transition in the system.

(b) $(\lambda/t)\,\phi^{1/2} \ll 1$ and $tL/\lambda \ll 1$: In this case, Eq. (4.10) gives [up to exponentially small corrections, c.f. with Eq. (4.6)]

$$0 = \delta\lambda + \frac{t}{\lambda} \frac{L^{d'-d}}{(4\pi)^{d'/2}} \Gamma\left(1-\frac{d'}{2}\right) \phi^{(d'-2)/2} + \frac{L^{1-d}}{4\pi^{(d+1)/2}} \Gamma\left(\frac{d-1}{2}\right) \sum_{l(d-d')}' |l|^{1-d} + \frac{L^{d'-d}}{2\pi^{(d'+1)/2}} \Gamma\left(\frac{d'-1}{2}\right) \left(\frac{t}{\lambda}\right)^{d'-1} \zeta(d'-1). \tag{4.14}$$

Here we find that the solutions of Eq. (4.14) depend upon that whether the dimensionality $d'<1$ or $d'>1$.

At $\lambda=\lambda_c$ and for $1<d'<2$, Eq. (4.14) has

$$\chi = L^2 \left(\frac{\lambda_c}{tL}\right)^{2/(2-d')} \times \left[ \frac{2^{d'-2}}{\pi^{(d-d'+1)/2}} \frac{\Gamma[(d-1)/2]}{\Gamma(1-d'/2)} \sum_{l(d-d')}' |l|^{1-d} \right]^{2/(2-d')} \tag{4.15}$$

as a solution. For $0 \leqslant d' < 1$, however, it has no solution obeying the initially imposed restriction $(\lambda_c/t)\,\phi^{1/2} \ll 1$.

At the shifted critical quantum parameter $\lambda_c(L)$ given by[28]

$$\frac{1}{\lambda} - \frac{1}{\lambda_c(L)} = \frac{L^{1-d}}{4\pi^{(d+1)/2}} \Gamma\left(\frac{d-1}{2}\right) \sum_{l(d-d')}' |l|^{1-d},$$

Equation (4.14) has a solution obeying the initial condition $(\lambda_c/t)\,\phi^{1/2} \ll 1$ only for $d'=1+\varepsilon$ and in this case the susceptibility behaves like

$$\chi = \frac{1}{(\pi\varepsilon)^2} \frac{\lambda_c^2}{t^2} \left[ 1 - \varepsilon\left( \gamma_E + \ln\frac{\varepsilon}{2} \right) \right]^2. \tag{4.16}$$

(2) For $L\phi^{1/2} \ll 1$, from Eqs. (A14) and Eq. (B9) we get once again Eq. (4.11). In spite of the fact that we have the same equation as in the case $(\lambda/t)\,\phi^{1/2} \ll 1$, the expected solutions for the susceptibility may be different because of the new imposed condition. Here also we will consider the two limiting cases of low-temperature and very low-temperature regimes.

(a) $L\phi^{1/2} \ll 1$ and $tL/\lambda \gg 1$: In this case Eq. (4.10) again is transformed into Eq. (4.11) and we obtain at $\lambda=\lambda_c$ the solution given by Eq. (4.12), which is valid only for $d'<2<d<3$, i.e., we have the same solution as in the previous case, i.e., $(\lambda/t)\,\phi^{1/2} \ll 1$.

At $\lambda=\lambda_c(t)$, we formally obtain Eq. (4.13) which, however, may be considered as a solution only in the neighborhood of the lower classical critical dimension $d=2$. For the cylindric geometry ($d'=1$ and $d=2+\varepsilon$) we get

$$\chi = \frac{L^2}{(\pi\varepsilon)^2} \left[ 1 - \frac{\varepsilon}{2} (\gamma_E - \ln 4\pi) \right]^2. \tag{4.17}$$

This result is contained in Eq. (30.109) of Ref. 34 in the large-$n$-limit case for the NLQ$\sigma$M.

In the case of slab geometry $d-d'=1$ ($d=2+\varepsilon, d'=1+\varepsilon$) instead of Eq. (4.17) we obtain

$$\chi = \frac{L^2}{(\pi\varepsilon)^2} [1 - \varepsilon(\gamma_E - \ln 2) - \varepsilon \ln\varepsilon]^2. \tag{4.18}$$

In the case of a bloc geometry ($d=2+\varepsilon$ and $d'=0$) we find the following behavior for the susceptibility

$$\chi = \frac{L^2}{2\pi\varepsilon} \left[ 1 - \frac{\varepsilon}{4} \left( \gamma_E - \ln\frac{[\Gamma(1/4)]^4}{4\pi^2} \right) \right]^2. \tag{4.19}$$

For the case of "quasibloc geometry" ($d=2+\varepsilon$ and $d'=\varepsilon$) we get

$$\chi = \frac{L^2}{2\pi\varepsilon} \left[ 1 - \frac{\varepsilon}{4} \left( 2\gamma_E + \ln\frac{\pi\varepsilon}{2} - 2\ln\frac{[\Gamma(1/4)]^2}{2\sqrt{\pi}} \right) \right]^2. \tag{4.20}$$

The appearance of $\varepsilon$ in the denominator in formulas (4.16)–(4.20) signalizes that the scaling in its simple form will fail at $\varepsilon=0$.

(b) $L\phi^{1/2} \ll 1$ and $tL/\lambda \ll 1$: Here we find that Eq. (4.14) is valid, and it has Eq. (4.15) as a solution at $\lambda=\lambda_c$ and for $0 \leqslant d' < 1$. For $1<d'<2$ the susceptibility is given by

$$\chi = \left(\frac{\lambda_c}{2t}\right)^2 \left[ \frac{2}{\pi^{1/2}} \frac{\Gamma[(d'-1)/2]}{\Gamma(1-d'/2)} \zeta(d'-1) \right]^{2/(2-d')}. \tag{4.21}$$

At the shifted critical point $\lambda_c(L)$, for the susceptibility we obtain Eq. (4.21) under the restriction $2>d'>1$, which guarantees the positiveness of the quantity under brackets.

