Nature of the driving force on an Abrikosov vortex

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From an energy and force analysis based on known calculations on one and two Abrikosov vortices (AV's) using the London equation, it is found that the driving force on an AV is not a magnetic Lorentz force as widely believed. In the low-ξ/λ limit, the force is dominated by a kinetic rather than magnetic interaction, and is proportional to the local densities of AV current and the driving current around the hard core. Some simulated and experimental results published in the literature on vortex depinning in Josephson-junction arrays can be interpreted in terms of this concept. [S0163-1829(98)01510-0]

Soon after the critical-state model was proposed by Bean for hard superconductors,[1] it was used for the volume supercurrents in type-II superconductors by Kim *et al.*,[2] and the mechanism of critical-current density J_c was explained by Anderson and Kim's theory of Abrikosov vortex (AV) depinning.[3,4] According to this theory, when a volume current flows in a sample containing AV's, an AV of unit length will experience a driving Lorentz force $\mathbf{J}\times\mathbf{\Phi}_0$ from the current of mean density $\mathbf{J}$ and a pinning force $\mathbf{p}$ from the defects, and J_c at 0 K is calculated from the following force balance equation just before depinning:

$$\mathbf{J}_c\times\mathbf{\Phi}_0+\mathbf{p}_{\max}=0, \tag{1}$$

where $\mathbf{\Phi}_0$ is the total flux carried by the AV.

Over the years, on the one hand, the correctness of Eq. (1) has been well proved experimentally in many type-II superconductors, but on the other hand, the nature of the driving force has long been a question. In electrodynamics, the Lorentz force $q\mathbf{v}\times\mathbf{B}$ is defined as the force experienced by a moving charge q with a velocity $\mathbf{v}$ in a field $\mathbf{B}$. Translating $q\mathbf{v}$ and $\mathbf{B}$ into $\mathbf{J}$ and $\mathbf{\Phi}_0$ in the AV case, one will find that $\mathbf{J}\times\mathbf{\Phi}_0$ is the force acting on the current, and therefore, the driving force on the AV should be $\mathbf{\Phi}_0\times\mathbf{J}$, which has the opposite direction to that in Eq. (1). Related to this, when using Eq. (1) in their theoretical treatments, some authors have changed the sign of the first term without necessary arguments (see, e.g., Refs. 5–8). After such a change, however, Eq. (1) will no longer be qualitatively consistent with all the well-known experimental results. For example, after the first AV's enter, the screening current will push them out but not in, as it should be.

Thus, there must be something unusual behind the driving force on the vortex. In 1993, Ao and Thouless pointed out that the name ''Lorentz force'' was improper because the Magnus force (which included the Lorentz force) was not a consequence of electromagnetic effects on a vortex.[9] However, this statement has apparently not been accepted by a wide audience; the sign change mentioned above was made even after their work, and in some recent books on superconductivity, the conventional meaning of the Lorentz force is clearly stated and applied, as can be seen, for example, in Ref. 10. Thus, the question is still open.

The present paper is devoted to further dealing with this problem. In contrast with Ao and Thouless's geometric phase approach on the Magnus force, we will concentrate on the Lorentz force itself, assuming the AV to be well pinned so that the more general Magnus force is reduced to the Lorentz force and there is no effect from normal currents. We will review well-accepted solutions of a single AV and two AV's in the low-ξ/λ limit, ξ and λ being the coherence length and the London penetration depth, respectively, from which the energy of AV's and the nature of the driving force acting on an AV are discussed. For this purpose, we will mainly follow de Gennes' classical book (see Ref. 11, where relevant page numbers are listed corresponding to our notes in the text below). We will show that although this force can be formally expressed as that in Eq. (1), it is not a Lorentz force, so that the contradiction mentioned above becomes understandable.

