Vortex motion in charged and neutral superfluids: A hydrodynamic approach

E. B. Sonin
*Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
and Ioffe Physical Technical Institute, 194021 St. Petersburg, Russia*

V. B. Geshkenbein
*Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
and L. D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia*

A. van Otterlo
*Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
and Physics Department, University of California, Davis, California 95616*

G. Blatter
Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
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We derive a Galilean invariant expression for the electric field induced by a vortex moving through a charged superfluid at $T=0$, which holds for any superconductor, from the dirty to the superclean limit. The contribution of different areas around the vortex to the average electric field and to the charge distribution is analyzed. The results are extended to a neutral system, where the chemical potential takes over the role of the electrostatic potential in the charged situation. Different contributions to the vortex mass in charged and neutral superfluids are brought together for comparison and discussion. [S0163-1829(98)03301-3]

I. INTRODUCTION

Dynamical properties of vortices in type-II superconductors have recently attracted a lot of attention.[1] The real-time dynamics of the vortex lines determines the dissipation and the Hall effect in current driven superconductors, with interesting findings in the high-temperature superconductors, such as the peculiar sign change in the Hall effect close to the transition to the normal state.[2] The imaginary-time dynamics of the flux lines determines the low-temperature thermodynamic properties of the vortex system,[3] such as the specific heat,[4] as well as the low-temperature quantum creep in a driven system.[5,6] The vortex motion is associated with the generation of an electric field, giving rise to dissipation if the electric field is oriented parallel to the driving current density **j** (vortices moving transverse to **j**), or producing a finite Hall voltage in the case where the vortex moves with the superfluid. Although the problem has been studied since the early times of vortex dynamics in superconductors[7] and has been properly discussed in the textbooks,[8] little attention has been paid to the general case where the vortex moves under the action of both dissipative and Hall forces and it is the main purpose of the present paper to fill this gap. In particular, we will be concerned with the question of which velocity (the vortex velocity measured with respect to the laboratory frame or with respect to the superflow) shows up in the expression for the electric field and which area around the vortex line predominantly contributes to its average value. In order to reply on these questions we will study the vortex dynamics within a hydrodynamic description, which produces simple and transparent results on the interplay between vortex motion and the generation of local and average electric fields.

A central element in our analysis is played by the vortex equation of motion, which for the case of uniform motion comprises two terms,[9,10] the dissipative and Hall terms $-\eta \mathbf{v}_L$ and $\eta' \mathbf{v}_L \times \mathbf{n}$, both linear in the vortex velocity $\mathbf{v}_L$ as measured with respect to the laboratory frame of reference ($\mathbf{n}$ denotes the direction of the vortex line). An important point to be discussed in this context is the relative importance of these two contributions. The relevant parameter addressing this question is the purity of the sample as quantified by the dimensionless product $\omega_0 \tau_r$, where $\omega_0 \sim \Delta^2/\varepsilon_F$ denotes the minigap separating quasiparticle states trapped in the vortex core[11] and τ_r is the relaxation time (ε_F is the Fermi energy, Δ denotes the energy gap). In usual superconductors the dissipative term is dominant, however, in the superclean limit $\omega_0 \tau_r > 1$, where the quasiparticle states are well defined, the Hall term takes over the leading role (in terms of the mean free path l the criterion for the superclean limit takes the form $l > \xi \varepsilon_F/\Delta$, where ξ denotes the coherence length of the superconductor). As a result, the vortices move transverse to the applied current density in the dirty case and move with the superfluid in superclean superconductors.

The electric field generated by the moving vortex can be split into two terms associated with the longitudinal (scalar) and transverse (vector) potentials V and $\mathbf{A}$. The dipolar term deriving from the scalar potential V is governed by the vortex velocity in the frame moving with the superfluid. It produces local electric fields both in the vicinity of the vortex core and at distances of the order of the London penetration depth λ away from the vortex line, however, its large-scale average vanishes. The local electric fields around the vortex core are quite large in the dirty limit, of order $(v_L/c)H_{c_2}$ where H_{c_2} denotes the upper critical field, but vanish in the superclean limit. The electric field deriving from the transverse potential $\mathbf{A}$ is associated with the moving flux of the vortex line ($\mathbf{B}=\nabla\times\mathbf{A}$) and hence involves the velocity v_L measured in the laboratory frame. In the end it is this term that determines the average electric field.

Recently, the electric fields and charges associated with a stationary vortex have attracted a great deal of attention within the context of the sign change in the Hall coefficient (the Hall anomaly)[12,13] and with respect to a possible direct observation of the electric field associated with the vortex charge.[14] Here, we will only be concerned with the corresponding effects associated with the uniform motion of the vortex.