When $\lambda<\lambda_c$ for $1<d<3$ and $d'<2$, i.e., when there is no phase transition in the system, we obtain

$$\chi = \left[ \frac{(4\pi)^{d'/2}}{\Gamma(1-d'/2)} \left(1 \frac{\lambda}{\lambda_c}\right) \right]^{2/(2-d')} t^{-2/(2-d')} L^{2(d-d')/(2-d')}. \tag{4.22}$$

If $d'>2$ there is a phase transition in the system at the shifted value of the critical quantum parameter $\lambda_{tL}$ (the shift in this case is due to the quantum and finite-size effects) and Eq. (4.10) transforms to

$$1 - \frac{\lambda}{\lambda_{tL}} = t \frac{L^{d'-d}}{(4\pi)^{d/2}} \Gamma\big(1-d'/2\big) \phi^{(d'-2)/2}, \tag{4.23}$$

which has the following solutions:

$$\chi = \begin{cases} \left[ \frac{(4\pi)^{d'/2}}{\Gamma(1-d'/2)} \left(1 - \frac{\lambda}{\lambda_{tL}}\right) \right]^{2/(2-d')} t^{-2/(2-d')} L^{2(d-d')/(2-d')}, & \lambda > \lambda_{tL} \\ \infty, & \lambda \leqslant \lambda_{tL}. \end{cases} \tag{4.24}$$

Let us notice that Eqs. (4.22) and (4.24) are the finite-size forms, for the susceptibility, of Eqs. (3.9) and (3.10), respectively, found for the bulk system.

### C. Two-dimensional case

The two-dimensional case needs special treatment because of its physical reasonability and the increasing interest in the context of the quantum critical phenomena.[5–10] From Eq. (4.1) for $d=2$ and in the absence of a magnetic field $h=0$ we get

$$\delta\lambda = \frac{\phi^{1/2}}{4\pi} - \frac{1}{4\pi} \sum_{m,\boldsymbol{l}(2-d')}{}' \frac{\exp\left[ -\phi^{1/2} \left( \frac{\lambda^2}{t^2} m^2 + L^2 \boldsymbol{l}^2 \right)^{1/2} \right]}{[(\lambda^2/t^2)\, m^2 + L^2 \boldsymbol{l}^2]^{1/2}}. \tag{4.25}$$

Introducing the scaling functions $Y_t^{d'} = (\lambda/t)\,\phi^{1/2}$ and $Y_L^{d'} = L\phi^{1/2}$, where the superscript $d'$ denotes the number of infinite dimensions in the system, and the scaling variable $a = tL/\lambda$, it is easy to write Eq. (4.25) in the scaling forms given in Eqs. (A14) and (A15). The solutions of the obtained scaling equations will depend on the number of the infinite dimensions in the system. Here we will consider the two most important particular cases: strip geometry $d'=1$ and bloc geometry $d'=0$. Our analysis will be confined to the study of the behavior of the scaling functions at the critical value of the quantum parameter $\lambda_c$, and at the shifted critical quantum parameter $\lambda_{tL}$ (see Sec. IV A). It is difficult to solve Eq. (4.25) by using an analytic approach; that is why we will give a numerical treatment of the problem. It is, however, possible to consider the two limits: $a \gg 1$, i.e., the low-temperature regime and $a \ll 1$, i.e., the very low-temperature regime.

*Strip geometry* $(d'=1)$: In this case in the rhs of Eq. (4.25) we have a twofold sum which permits a numerical analysis of the geometry under consideration. Figure 2 graphs the variation of the scaling functions $Y_t^1$ and $Y_L^1$ against the variable $a$ at $\lambda=\lambda_c$. This shows that for comparatively small value of the scaling variable $a \sim 5$ the finite-size behavior [see the curve of the function $Y_t^1(a)$] merges in the low-temperature bulk one, while the behavior of $Y_L^1(a)$ shows that for relatively not very low temperatures ($a \sim \frac{1}{5}$, $L$-fixed) the system simulates the behavior of a three-dimensional classical spherical model with one finite dimension. The mathematical reasons for this are the exponentially small values of the corrections, as we will show below.

*Bloc geometry* $(d'=0)$: In this case the threefold sum in the rhs of Eq. (4.25) is not an obstacle to analyzing it numerically. For $\lambda=\lambda_c$ the behavior of the scaling functions $Y_L^0(a)$ and $Y_t^0(a)$ is presented in Fig. 2. They have the same qualitative behavior as in the strip geometry, the only difference is the appearance of a universal number for $t=0$, i.e., $\Omega$, instead of the constant $\Theta$ as a consequence of the asymmetry of the sum in the low-temperature and the very low-temperature regimes.

Now, let us consider analytically Eq. (4.25). To this end, we will first fix the quantum parameter at its critical value $\lambda_c$. For arbitrary values of the number of infinite dimensions $d'$, in the *low-temperature regime* $(a \gg 1)$, Eq. (4.25) can be transformed into [up to small corrections $\mathcal{O}(e^{-2\pi a})$]

$$\delta\lambda = \frac{1}{2\pi\lambda} \ln 2 \sinh \frac{\lambda}{2t} \phi^{1/2} - \frac{1}{2\pi\lambda} \sum_{\boldsymbol{l}(2-d')}{}' K_0(L\phi^{1/2}|\boldsymbol{l}|). \tag{4.26}$$

For $\lambda=\lambda_c$ Eq. (4.26) has the solution

$$\chi^{-1/2} \approx \frac{t}{\lambda_c}\Theta + (2-d') \sqrt{\frac{2\pi}{5\Theta}} \left(\frac{t}{L\lambda_c}\right)^{1/2} \exp\left(-\frac{tL}{\lambda_c}\Theta\right), \tag{4.27}$$

i.e., the finite-size corrections to the bulk behavior are exponentially small.