For a superconductor in which the supercurrent density $\mathbf{J}$ and magnetic field $\mathbf{H}$ have a slow variation in space, the London equation is derived from minimizing the free energy which includes the condensation, kinetic, and magnetic field energies. The magnetic energy density $\epsilon_h=\mu_0H^2/2$ and the kinetic energy density $\epsilon_k=nmv^2/2$, where n, m, $-e$, and $\mathbf{v}$ are the number density, mass, electrical charge, and velocity of superconducting electrons, respectively. Since $\mathbf{v}$ is related to $\mathbf{J}$ by

$$\mathbf{J}=-ne\mathbf{v} \tag{2}$$

and $\mathbf{J}$ is related to $\mathbf{H}$ by the Ampère law, both energies can be expressed in terms of $\mathbf{H}$. Thus, a variational minimum of the free energy leads to the London equation[11]

$$\mathbf{H}+\lambda^2\nabla\times\nabla\times\mathbf{H}=0, \tag{3}$$

where the penetration depth follows:

$$\lambda^2 = \frac{m}{\mu_0 e^2 n}. \tag{4}$$

In an extreme type-II superconductor, the field distribution in an AV with a hard core of very small radius ξ located along the z axis can be obtained by solving

$$\mathbf{H} + \lambda^2 \nabla \times \nabla \times \mathbf{H} = \mathbf{\Phi}_0 \delta(\mathbf{r})/\mu_0, \tag{5}$$

where $\mathbf{\Phi}_0$ is a vector in the field direction of the AV and $\delta(\mathbf{r})$ is the two-dimensional delta function.[11] Equation (5) is obtained from Eq. (3) with a singularity at the core. Integrating Eq. (5) over the interior surface of a circle C of radius r and using the curl formula, we have

$$\int \mathbf{H} \cdot d\mathbf{s} + \lambda^2 \oint \nabla \times \mathbf{H} \cdot d\mathbf{l} = \Phi_0/\mu_0. \tag{6}$$

Since H is finite, the first integral of Eq. (6) is negligible when $r = \xi$ compared with the second, and we have

$$2\pi\lambda^2 r |\nabla \times \mathbf{H}| = \Phi_0/\mu_0 \qquad (r = \xi). \tag{7}$$

If the superconductor is a long cylinder coaxial with the z axis, the solution to Eq. (5) for the only z component of $\mathbf{H}$ under the boundary condition, Eq. (7), is

$$H(r) = \frac{\Phi_0}{2\pi\mu_0\lambda^2} K_0\left(\frac{r}{\lambda}\right), \tag{8}$$

where K_0 is the zero-order second-kind modified Bessel function.

Having this $\mathbf{H}$ and neglecting the contribution within the core, the energy E of the AV of unit length is calculated as

$$\begin{aligned} E &= \int \left[\frac{\mu_0}{2}\mathbf{H}^2 + \frac{\mu_0\lambda^2}{2}(\nabla \times \mathbf{H})^2\right] dV \\ &= \frac{\mu_0\lambda^2}{2}\int d\mathbf{s} \cdot \mathbf{H} \times \nabla \times \mathbf{H} \\ &= \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} K_0\left(\frac{\xi}{\lambda}\right) \\ &= \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} \ln\left(\frac{\lambda}{\xi}\right), \end{aligned} \tag{9}$$

where the volume integration is performed outside the core with the first and second terms corresponding to the field energy and the kinetic energy, E_h and E_k, respectively, and the surface integration is performed over the tubular surface of the core with $d\mathbf{s}$ directed inwards. In deriving the second equality, Eq. (5) has been used;[12] in deriving the last equality, Eq. (7) and the low-r/λ limit of Eq. (8) have been used.