A further point often discussed within the context of vortex dynamics is the vortex mass, contributing the inertial term $-\mu\partial_t\mathbf{v}_L$ to the vortex equation of motion. Various contributions to the vortex mass have been discussed in the past, starting from the work of Suhl,[15] who, based on the time-dependent Ginzburg-Landau (TDGL) theory, determined the inertial forces due to the quasiparticle states trapped within the vortex core ($\rightarrow\mu_{\mathrm{core}}$) and arising from the energy stored in the electromagnetic fields around the moving vortex ($\rightarrow\mu_{\mathrm{em}}$). With $\mu_{\mathrm{core}}\sim mk_F$ and $\mu_{\mathrm{em}}\sim(\Phi_0/c\xi)^2$ the core mass is by a factor $(\xi/r_D)^2$ larger than the electromagnetic one. Here, k_F is the Fermi wave vector and r_D is the Debye screening length, $\Phi_0=hc/2e$ denotes the flux unit and ξ is the coherence length of the superconductor. These results have later been confirmed by Kuprianov and Likharev[16] and by Duan and Leggett.[17]

The vortex mass due to the trapped quasiparticles depends on the purity of the superconductor. In fact, Suhl's result[15] for the vortex core mass applies to the dirty regime, where the vortex core can be described in terms of a normal metallic cylinder. In the superclean regime the quasiparticles trapped in the core have to be treated more accurately. Based on the work of Kopnin and Kravtsov,[9] Kopnin[18] derived the vortex mass in the superclean limit and found a very large core mass $\mu_{\mathrm{core}}\sim mk_F(\varepsilon_F/\Delta)^2$. Recent work[10,19–21] discussing the core mass in terms of the core bound states has confirmed this result and the crossover from the dirty limit to the superclean limit results has been analyzed in Ref. 22.

A third contribution to the vortex mass is due to the volume difference of the metal in the normal and superconducting states, producing a polaron-type mass due to the lattice deformation accompaning the motion of the vortex, see work by Coffey[23] and by Duan and Šimánek.[24] Rough estimates place this contribution to the vortex mass in the range of the electromagnetic one. Finally, the mass of vortices in other systems such as the neutral superfluid ^{4}He and Josephson junction arrays has been discussed by Baym and Chandler[25] and by Šimánek and by Eckern and Schmid.[26]

Despite the appreciable size of the vortex mass in particular situations, e.g., the large core mass in the superclean limit, the low-frequency dynamics of the vortex is always dominated either by the dissipative or by the Hall force (note that the description of the vortex dynamics in terms of the nondispersive transport coefficients η, η', and μ breaks down at higher frequencies $\omega>\omega_0$). A notable exception is found in the vortex motion in Josephson junction arrays, where the electromagnetic mass plays an important role. However, in order to decide upon the irrelevance of the vortex mass, we still need to know in detail the size of its various contributions, and we will make use of our hydrodynamic analysis to provide insight into the origin and size of various mass terms in the equation of motion. In particular, we will study the electromagnetic mass arising from the electric fields around the vortex core and discuss its crossover to the compressibility mass in the limit of an uncharged superfluid. Furthermore, we will discuss the core and backflow masses and comment on their interrelation.

We give a short outline: In Sec. II we discuss the electric field and the charge arising from a vortex moving through a superfluid at $T=0$ in both the screened ($\xi\gg r_D$) and unscreened ($\xi\ll r_D$) regimes. In Sec. III we summarize the situation concerning the vortex mass and derive various contributions to the mass based on hydrodynamic (London electrodynamic) considerations.

II. ELECTRIC FIELD AND CHARGE DENSITY AROUND A MOVING VORTEX

Let us consider a straight vortex line moving in a charged superfluid (a superconductor). The superfluid may be treated as an ideal fluid with its motion described by the Euler equation

$$m\frac{\partial\mathbf{v}_s}{\partial t}+m(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s=e\mathbf{E}+\frac{e}{c}[\mathbf{v}_s\times\mathbf{B}]-\nabla\mu+\frac{1}{n}\mathbf{F}_\Sigma\,\delta(\mathbf{r}-\mathbf{r}_L). \tag{1}$$

Here, $\mathbf{v}_s=(\hbar/2m)[\nabla\phi-(2\pi/\Phi_0)\mathbf{A}]$ is the gauge invariant superfluid velocity, $\mu=\partial_n F$ is the chemical potential measured in the frame moving with the superfluid velocity $\mathbf{v}_s$, with F the free-energy and n the electron density. In order to avoid complications with the choice of mass m, we assume a nearly free-electron model such that m is the free-electron mass, whereas $2m$ is the mass of the Cooper pair with charge $2e$. We restrict our analysis to the $T=0$ case, but will retain the subscript s for the velocity $\mathbf{v}_s$ in order to emphasize its superfluid nature.