In the *very low-temperature regime* $(a \ll 1)$, Eq. (4.25) reads [up to $\mathcal{O}(e^{-2\pi/a})$]

$$\delta\lambda = \frac{\phi^{1/2}}{4\pi} - \frac{L^{-1}}{4\pi} \sum_{\boldsymbol{l}(2-d')}{}' \frac{\exp(-L\phi^{1/2}|\boldsymbol{l}|)}{|\boldsymbol{l}|}$$

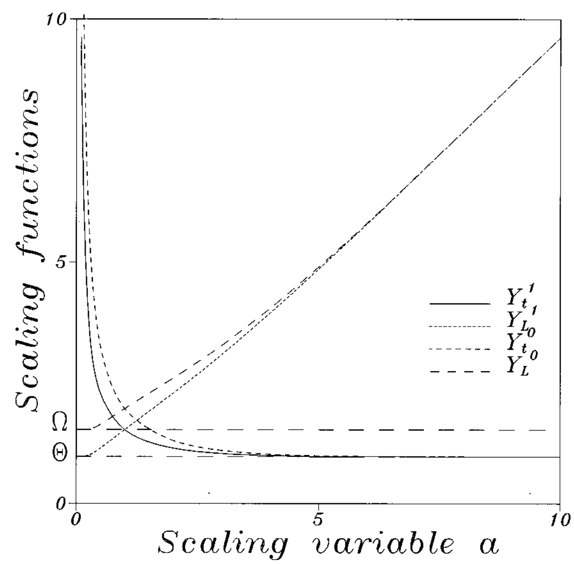


FIG. 2. The effects of the finite-size geometry on the bulk behavior of $\phi^{1/2}$ for the two-dimensional case at $\lambda=\lambda_c$. The superscript $d'$ in $Y_L^{d'} = L\phi^{1/2}$ and $Y_t^{d'} = (\lambda_c/t)\,\phi^{1/2}$ indicates the number of infinite dimensions in the system. The scaling variable $a = tL/\lambda_c$. The universal numbers are $\Theta = 0.962\,424\ldots$ [see Eq. (3.13c)] and $\Omega = 1.511\,955\ldots$ .

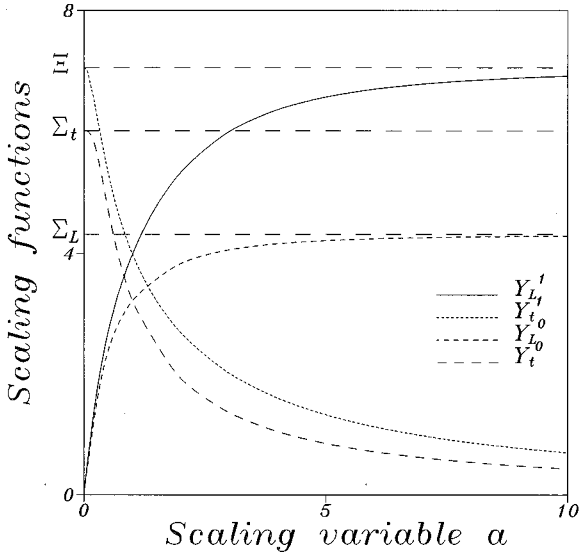


FIG. 3. The same as in Fig. 2 but for $\lambda=\lambda_{tL}$ and $a=tL/\lambda_{tL}$. The universal numbers are $\Xi=7.061\,132\ldots$, $\Sigma_t=6.028\,966\ldots$ and $\Sigma_L=4.317\,795\ldots$.

$$-\frac{L^{d'-2}}{\pi^{(d'+1)/2}}\left(2\frac{\lambda}{t}\right)^{(1-d')/2}\sum_{m=1}^{\infty}K_{(d'-1)/2}\left(\frac{\lambda}{t}\phi^{1/2}m\right), \tag{4.28}$$

which has the solutions

$$\chi^{-1/2}\approx\frac{1}{L}\Theta+\sqrt{\frac{2\pi}{5\Theta}}\left(\frac{L\lambda_c}{t}\right)^{1/2}\exp\left(-\frac{\lambda_c}{tL}\Theta\right) \tag{4.29}$$

for $d'=1$, and

$$\chi^{-1/2}\approx\frac{1}{L}\Omega+\frac{1}{L}\left\{\frac{1}{2\Omega}+\frac{\Omega}{2}\sum_{\boldsymbol{l}(2)}{}'\ (\Omega^2+4\pi^2\boldsymbol{l}^2)^{-3/2}\right\}^{-1}\times\exp\left(-\Omega\frac{\lambda_c}{tL}\right) \tag{4.30}$$

for $d'=0$. Here $\Omega=1.511\,955\ldots$ is a universal constant.

In Sec. IV A an analytic continuation of the shift of the critical quantum parameter for $d=2$ was presented. It is possible to consider the solutions of Eq. (4.25) at $\lambda=\lambda_{tL}$ [from Eqs. (4.5), (4.7), and (4.8)] and for different geometries. In this case the scaling functions $Y_t^1$, $Y_L^1$, $Y_t^0$, and $Y_L^0$ are graphed in Fig. 3. For $d'=1$ again we see that a symmetry between the two limits $a\ll 1$ and $a\gg 1$ take place, since the scaling functions $Y_t^1$ and $Y_L^1$ are limited by the universal constant $\Xi$. The asymmetric case $d'=0$ has two different constants $\Sigma_t$ and $\Sigma_L$, limiting the solutions of $Y_t^0$ and $Y_L^1$ from above.

The constants $\Xi$, $\Sigma_t$ and $\Sigma_L$ are obtained from the asymptotic analysis (with respect to $a$) of Eq. (4.25) for $\lambda=\lambda_{tL}$. In the limit $a\gg 1$ for arbitrary values of $d'$ we get [from Eq. (4.26)]

$$B_0+\gamma_E+\ln\frac{L\phi^{1/2}}{2}=\sum_{\boldsymbol{l}(2-d')}{}'\ K_0(L\phi^{1/2}|\boldsymbol{l}|), \tag{4.31}$$

where the equation of $\lambda_{tL}$ from Eq. (4.5) is used. Equation (4.31) has the solutions

$$L\chi^{-1/2}=\begin{cases}\Xi & \text{for } d'=1,\\ \Sigma_L & \text{for } d'=0,\end{cases} \tag{4.32}$$

where the universal numbers $\Xi=7.061\,132\ldots$ and $\Sigma_L=4.317\,795\ldots$ are the solutions of the scaling equation (4.31) for $d'=1$ and $d'=0$, respectively.