Without the δ-function term on the right-hand side, Eq. (1) describes the motion of a charged ideal liquid subject to an electromagnetic field in the absence of any further external forces acting on the liquid. The δ-function term in Eq. (1) then represents the *total* external force $\mathbf{F}_\Sigma$ acting on the superfluid through the presence of a vortex line at the position $\mathbf{r}_L$ and includes forces arising from quasiparticle scattering in the vortex core as well as pinning forces. In the flux-flow regime at weak magnetic fields $B\ll H_{c2}$, where there is no pinning, and neglecting effects of normal currents, the most general expression for the external force in an axisymmetric medium is

$$\mathbf{F}_\Sigma=\eta\mathbf{v}_L-\eta'[\mathbf{v}_L\times\mathbf{n}]. \tag{2}$$

The force on the right-hand side arises from scattering of quasiparticles in the vortex core, e.g., impurity scattering, see Ref. 9. Since the impurities are at rest with respect to the crystal, they move with velocity $-\mathbf{v}_L$ with respect to the vortex, where $\mathbf{v}_L=\partial_t\mathbf{r}_L$ denotes the vortex velocity in the reference frame of the crystal. In reality, the force $\mathbf{F}_\Sigma$ is distributed over a finite area around the vortex line, at least the core area or even a larger one, in which the scattering of quasiparticles by the vortex field occurs. Still, the δ-function force is a good approximation as long as the dimension of this area is less than other spatial scales involved in the problem, such as the London penetration depth λ or the intervortex distance.

The external force $\mathbf{F}_\Sigma$ applied to the vortex is balanced by the *hydrodynamic* Magnus force[27,28]

$$en_s\frac{\Phi_0}{c}[(\mathbf{v}_L-\mathbf{v}_0)\times\mathbf{n}]=\mathbf{F}_\Sigma\,. \tag{3}$$

The total superfluid velocity field around the vortex line consists of the transport velocity $\mathbf{v}_0$ of the flow past the vortex, superimposed on the circular velocity field $\mathbf{v}_v$ around the stationary vortex,

$$\mathbf{v}_s(\mathbf{r})=\mathbf{v}_0+\mathbf{v}_v(\mathbf{r}). \tag{4}$$

The velocity $\mathbf{v}_0$ determines the transport supercurrent $\mathbf{j}=en_s\mathbf{v}_0$, which is assumed to be spatially uniform near the vortex line. In fact, due to backflow effects, this assumption fails very close to the vortex core, see Sec. III D, however, since all such perturbations decrease faster than $v_v\propto 1/r$ away from the vortex they are irrelevant for the present analysis based on the Magnus-force relation (3). At distances less than λ from the vortex line, the flow $\mathbf{v}_v$ is given by

$$\mathbf{v}_v(\mathbf{r})=\frac{\hbar}{2m}\frac{\mathbf{n}\times(\mathbf{r}-\mathbf{r}_L)}{|\mathbf{r}-\mathbf{r}_L|^2}. \tag{5}$$

Note that with the specification ''hydrodynamic'' Magnus force we refer to the left-hand side of Eq. (3), in contrast to the *superfluid* Magnus force in the superconductivity theory, which usually refers to the contribution proportional to the vortex velocity only. The second contribution proportional to the transport velocity $\mathbf{v}_0$ is the Lorentz force. The external force given by Eq. (2) also contains a component $\propto\eta'$ transverse to the vortex velocity $\mathbf{v}_L$. In the force balance equation for the vortex [combine Eqs. (2) and (3)] one may unite this force with the bare superfluid Magnus force $\propto n_s$ into one term, usually called the Hall or *effective* Magnus force. The latter then determines all transverse dynamical processes such as the Hall effect or quantum Hall tunneling of the vortices.[28]

For the present analysis, which concentrates on the electric field and the vortex mass, we do not need to use the vortex force balance equation, however. What we need to know is the vortex velocity $\mathbf{v}_L$ resulting from this equation, rendering our analysis quite general. In particular, we are not restricted to any special regime of vortex motion, such as the flux-flow regime for which Eq. (2) holds. Instead, the external force may include the pinning force as well.

We transform the Euler equation (1) using Eq. (3) and the following vector identities for the gauge invariant velocity $\mathbf{v}_s=(\hbar/2m)[\nabla\phi-(2\pi/\Phi_0)\mathbf{A}]$,

$$(\mathbf{v}_s\cdot\boldsymbol{\nabla})\mathbf{v}_s=\nabla\frac{v_s^2}{2}-[\mathbf{v}_s\times[\boldsymbol{\nabla}\times\mathbf{v}_s]], \tag{6}$$

$$\boldsymbol{\nabla}\times\mathbf{v}_s=\frac{e}{m}\frac{\Phi_0}{c}\mathbf{n}\delta(\mathbf{r}-\mathbf{r}_L)-\frac{e}{mc}\mathbf{B}. \tag{7}$$

Then the Euler equation may be written as

$$m\frac{\partial\mathbf{v}_s}{\partial t}=e\mathbf{E}-\nabla\left(\mu+\frac{mv_s^2}{2}\right)+\frac{e\Phi_0}{c}(\mathbf{v}_L\times\mathbf{n})\,\delta(\mathbf{r}-\mathbf{r}_L). \tag{8}$$

Outside the core Eq. (8) is just Newton's second law applied to the superfluid electrons and is equivalent to the Josephson relation: indeed, bearing in mind that $\mathbf{E}=-\nabla V-\partial_t\mathbf{A}/c$, where V is the electrostatic potential, one obtains (neglecting the δ-function term)

$$-\hbar\frac{\partial\phi}{\partial t}=2eV+2\left(\mu+\frac{mv_s^2}{2}\right). \tag{9}$$

The right-hand side of this equation is the total electrochemical potential measured in the laboratory frame of reference. On the other hand, the δ-function term on the right-hand side of Eq. (8) represents the effect of phase slips due to the flow of singular vortex lines: as a vortex line crosses the line connecting two close points in the liquid, the phase difference between these two points jumps by 2π.

For a stationary vortex motion we can replace the time derivative by $\partial_t\equiv-\mathbf{v}_L\cdot\boldsymbol{\nabla}$. Then

$$\frac{\partial\mathbf{v}_s}{\partial t}=-(\mathbf{v}_L\cdot\boldsymbol{\nabla})\mathbf{v}_s=-\nabla(\mathbf{v}_L\cdot\mathbf{v}_s)+[\mathbf{v}_L\times[\boldsymbol{\nabla}\times\mathbf{v}_s]], \tag{10}$$

and the Euler equation (8) yields the desired relation giving the electric field and the chemical potential in terms of the vortex velocity $\mathbf{v}_L$,

$$e\mathbf{E}-\nabla\left(\mu+\frac{mv_v^2}{2}\right)+m\nabla[(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v]+\frac{e}{c}[\mathbf{v}_L\times\mathbf{B}]=0, \tag{11}$$

where Eq. (11) is valid in both charged and neutral superfluids. Further on, we are interested in the fields produced by the vortex motion and therefore ignore the contribution $mv_v^2/2$ responsible for the electrostatic fields and the charge of the vortex at rest (the latter have been analyzed in Refs. 12 and 14).

The variation in the chemical potential $\delta\mu$ relates to the density modulation δn via

$$\delta\mu=\frac{\partial\mu}{\partial n}\delta n=\frac{ms^2}{n}\delta n, \tag{12}$$

where s is the sound velocity. In superconductors (i.e., charged superfluids) δn is connected with the electric field via Poisson's equation,

$$\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi e\,\delta n, \tag{13}$$

and one can rewrite Eq. (11) as a differential equation for the electric field,

$$e[\mathbf{E}-r_D^2\nabla(\boldsymbol{\nabla}\cdot\mathbf{E})]+m\nabla[(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v]+\frac{e}{c}[\mathbf{v}_L\times\mathbf{B}]=0, \tag{14}$$

with the Debye radius r_D given by

$$r_D^2 = \frac{1}{4\pi e^2}\frac{\partial \mu}{\partial n} = \frac{ms^2}{4\pi e^2 n}. \qquad (15)$$

A. Debye screening

At distances away from the vortex line exceeding the Debye radius r_D the condition of quasineutrality holds and we can neglect variations in the chemical potential, i.e., we can ignore the term $\propto r_D^2$ in Eq. (14). Solving for the electric field we obtain

$$\mathbf{E} = -\frac{m}{e}\nabla[(\mathbf{v}_L - \mathbf{v}_0)\cdot \mathbf{v}_v] - \frac{1}{c}[\mathbf{v}_L \times \mathbf{B}], \qquad (16)$$

a result that is valid outside the core region. The result (16) is Galilean invariant. In the limit of an ideal fluid (a superclean superconductor) the vortex moves with the fluid, $\mathbf{v}_L = \mathbf{v}_0$, and only the second term describing the moving flux survives. On the other hand, in a dirty superconductor the vortex velocity $\mathbf{v}_L$ by far exceeds the transport velocity $\mathbf{v}_0$ and the first term, which is singular at small r, becomes important. As a result, the electric field in the core is quite large in this case, of the order of $H_{c_2} v_L / c$, see Ref. 8. This large electric field is needed to push the normal current through the core region.