In the opposite limit $a\ll 1$, for $d'=1$, we get from Eqs. (4.7) and (4.28) the equation

$$\gamma_E+\ln\frac{\lambda_{tL}\phi^{1/2}}{4\pi t}=2\sum_{m=1}^{\infty}K_0\left(\frac{\lambda_{tL}}{t}\phi^{1/2}m\right), \tag{4.33}$$

which has

$$\frac{\lambda_{tL}}{t}\chi^{-1/2}=\Xi, \tag{4.34}$$

as a universal solution. For $d'=0$ we have

$$\left(\frac{\lambda_{tL}}{t}\phi^{1/2}-6\right)\exp\left(\frac{\lambda_{tL}}{t}\phi^{1/2}\right)-\frac{\lambda_{tL}}{t}\phi^{1/2}-6=0 \tag{4.35}$$

obtained from Eqs. (4.8) and (4.28), where we have used the identity (B11).

From Eq. (4.35) we obtain the universal result

$$\frac{\lambda_{tL}}{t}\chi^{-1/2}=\Sigma_t=6.028\,966\ldots. \tag{4.36}$$

We finally conclude that if we take $\lambda=\lambda_c$ the scaling functions $Y_t^{d'}$ and $Y_L^{d'}$ have similar qualitative behavior weakly depending on the geometry (i.e., bloc $d'=0$ or strip $d'=1$) of the system. However, for a given geometry one distinguishes quite different quantitative behavior of the scaling functions depending on whether the quantum parameter $\lambda$ is fixed at its critical value, i.e., $\lambda=\lambda_c$, or takes "running" values $\lambda_{tL}$ obtained from the "shift equations" (4.5), (4.7), or (4.8).

### D. Equation of state

The equation of state of the model Hamiltonian (2.1) for dimensionalities $1<d<3$ is given by [see Eqs. (3.14) and (4.1)]

$$0=\delta\lambda-(4\pi)^{-(d+1)/2}\left|\Gamma\left(\frac{1-d}{2}\right)\right|\left(\frac{h}{\mathcal{M}}\right)^{(d-1)/2}+\frac{(h/\mathcal{M})^{(d-1)/2}}{(2\pi)^{(d+1)/2}}\times\sum_{m,\boldsymbol{l}(d-d')}{}'\ \frac{K_{(d-1)/2}\{(h/\mathcal{M})^{1/2}[(\lambda m/t)^2+(L|\boldsymbol{l}|)^2]^{1/2}\}}{\{(h/\mathcal{M})^{1/2}[(\lambda m/t)^2+(L|\boldsymbol{l}|)^2]^{1/2}\}^{(d-1)/2}}+\mathcal{M}^2. \tag{4.37}$$

It is straightforward to write this equation in a similar form as in Eq. (A14) or Eq. (A15), i.e.,

$$h=\mathcal{M}^{\delta}f_h^L\left\{\delta\lambda\mathcal{M}^{-1/\beta},\frac{tL}{\lambda},L^{-1/\nu}\mathcal{M}^{-1/\beta}\right\},\qquad(4.38a)$$

or

$$h=\mathcal{M}^{\delta}f_h^t\left\{\delta\lambda\mathcal{M}^{-1/\beta},\frac{tL}{\lambda},\left(\frac{t}{\lambda}\right)^{1/\nu}\mathcal{M}^{-1/\beta}\right\}.\qquad(4.38b)$$

Equations (4.38) are generalizations of Eqs. (3.16) in the case of systems confined to a finite geometry. The appearance of an additional variable $tL/\lambda$ is a consequence of the fact that the system under consideration may be regarded as an ''hyperparallelepiped'' (in not necessary a Euclidean space) of linear size $L$ in $d-d'$ directions and of linear size $L_\tau$ in one direction with periodic boundary conditions.

## V. SUMMARY AND DISCUSSION

Since exact solvability is a rare event in statistical physics,[35] the model under consideration yields a conspicuous possibility to investigate the interplay of quantum and classical fluctuations as a function of the dimensionality $d$, the external field $h$, and the geometry of the system in an exact manner. Equations of the type (3.1) are specific for a *closed-form approximation* (in the $d$-dimensional case) in the theory of phase transitions. They reflect the availability of spherical constraints[11,20–22] or self-consistent equations[26–29] and so generate similar critical behavior for various physical phenomena. The central role of this type of equations can be confirmed by a more sophisticated large-$n$ limit analysis.[6] For this reason it is not a surprise that the bulk low-temperature properties, of the SQRM (see Sec. III) are similar to those obtained by saddle-point calculation for the QNL $\sigma$M; the main analytical model in the theory of quantum critical phenomena. An attractive feature of the present model is the lattice formulation, which seems to be more transparent in the finite-size case, since no ultraviolet regularization is necessary and there are no ambiguities associated with taking the continuum limit.

The discussion of the obtained results in Sec. III, serves as a basis for the further FSS investigations. Identifying the temperature, which governs the crossover between the classical and the quantum fluctuations as an additional temporal dimension one makes possible the use of the methods of FSS theory in a very effective way.