Let us analyze the two terms in Eq. (16) in more detail: we may express the electric field in Eq. (16) via scalar and vector potentials, $\mathbf{E} = -\nabla V - \partial_t \mathbf{A}/c$. A convenient choice for these potentials is to refer to the vector potential $\mathbf{A}$ as that of the equilibrium magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ around the stationary vortex line. Then the second term in Eq. (16) originates from this vector potential via the vortex motion: $\partial_t \mathbf{A} = -(\mathbf{v}_L \cdot \nabla)\mathbf{A} = \mathbf{v}_L \times \mathbf{B}$. Second, the gradient term in Eq. (16) corresponds to the scalar potential $V = (m/e) \times (\mathbf{v}_L - \mathbf{v}_0)\cdot \mathbf{v}_v$. At distances small compared to λ the velocity $\mathbf{v}_v$ is given by Eq. (5) and our scalar potential V is simply the electrostatic potential $V = (2\mathbf{p}\cdot \mathbf{r})/r^2$ of a line of dipoles with dipole moment $\mathbf{p} = (\hbar/4e)[(\mathbf{v}_L - \mathbf{v}_0)\times \mathbf{n}]$ per unit length [here and later on we assume that in Eq. (5) $\mathbf{r}_L = 0$; note that the line charge q generates the potential $V_q(r) = 2q \ln r$ and the line dipole $\mathbf{p} = q\mathbf{u}$ produces the above dipole potential including the factor 2].

In order to obtain the single-vortex contribution to the average electric field we integrate the expression (16) over the area around the vortex line. Let us integrate the potential part $-\nabla V$ over the area restricted by the radius r_0 with $r_c < r_0 < \lambda$ intermediate between the core radius r_c and the transverse screening length λ. In this case the integral does not depend on r_0 and is determined by the line dipole moment $\mathbf{p}$ [for the analogous three-dimensional (3D) electrostatic problem, see Ref. 29],

$$-\int_{r<r_0} d^2 r\, \nabla V = -\frac{m}{e}\int_{r=r_0} d\varphi [(\mathbf{v}_L - \mathbf{v}_0)\cdot \mathbf{v}_v]\mathbf{r}$$

$$= -2\pi \mathbf{p} = -\frac{\Phi_0}{2c}[(\mathbf{v}_L - \mathbf{v}_0)\times \mathbf{n}]. \qquad (17)$$

The independence from the choice of r_0 tells us that the integral in Eq. (17) draws its weight mostly from the core area, whereas the contribution from large distances $r > r_c$ is not essential. Thus Eq. (17) represents the contribution of the core electric fields to the average electric field, even though it is evident that inside the core the electric scalar potential is different from the potential $V = (m/e)(\mathbf{v}_L - \mathbf{v}_0)\cdot \mathbf{v}_v$ produced by the line of dipoles: in making use of Gauss' theorem it is sufficient to know that the scalar potential is continuous everywhere inside the core region.

However, the scalar potential differs from the dipole potential not only at small distances on the order of r_c, but also at large distances on the order of λ, where the velocity field $\mathbf{v}_v$ of the vortex decreases exponentially. At distances from the vortex line much larger than the London penetration depth λ all the fields, including V, vanish, thus if the radius r_0 in Eq. (17) exceeds λ the integral also vanishes. We thus conclude that the scalar potential part of the electric field induced by the vortex motion *does not* contribute to the *average electric field* at all: the large electric field inside the core is compensated by the scalar field at $r \sim \lambda$, which, though much smaller, is distributed over the much larger area $\sim \lambda^2$.

In summary, we may represent the integral of the electric field as consisting of three terms:

$$\int_{r<\infty} d^2 r\, \mathbf{E} = -\underbrace{\frac{\Phi_\circ}{2c}[(\mathbf{v}_L - \mathbf{v}_0)\times \mathbf{n}]}_{I}$$

$$+\underbrace{\frac{\Phi_\circ}{2c}[(\mathbf{v}_L - \mathbf{v}_0)\times \mathbf{n}]}_{II} - \underbrace{\frac{\Phi_\circ}{c}[\mathbf{v}_L \times \mathbf{n}]}_{III}. \qquad (18)$$

The term I is the scalar potential from the core (originating from small distances), the term II is due to the scalar potential arising from distances $\sim \lambda$, and the term III is due to the vector potential part at large distances. Rearranging terms in the above manner, one then concludes that the average electric field generated by a moving vortex is not originating from the core: the core contribution I yields only one-half of the overall average field (which is compensated to zero further away from the core) in the dirty limit and is not important at all in the superclean limit. In the end, only the part due to the vector potential [the contribution III in Eq. (18)] survives the averaging process and thus determines the average electric field. This interpretation is at variance with the more traditional one as found, e.g., in Ref. 8, where the average electric field comes only from the core.

Finally, assuming a vortex array with density $n_v = B/\Phi_0$ the average electric field is still given by the well-known expression

$$\langle \mathbf{E} \rangle = -\frac{1}{c}[\mathbf{v}_L \times \langle \mathbf{B} \rangle]. \qquad (19)$$

Let us note here that starting from the paper by Bardeen and Stephen[7] it has been assumed that the dipole electric field is proportional to the vortex velocity $\mathbf{v}_L$ measured in the laboratory frame of reference (see, e.g., Ref. 8). Our result (16) shows that instead the velocity measured with respect to the moving superfluid, $\mathbf{v}_L - \mathbf{v}_0$, is the relevant quantity. The origin of this discrepancy is found in the term $mv_s^2/2$ in the Euler equation, which is essential for the Galilean invariance of the theory and which has been ignored in previous work.

Nevertheless, one may neglect the transport velocity $\mathbf{v}_0$ in the dirty limit when $v_L \gg v_0$, which is the most relevant situation in conventional superconductors. In general, however, Eq. (16) should be used, which shows that the dipole field vanishes in the superclean limit when $\mathbf{v}_L = \mathbf{v}_0$.

B. Charge distribution

Next, let us determine the charge distribution arising from the moving vortex. In order to find this charge density, we insert Eq. (16) into the Poisson equation (13). In the absence of London screening the dipole field proportional to the relative velocity $\mathbf{v}_L - \mathbf{v}_0$ is divergence free and therefore does not produce a charge. However, including London screening, small relativistic corrections arise. In order to find them we have to take into account that $\nabla^2 \mathbf{v}_v = (1/en_s)\nabla^2 \mathbf{j}_v = (e/mc^2)\ \mathbf{j}_v$, where $\mathbf{j}_v = en_s \mathbf{v}_v = (c/4\pi)[\boldsymbol{\nabla} \times \mathbf{B}]$ is the equilibrium circular current density around the vortex. Making use of Eqs. (13) and (16) we arrive at the charge density

$$\begin{aligned}\rho_q &= \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E} = -\frac{m}{4\pi e}\nabla^2[(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v] - \frac{1}{4\pi c}\boldsymbol{\nabla}\cdot[\mathbf{v}_L\times\mathbf{B}] \\ &= -\frac{1}{4\pi c}(\mathbf{v}_L-\mathbf{v}_0)\cdot[\boldsymbol{\nabla}\times\mathbf{B}] + \frac{1}{4\pi c}\mathbf{v}_L\cdot[\boldsymbol{\nabla}\times\mathbf{B}] \\ &= \frac{1}{c^2}\mathbf{j}_v\cdot\mathbf{v}_0 .\end{aligned} \tag{20}$$

Note that all the charges considered above are local, i.e., the fluxon has no total charge. The same is true for a uniform array of vortices: the average charge in the unit cell of a vortex array vanishes exactly. However, if the vortex density or vortex velocity varies in space, a small average charge per vortex might arise, in analogy with the corresponding electrostatic effect in an insulator: the spatial variation of the polarization $\mathbf{P}$ induces a polarization charge $-\boldsymbol{\nabla}\cdot\mathbf{P}$, though there is no charge density in the uniform state.

C. Uncharged superfluids: chemical potential and density variations

A second case of interest is the vortex in an uncharged liquid (or, equivalently, in a ''weakly'' charged superconductor at distances from the vortex line much smaller than the Debye radius r_D). In this case we can neglect the electric and magnetic fields in Eq. (11) and obtain an expression for the chemical potential μ. The term $\propto v_v^2$ is important only for the density variation around the stationary vortex (which we ignore here, see Ref. 14 for a discussion of the stationary vortex charge). The additional variations of the chemical potential and the density originating from the vortex motion then are given by

$$\delta\mu = \frac{\partial\mu}{\partial n}\,\delta n = \frac{ms^2}{n}\,\delta n = m(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v . \tag{21}$$

We observe that in the uncharged liquid the chemical potential μ takes over the role of the electrostatic potential in the charged liquid. According to Eq. (21), the vortex motion produces a density variation $\propto 1/r$, yielding a logarithmic contribution to the vortex mass (see Sec. III B).