A quantum analog of the Privman-Fisher hypothesis[36] for the FSS in the presence of a magnetic field $h$ was shown to be consistent with exact results obtained in Secs. III, Eq. (3.4), and IV, Eq. (4.9). We mention that in the case of geometry $L^{d-d'}\times\infty^{d'}\times L_\tau$ the scaling functions depend on $hL^{\Delta/\nu}$ or $hL_\tau^{\Delta/\nu}$ and on the shape factor $L/L_\tau$ which provides different regimes: low-temperature ($L\gg L_\tau$) and very low-temperature ($L\ll L_\tau$). The analogy between the model (2.1) and the QNL$\sigma$M in the large-$n$ limit was already noticed. The external field dependence of the thermodynamics of the last model has been studied in Ref. 7. The key element of this treatment is the specific orientation of the magnetic field to facilitate a simple large-$n$ limit. In particular, for the case when $h$ couples to a ''conserved charge'' the equality between the scaling dimensions of the field and the temperature was obtained.[7] A phenomenological study of the nonlinear field dependence of a system, without having a total conserved charge, has been presented in Ref. 37. These phenomenological ideas are illustrated in the framework of the concrete model (2.1). Let us note an important difference in symmetry between the model (2.1) with its discrete $\mathbb{Z}_2$ symmetry and the $\mathcal{O}(n)$ symmetry of the basic rotors model.

In Sec. IV A the shift of the critical quantum parameter $\lambda$ as a consequence of the quantum and finite-size effects is obtained. In comparison with the classical case (for details see Ref. 19, and references therein) here the problem is rather complicated by the presence of the two finite characteristic lengths $L$ and $L_\tau$. We observe a competition between finite-size and quantum effects which reflects the appearance of the two regimes: low-temperature and very low-temperature. The behavior of the shift is analyzed in some actual cases of concrete geometries, e.g., strip and bloc.

In the parameter space (temperature $t$ and quantum parameter $\lambda$), where quantum zero-point fluctuations are relevant, there are three distinct regions named ''renormalized classical,'' ''quantum critical,'' and ''quantum disordered.'' The existence of these regions in conjunction with both regimes: low-temperature and very low-temperature, is an intrinsic feature of the physics near the quantum critical point and makes the model a useful tool for the exploration of the qualitative behavior of a large class of systems.

In Sec. IV B the susceptibility (or the correlation length) is calculated and the critical behavior of the system in different regimes and geometries is analyzed. We have studied the model (2.1) via $\varepsilon$ expansion in order to illustrate the effects of the dimensionality $d$ on the existence and properties of the ordered phase. An indicative example is given by Eqs. (4.17) and (4.18), while the former is known (see Ref. 34), the last one is quite different and new. These shows that one must be accurate in taking the limit $\varepsilon\to0^+$. The relation with the QNL$\sigma$M in the $n\to\infty$ limit may serve as an illustration of Stanley's arguments of the relevance of the spherical approximations in the quantum case. Let us note, however, that the use of such arguments needs an additional more subtle treatment in the finite-size case.

In Sec. IV C, special attention is paid to the two-dimensional case. The two important cases of strip and bloc geometries are considered. The universal constant $\Theta$ given by Eq. (3.13c), which characterizes the bulk system, is changed to a set of universal constants: $\Omega$ [see Eq. (4.30)], $\Xi$ and $\Sigma_L$ [see Eq. (4.32)], and $\Sigma_t$ [see Eq. (4.36)]. The appearance of universal constants reflects the new situation, when there are two relevant values of the quantum parameter $\lambda$: $\lambda=\lambda_c$ in the bulk case and $\lambda=\lambda_{tL}$ in the case of finite geometries. Due to their universality these constants may play an important role even in studying more complicated model Hamiltonians. The behaviors of the scaling functions at the bulk critical quantum parameter $\lambda_c$ and the shifted critical quantum parameter $\lambda_{tL}$ are given in Figs. 2 and 3. $L_\tau$ is the main characteristic length and the $1/L$ corrections are exponentially small in the case of low-temperature regime, and vice versa in the case of the very low-temperature regime.

The equation of state, for the system confined to the general geometry $L^{d-d'}\times\infty^{d'}\times L_\tau$, is obtained in Sec. IV D. This reflects the modifications of the scaling functions as a consequence of the finite sizes and the temperature.

It is a common wisdom that the spherical limit models are not free of any pathologies. So some really interesting problems come if one goes beyond the spherical approximation. One can see from Eq. (2.1) that in the absence of the spherical constraint (2.2), if $A\equiv \mu/4+ \boldsymbol{J}d/4<0$, such a system is thermodynamically unstable, i.e., the parameter $A<0$ defines the frequency of an unstable mode suggesting that an appropriate stabilization (for example, by adding the term $B\mathcal{S}_{\ell}^4$) of the system can again creates a gap in the spectrum. Such Hamiltonians are frequently used in the theory of structural phase transitions (see, e.g., Refs. 38, 39, and references therein). A relaxed version of the spherical constraint in conjunction with exact solvability may be obtained in this case by the ansatz $\mathcal{S}_{\ell}^2 \Rightarrow 1/N(\Sigma_{\ell}\mathcal{S}_{\ell}^2)$ (see Refs. 26–29,40). The model obtained in this way is a quantum counterpart of the ''soft'' classical mean spherical model studied in Ref. 41 in the context of the FSS theory. Strictly speaking, in order to obtain *exact finite-size corrections*, even this Hamiltonian with ''truncated fluctuations'' is analytically hard tractable despite that it belongs to the *bulk* universality class of the model (2.1). For example it is not obvious how to obtain the corrections to the bulk result, since both the $1/L$ and $1/N$ parts enter. That this is a nontrivial problem, even in the simplest case of the classical Husimi-Temperely spherical model, was demonstrated in Ref. 42. In the quantum case, where the situation is much more difficult, up to now this is an open problem. Certainly, if we discard the problem of the status of the approximation scheme [in the last case it is equivalent to the well-known self-consistent phonon approximation (see, e.g., Ref. 39)] then our treatment is not restricted only to the Hamiltonian (2.1), but it can be applied to a wide class of finite lattice models (e.g., directly to the anharmonic crystal model, see Refs. 26–29) and it can also provide a methodology for seeking different quantum finite-size effects in such systems.