III. VORTEX MASS

The variation of the electric field and the chemical potential around the moving vortex is associated with an additional kinetic energy $\mu_{\mathrm{v}}(\mathbf{v}_L-\mathbf{v}_0)^2/2$ in the frame moving with the fluid. In order to find an expression for the vortex mass μ_{v} we have to estimate this energy. Such a program was carried out before in a number of works using the TDGL theory;[15,17] however, such an approach suffers from the restricted regime of applicability of the TDGL approach. Luckily, the contributions to the vortex mass originating from outside the vortex core can be calculated without invoking the TDGL theory, as we may use simple London electrodynamics that is free from the restrictions imposed on the application of the TDGL theory.

A. Electromagnetic mass in a superconductor

In usual superconductors the core radius $r_c \approx \xi$ exceeds the Debye radius r_D (as usual, ξ denotes the coherence length). In this case, the contribution to the electromagnetic vortex mass from the area outside the core is mainly due to the electric dipole field,

$$\begin{aligned}\mu_{\mathrm{em}}\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2} &= \int_{r>r_c} d^2r\frac{E^2}{8\pi} \\ &= \frac{1}{8\pi}\left(\frac{m}{e}\right)^2\int_{r>r_c} d^2r[\nabla(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v]^2 \\ &= \left(\frac{\Phi_0^2}{c^2\xi^2}\right)\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2},\end{aligned} \tag{22}$$

where we have made use of Eq. (16). This result agrees with the mass as calculated by Suhl[15] and by Duan and Leggett.[17] Note that additional corrections to μ_{em} of the order of $(\xi/\lambda)^2$ arise from the magnetic field term in Eq. (16) as well as from cutting off the dipole field at the distance λ.

B. Compressibility mass in a neutral superfluid

In a neutral superfluid the contribution to the vortex mass from outside the core area is due to the variation in the particle density,

$$\begin{aligned}\mu_c\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2} &= \int_{r>r_c} d^2r\frac{\partial\mu}{\partial n}\,\frac{\delta n^2}{2} \\ &= \frac{m^2}{2}\,\frac{\partial n}{\partial\mu}\int_{r>r_c} d^2r[(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{v}_v]^2 \\ &= \frac{mn}{4\pi s^2}\left(\frac{\hbar}{2m}\right)^2 \ln\frac{R}{r_c}\,\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2},\end{aligned} \tag{23}$$

where we have made use of Eq. (21). Thus the compressibility mass is given as the ratio between the static vortex energy and the square of sound velocity (see Refs. 17 and 30 and references therein). As the static vortex energy, the vortex mass involves a logarithmic divergence, which has to be cut at some hydrodynamic scale R, e.g., the intervortex distance. Note that the compressibility mass μ_c in a Fermi superfluid differs essentially from the one in the weakly interacting

Bose superfluid. In the former case the compressibility $\partial\mu/\partial n = ms^2/n$ is large with a sound velocity of the order of the Fermi velocity, $s = v_F/\sqrt{3}$. Thus, apart from the logarithmic factor (which may be large, indeed), the compressibility mass is of the same order as the mass, $\sim mk_F \sim mn^{-1/3}$ obtained by Suhl from TDGL theory.[15] However, within the Gross-Pitaevskii theory[31] for the weakly interacting Bose gas the chemical potential is $\mu = V_{\rm int} n = \hbar^2/2m\xi^2$, where $V_{\rm int}$ quantifies the interaction and ξ is the coherence length, which is much larger than the interparticle distance $n^{-1/3}$ and even becomes infinite in the ideal Bose gas. Thus, the compressibility and, correspondingly, the sound velocity $s = \sqrt{(n/m)\partial\mu/\partial n} \sim \hbar/m\xi$ are small and as a consequence, the compressibility mass is quite large, of the order of $\sim mn\xi^2$. This result agrees with the expression obtained by Baym and Chandler.[25]

For a better understanding of the crossover between the charged and uncharged superfluids it is useful to consider a ''weakly'' charged superfluid with a Debye radius r_D essentially exceeding the core radius r_c. Then the total mass from outside the core can roughly be estimated as the sum of the compressibility mass (from distances $r < r_D$) and the electric field mass originating from large distances $r > r_D$. The Debye radius r_D then appears as the upper cutoff in the compressibility mass and the lower cutoff in the electric mass and the final result reads

$$\mu_{\rm mix} \approx \frac{\Phi_0^2}{c^2}\left(\frac{1}{r_D^2}\ln\frac{r_D}{r_c} + \frac{1}{r_D^2}\right). \tag{24}$$

C. Core mass

The above mechanisms responsible for the vortex mass outside the core can be extrapolated onto the core itself, assuming that the divergent growth of the electric fields or the chemical potential is cut off in the core. This procedure yields a core contribution to the electric mass that is of the same order as that from the London region, Eq. (22), whereas the core contribution to the compressibility mass, Eq. (23), merely adds a numerical factor to the argument of logarithm and thus is not important if the logarithm itself is large.