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## APPENDIX A

In this appendix we will derive Eqs. (3.2) and (4.1) of the shifted spherical field $\phi$ for the model Hamiltonian (2.1) confined to the general geometry $L^{d-d'}\times\infty^{d'}\times L_\tau$, with periodic boundary conditions, in the low-temperature regime. To achieve that, let us start with Eq. (2.3)

$$1=\mathcal{W}_d(\phi,L,t)+\frac{h^2}{\phi^2}, \tag{A1a}$$

where we have used the notation

$$\mathcal{W}_d(\phi,L,t)=\frac{t}{N}\sum_{m=-\infty}^{\infty}\sum_{\boldsymbol{q}}\times\frac{1}{\phi+2\sum_{i=1}^{d}(1-\cos q_i)+(2\pi t/\lambda)^2m^2}. \tag{A1b}$$

Now if we assume that the system is infinite in $d'$ dimensions, then we may write Eq. (A1b) in the following form:

$$\mathcal{W}_d(\phi,L,t)=\frac{tL^{d'-d}}{(2\pi)^{d'}}\sum_{\boldsymbol{q}(d-d')}\sum_{m=-\infty}^{\infty}\int_{-\pi}^{\pi}d^{d'}\boldsymbol{q}\int_0^\infty dx \times\exp\left\{-x\left[\phi+2\sum_i(1-\cos q_i)+\left(\frac{2\pi t}{\lambda}\right)^2m^2\right]\right\}, \tag{A2}$$

To obtain the last expression use has been made of the representation

$$\frac{1}{z}=\int_0^\infty \exp(-zx)dx, \tag{A3}$$

and that $N=L^d$.

Now by rearranging it is possible to write Eq. (A2) in the following form:

$$\mathcal{W}_d(\phi,L,t)=t\sum_{m=\infty}^{\infty}\int_0^\infty dx\exp\left[-x\left\{\phi+2d+\left(\frac{2\pi t}{\lambda}\right)^2m^2\right\}\right]\times[I_0(2x)]^{d'}\left[\frac{1}{L}\sum_q\exp(2x\cos q)\right]^{d-d'}. \tag{A4}$$

Here $I_0(x)$ is the modified Bessel function.

The use of Poisson summation formula

$$\frac{1}{L}\sum_{n=-(L-1)/2}^{n=(L-1)/2}G\left(\frac{2\pi n}{L}\right)=\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi}\frac{dq}{2\pi}G(q)\exp(iqlL), \tag{A5}$$

where $G(q)$ is a periodic function, allows us to continue the sum over the wave vector $q=2\pi n/L$ $(n\in[-L/2,L/2])$ to the rest of the real line periodically. With the aid of Eq. (A5) we can transform Eq. (A4) into

$$\mathcal{W}_d(\phi,L,t)=t\sum_{m=\infty}^{\infty}\int_0^\infty dx\exp\left[-x\left\{\phi+2d+\left(\frac{2\pi t}{\lambda}\right)^2m^2\right\}\right]\times[I_0(2x)]^{d'}\left[\sum_{l=-\infty}^{\infty}I_{lL}(2x)\right]^{d-d'}. \tag{A6}$$

In order to investigate the low-temperature effects for the model Hamiltonian (2.1) we use the Jacobi identity

$$\sum_{m=-\infty}^{\infty} \exp(-um^2) = \left(\frac{\pi}{u}\right)^{1/2} \sum_{m=-\infty}^{\infty} \exp\left(-m^2\frac{\pi^2}{u}\right), \tag{A7}$$

which applied to Eq. (A6) gives

$$\mathcal{W}_d(\phi,L,t) = \lambda\mathcal{W}_d(\phi) + \frac{\lambda}{2\pi^{1/2}} \sum_{m,\boldsymbol{l}(d-d')}{}' \int_0^\infty \frac{dx}{x^{1/2}} \times \exp\left[-x(\phi+2d) - \left(\frac{\lambda}{2tx}\right)^2 m^2\right] \times [I_0(2x)]^{d'} I_{\boldsymbol{l}L}(2x), \tag{A8}$$

where we have used the formal notations

$$\sum_{\boldsymbol{l}(d-d')} I_{\boldsymbol{l}L}(2x) = \left[\sum_{l} I_{lL}(2x)\right]^{d-d'}, \boldsymbol{l}^2 = l_1^2 + \cdots + l_{d-d'}^2,$$

$$\mathcal{W}_d(\phi) = \frac{1}{2(2\pi)^d} \int_{-\pi}^{\pi} dq_1 \cdots \int_{-\pi}^{\pi} dq_d \left(\phi + 2\sum_{i=1}^{d} (1-\cos q_i)\right)^{-1/2}. \tag{A9}$$

The prime means that the vector with components $m = l_1 = \cdots = l_{d-d'} = 0$ is omitted.

At sufficiently low temperature ($\lambda/t \gg 1$) and large enough size ($L \gg 1$), we can use the asymptotic form for the Bessel functions[12]

$$I_\nu(x) \approx \frac{e^{x-\nu^2/2x}}{\sqrt{2\pi x}} \left[1 + \frac{1}{8x} + \frac{9-32\nu^2}{2!(8x)^2} + \cdots\right], \tag{A10}$$

in order to get after substitution in Eq. (A1a)

$$1 = \lambda\mathcal{W}_d(\phi) + \frac{\lambda\phi^{(d-1)/2}}{(4\pi)^{(d+1)/2}} \times \sum_{m,\boldsymbol{l}(d-d')}{}' \frac{K_{(d-1)/2}\{\phi^{1/2}[(\lambda/t)^2m^2 + L^2\boldsymbol{l}^2]^{1/2}\}}{\{\phi^{1/2}[(\lambda/t)^2m^2 + L^2\boldsymbol{l}^2]^{1/2}\}^{(d-1)/2}}. \tag{A11}$$

The Watson-type integral $\mathcal{W}_d(\phi)$ [see Eq. (A9)] has been studied in considerable details;[27] for $1<d<3$, it can be approximated by ($\phi \ll 1$),

$$\mathcal{W}_d(\phi) \approx \mathcal{W}_d(0) - (4\pi)^{-(d+1)/2} \left|\Gamma\left(\frac{1-d}{2}\right)\right| \phi^{(d-1)/2}, \tag{A12}$$

which leads one to conclude that at zero temperature the system exhibits a phase transition driven by the parameter $\lambda$ at the quantum critical point:

$$\lambda_c = \frac{1}{\mathcal{W}_d(0)}. \tag{A13}$$

Finally, substituting Eq. (A12) in Eq. (A11) we obtain Eq. (4.1). Equation (3.2) is obtained by setting $d=d'$.

It is possible to transform Eq. (4.1) in the following equivalent forms:

$$L^{d-1}\delta\lambda + \left(\frac{hL^{(d+3)/2}}{L^2\phi}\right)^2 = \frac{(L\phi^{1/2})^{d-1}}{(4\pi)^{(d+1)/2}} \left[\left|\Gamma\left(\frac{1-d}{2}\right)\right| - 2\mathcal{K}_{\lambda/tL}\left(\frac{d-1}{2}\middle| d-d'+1, \frac{L\phi^{1/2}}{2}\right)\right], \tag{A14a}$$

where

$$\mathcal{K}_a(\nu|p,y) = \sum_{m,\boldsymbol{l}(p-1)}{}' \frac{K_\nu(2y\sqrt{\boldsymbol{l}^2 + a^2m^2})}{(y\sqrt{\boldsymbol{l}^2 + a^2m^2})^\nu},$$

$$y>0, \quad \boldsymbol{l}^2 = l_1^2 + l_2^2 + \cdots + l_{p-1}^2. \tag{A14b}$$

or

$$\left(\frac{t}{\lambda}\right)^{1-d} \delta\lambda + \left[\frac{h}{\phi}\left(\frac{t}{\lambda}\right)^2 \left(\frac{\lambda}{t}\right)^{(d+3)/2}\right]^2 = \frac{(\lambda\phi^{1/2}/t)^{d-1}}{(4\pi)^{(d+1)/2}} \left[\left|\Gamma\left(\frac{1-d}{2}\right)\right| - 2\widetilde{\mathcal{K}}_{tL/\lambda}\left(\frac{d-1}{2}\middle| d-d'+1, \frac{\lambda\phi^{1/2}}{2t}\right)\right], \tag{A15a}$$

where

$$\widetilde{\mathcal{K}}_a(\nu|p,y) = \mathcal{K}_{1/a}(\nu|p;ay) = \sum_{m,\boldsymbol{l}(p-1)}{}' \frac{K_\nu(2y\sqrt{a^2\boldsymbol{l}^2 + m^2})}{(y\sqrt{a^2\boldsymbol{l}^2 + m^2})^\nu}, \quad y>0. \tag{A15b}$$

The functions $\mathcal{K}_a(\nu|p;y)$ and $\widetilde{\mathcal{K}}_a(\nu|p;y)$ are anisotropic generalizations of the $\mathcal{K}$ function introduced in Ref. 12.

## APPENDIX B

In this appendix we will sketch a way to find the asymptotic behavior of the functions $\mathcal{K}_a(\nu|p,y)$ defined in Sec. IV [see Eq. (A14)]. They have the following form:

$$\mathcal{K}_a(\nu|p,y) = \sum_{m,\boldsymbol{l}(p-1)}{}' \frac{K_\nu(2y\sqrt{\boldsymbol{l}^2 + a^2m^2})}{(y\sqrt{\boldsymbol{l}^2 + a^2m^2})^\nu}, \quad y>0, \tag{B1a}$$

where

$$\boldsymbol{l}^2 = l_1^2 + l_2^2 + \cdots + l_{p-1}^2. \tag{B1b}$$

By the use of the integral representation of the modified Bessel function

$$K_\nu(2\sqrt{zt}) = K_{-\nu}(2\sqrt{zt}) = \frac{1}{2}\left(\frac{z}{t}\right)^{\nu/2} \int_0^\infty x^{-\nu-1} e^{-tx-z/x} dx \tag{B2}$$

and the Jacobi identity for a $p$-dimensional lattice sum

$$\sum_{m,\boldsymbol{l}(p-1)} e^{-(\boldsymbol{l}^2+a^2m^2)t}=\frac{1}{a}\left(\frac{\pi}{t}\right)^{p/2}\sum_{m,\boldsymbol{l}(p-1)} e^{-\pi^2(\boldsymbol{l}^2+m^2/a^2)/t}, \tag{B3}$$

we may write Eq. (B1) as

$$\mathcal{K}_a(\nu|p,y)=\frac{\pi^{p/2}}{2a}\Gamma\left(\frac{p}{2}-\nu\right)y^{-p}+\frac{\pi^{2\nu-p/2}}{2a}y^{-2\nu}\int_0^\infty dx x^{(1/2)p-\nu-1}e^{-xy^2/\pi^2}\times\left[\sum_{m,\boldsymbol{l}(p-1)}{}' e^{-x(\boldsymbol{l}^2+m^2/a^2)}-a\left(\frac{\pi}{x}\right)^{p/2}\right]. \tag{B4}$$

Let us notice that the two terms in the square brackets in the last equality cannot be integrated separately, since they diverge. Nevertheless, in order to encounter this divergence, we can transform further Eq. (B4) by adding and subtracting the unity from $\exp(-xy^2/\pi^2)$, which enables us to write down (after some algebra) the result

$$\mathcal{K}_a(\nu|p,y)=\frac{\pi^{p/2}}{2a}\Gamma\left(\frac{p}{2}-\nu\right)y^{-p}+\frac{\pi^{2\nu-p/2}}{2a}y^{-2\nu}C_a(p|\nu)-\frac{1}{2}\Gamma(-\nu)+\frac{\pi^{2\nu-p/2}}{2a}\frac{\Gamma(p/2-\nu)}{y^{2\nu}}\times\sum_{m,\boldsymbol{l}(p-1)}{}'\left[\left(\boldsymbol{l}^2+\frac{m^2}{a^2}+\frac{y^2}{\pi^2}\right)^{\nu-p/2}-\left(\boldsymbol{l}^2+\frac{m^2}{a^2}\right)^{\nu-p/2}\right], \tag{B5}$$