More importantly, the core region generates a mass which is due to the quasiparticles localized within the core, the determination of which requires a microscopic analysis. Such an analysis was first carried out by Kopnin,[18] who solved the kinetic equation for those quasiparticles trapped in the core, and is at present vividly discussed.[10,19,20] In the superclean limit the core mass $\mu_{\rm core} \sim mn\xi^2$ is large, as if all particles inside the core move with the vortex velocity $\mathbf{v}_L$, thus contributing with their inertia to $\mu_{\rm core}$. On the other hand, in the dirty limit the core mass is quite small, of the order of the Suhl mass $\mu_{\rm core} \sim mk_F$ as discussed in Refs. 20 and 22.

D. Backflow mass

Within this context it is interesting to discuss one further contribution to the vortex mass, which admits a simple minded estimation within the hydrodynamic (London electrodynamic) approach. This contribution was first discussed by Baym and Chandler for the ^{4}He superfluid[25] and is due to the backflow induced by the moving vortex core: as the superfluid moves with respect to the vortex, the current, either completely or partly, flows around the vortex core since the superfluid density is suppressed in the core. For the simplest situation of a hard-core model (with no current penetrating through the core of radius r_c) a backflow arises that is given by the dipole velocity field[25]

$$\mathbf{v}_{\rm bf}(\mathbf{r}) = -r_c^2 \nabla\left[\frac{(\mathbf{v}_L - \mathbf{v}_0)\cdot\mathbf{r}}{r^2}\right]. \tag{25}$$

The kinetic energy of the backflow then contributes to the vortex mass with a term

$$\mu_{\rm bf}\, n\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2} = \frac{mnr_c^4}{2}\int_{r>r_c} d^2r \left|\nabla\left[\frac{(\mathbf{v}_L-\mathbf{v}_0)\cdot\mathbf{r}}{r^2}\right]\right|^2 = \pi r_c^2 mn\frac{(\mathbf{v}_L-\mathbf{v}_0)^2}{2}. \tag{26}$$

With $r_c \sim \xi$ we obtain a backflow mass $\mu_{\rm bf} \sim \pi mn\xi^2$. Comparing this mass to the compressibility mass Eq. (23) for the weakly interacting Bose gas we find that $\mu_{\rm bf}$ is smaller by a large logarithm factor. However, for the weak-coupling BCS superconductor the backflow mass for an *inpenetrable* core exceeds the core mass as obtained by Suhl by the large factor $\xi^2 k_F^2 \sim \epsilon_F^2/\Delta^2$ and matches up with the core mass due to the trapped quasiparticles as obtained in the superclean limit. It then seems that the assumption of an impenetrable core is applicable to the superclean limit, whereas in a dirty material the current will flow through the core, thus producing a reduced backflow field. Indeed, within the Bardeen-Stephen model[7] one assumes that the total current density flows homogeneously through the sample with no backflow arising around the core region. It then seems that the backflow mass around the core can be related to the vortex core mass due to the trapped quasiparticles through a core transparency that depends on the relaxation time of the quasiparticles.

IV. CONCLUSIONS

Summarizing, using a London electrodynamic/hydrodynamic approach we have derived expressions for the electric field/chemical potential and the charge/density modulation produced by a moving vortex. The results are general and refer to any superconductor, dirty, clean, or superclean. We have obtained a singular dipolar contribution to the electric field that is linear in the *relative* velocity $\mathbf{v}_L - \mathbf{v}_0$ of the vortex line with respect to the superfluid velocity $\mathbf{v}_0$ away from the vortex core. In contrast to previous analyses, which neglected the underlying superflow $\mathbf{v}_0$ (see Ref. 8 and references therein), our expression for the dipole electric field shows that this field vanishes in the superclean limit when $\mathbf{v}_L = \mathbf{v}_0$. Using the results for the electric field/chemical potential we have determined the associated vortex masses in charged and neutral superfluids and have compared our results to other contributions arising from the quasiparticles trapped within the vortex core and from the backflow around the core region. We find that the most important contribution in the charged superfluid originates from the core mass, whereas it is the logarithmically divergent compressibility mass that provides the most relevant term in a neutral superfluid.

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