where

$$C_a(p|\nu)=\lim_{\delta\to 0}\int_\delta^\infty dx x^{(1/2)p-\nu-1}\left[\sum_{m,\boldsymbol{l}(p-1)}{}' e^{-x(\boldsymbol{l}^2+m^2/a^2)}-a\left(\frac{\pi}{x}\right)^{p/2}\right], \tag{B6a}$$

$$=\lim_{\delta\to 0}\left\{\sum_{m,\boldsymbol{l}(p-1)}{}'\frac{\Gamma[p/2-\nu,\delta(\boldsymbol{l}^2+m^2/a^2)]}{(\boldsymbol{l}^2+m^2/a^2)^{\nu-p/2}}-\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dm d^{p-1}\boldsymbol{l}\frac{\Gamma[p/2-\nu,\delta(\boldsymbol{l}^2+m^2/a^2)]}{(\boldsymbol{l}^2+m^2/a^2)^{\nu-p/2}}\right\} \tag{B6b}$$

is the Madelung-type constant and $\Gamma[\alpha,x]$ is the incomplete gamma function.

We see from Eq. (B5) that the shift of the critical quantum parameter is given by the Madelung-type constant (B6) instead of the sum in Eq. (4.2). Indeed it is possible to show that these two representations are equivalent. This may be done, following Ref. 33, by starting from the Jacobi identity Eq. (B3), where we multiply the two sides by $\delta^{p/2-\nu-1}$ and integrating over $\delta$ to obtain the key equation

$$C_a(p|\nu)=\sum_{m,\boldsymbol{l}(p-1)}{}'\frac{\Gamma[p/2-\nu,\delta(\boldsymbol{l}^2+m^2/a^2)]}{(\boldsymbol{l}^2+m^2/a^2)^{p/2-\nu}}-\frac{\delta^{p/2-\nu}}{p/2-\nu}+a\pi^{p/2-2\nu}\sum_{m,\boldsymbol{l}(p-1)}{}'\frac{\Gamma[\nu,\pi^2(\boldsymbol{l}^2+a^2m^2)/\delta]}{(\boldsymbol{l}^2+a^2m^2)^{\nu}}-a\frac{\pi^{p/2}}{\nu\delta^{\nu}}. \tag{B7}$$

Finally from Eq. (B7) we see easily that the integration constant $C_a(p|\nu)$ may be written in two different forms. In the first case we take the limit $\delta\to\infty$ and obtain

$$C_a(p|\nu)=a\pi^{p/2-2\nu}\Gamma(\nu)\sum_{m,\boldsymbol{l}(p-1)}{}'\frac{1}{(\boldsymbol{l}^2+a^2m^2)^{\nu}}. \tag{B8}$$

In the other case we take the limit $\delta\to 0$, and then both the first and last terms in the rhs of Eq. (B7) yields Eq. (B6).

Using a similar procedure we find, for the functions $\tilde{\mathcal{K}}_a(\nu|p,y)$ defined in Eq. (A15b), the following expression:

$$\tilde{\mathcal{K}}_a(\nu|p,y)=\frac{\pi^{p/2}}{2a^{p-1}}\Gamma\left(\frac{p}{2}-\nu\right)y^{-p}+\frac{\pi^{2\nu-p/2}}{2a^{p-1}}y^{-2\nu}\tilde{C}_a(p|\nu)-\left(\frac{1}{2}\right)\Gamma(-\nu)+\frac{\pi^{2\nu-p/2}}{2a^{p-1}}\frac{\Gamma(p/2-\nu)}{y^{2\nu}}\times\sum_{m,\boldsymbol{l}(p-1)}{}'\left[\left(\frac{\boldsymbol{l}^2}{a^2}+m^2+\frac{y^2}{\pi^2}\right)^{\nu-p/2}-\left(\frac{\boldsymbol{l}^2}{a^2}+m^2\right)^{\nu-p/2}\right]. \tag{B9}$$

Here the Madelung-type constant is given by

$$\widetilde{C}_a(p|\nu)=\lim_{\delta\to 0}\int_\delta^\infty dx x^{(1/2)p-\nu-1}\left[\sum_{m,\boldsymbol{l}(p-1)}{}' e^{-x(\boldsymbol{l}^2/a^2+m^2)}-a^{p-1}\left(\frac{\pi}{x}\right)^{p/2}\right], \quad \text{(B10a)}$$

$$=\lim_{\delta\to 0}\left\{\sum_{m,\boldsymbol{l}(p-1)}{}' \frac{\Gamma[p/2-\nu,\delta(\boldsymbol{l}^2/a^2+m^2)}{(\boldsymbol{l}^2/a^2+m^2)^{\nu-p/2}}-\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dm d^{p-1}\boldsymbol{l}\frac{\Gamma[p/2-\nu,\delta(\boldsymbol{l}^2/a^2+m^2)]}{(\boldsymbol{l}^2/a^2+m^2)^{\nu-p/2}}\right\} \quad \text{(B10b)}$$

$$=a^{p-1}\pi^{p/2-2\nu}\Gamma(\nu)\sum_{m,\boldsymbol{l}(p-1)}{}' \frac{1}{(a^2\boldsymbol{l}^2+m^2)^\nu}. \quad \text{(B10c)}$$

Equations (B5) and (B9) are slight generalizations (for the anisotropic case $a\neq 1$) of the result obtained in Ref. 33 from one side, and are related to the Watson-type sums proposed earlier in Ref. 13 from the other (see also Ref. 19).

If we set in Eqs. (B5) or (B9) $d=2$, $d'=0$, and $a=1$ we obtain the identity

$$\sum_{l_1,l_2}{}' \frac{\exp(-y\sqrt{l_1^2+l_2^2})}{\sqrt{l_1^2+l_2^2}}=\frac{2\pi}{y}+4\zeta\left(\frac{1}{2}\right)\beta\left(\frac{1}{2}\right)+y+2\pi\sum_{l_1,l_2}{}' \left\{\frac{1}{\sqrt{y+4\pi^2(l_1^2+l_2^2)}}-\frac{1}{2\pi\sqrt{l_1^2+l_2^2}}\right\}. \quad \text{(B11)}$$

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