We next calculate in a large superconductor the interaction between two AV's parallel to the z axis located at $\mathbf{r}_1 = (x_1, 0)$ and $\mathbf{r}_2 = (x_2, 0)$ with $x_2 > x_1 > 0$ for convenience.[11] In this case, the field distribution is determined by

$$\mathbf{H} + \lambda^2 \nabla \times \nabla \times \mathbf{H} = \mathbf{\Phi}_0[\delta(\mathbf{r} - \mathbf{r}_1) + \delta(\mathbf{r} - \mathbf{r}_2)]/\mu_0. \tag{10}$$

Its solution $\mathbf{H}(\mathbf{r})$ is the superposition of two fields $\mathbf{H}_i(\mathbf{r})$, $i = 1$ and 2, of both AV's, in the z direction:

$$H_i(\mathbf{r}) = \frac{\Phi_0}{2\pi\mu_0\lambda^2} K_0\left(\frac{|\mathbf{r} - \mathbf{r}_i|}{\lambda}\right). \tag{11}$$

After expressing the total energy like in the first equality of Eq. (9), where $\mathbf{H}$ is replaced by $\mathbf{H}_1 + \mathbf{H}_2$ and the integration is made over the entire volume outside both cores, and considering the smallness of radia ξ, we obtain the interaction energy E_{12} between both AV's as

$$E_{12} = \frac{\mu_0\lambda^2}{2}\int d\mathbf{s}_1 \cdot \mathbf{H}_2 \times \nabla \times \mathbf{H}_1 + d\mathbf{s}_2 \cdot \mathbf{H}_1 \times \nabla \times \mathbf{H}_2, \tag{12}$$

where the integration is made over the tubular surfaces of the cores of both AV's. Since, similar to Eq. (7),

$$2\pi\lambda^2 |\mathbf{r} - \mathbf{r}_i| |\nabla \times \mathbf{H}_i| = \Phi_0/\mu_0 (|\mathbf{r} - \mathbf{r}_i| = \xi), \tag{13}$$

Eq. (12) can be written as

$$E_{12} = \Phi_0 H_{12}, \tag{14}$$

where

$$H_{12} = H_1(\mathbf{r}_2) = H_2(\mathbf{r}_1) = \frac{\Phi_0}{2\pi\mu_0\lambda^2} K_0\left(\frac{x_2 - x_1}{\lambda}\right). \tag{15}$$

E_{12} is a repulsive energy since it decreases with increasing the AV distance $x_2 - x_1$. The force experienced by the second AV has an x component only, which is calculated using the Ampère law as

$$F_{2x} = -\partial E_{12}/\partial x_2 = -\Phi_0 \partial H_1(\mathbf{r}_2)/\partial x_2 = \Phi_0 J_{1y}(\mathbf{r}_2). \tag{16}$$

In Eq. (16), $J_{1y}(\mathbf{r}_2)$ is the density of the current of the first AV at $\mathbf{r}_2$, where the small tubular core of the second AV is located. Checking the relation among the directions of the force, current, and $\mathbf{\Phi}_0$, we find that Eq. (16) can be written as

$$\mathbf{F}_2 = \mathbf{J}_1(\mathbf{r}_2) \times \mathbf{\Phi}_0, \tag{17}$$

which agrees formally with the driving force on an AV expressed in Eq. (1). We say the agreement to be formal since $\mathbf{J}_1(\mathbf{r}_2)$ here has a different meaning from the $\mathbf{J}_c$ in Eq. (1); $\mathbf{J}_1(\mathbf{r}_2)$ is the *local density* of the current of the first AV around the core of the second AV, whereas $\mathbf{J}_c$ is the *mean density* of the driving transport current over the entire AV. We should emphasize that although Eq. (17) has been derived in the case of the interaction between two AV's, when an AV in an external current is considered, the force of the current on the AV can be generally expressed by this equation with the density of the external current being the local one around the AV core. Another example can be found in the calculation of the surface barrier to an AV, where the force on the AV equals $\mathbf{J} \times \mathbf{\Phi}_0$, $\mathbf{J}$ being the local density around the AV core of the sum of both the screening current and image current.[11]

Up to now, Φ_0 has been introduced as a constant which signifies the singularity at the AV center, so that the physical meaning of the driving force expressed by Eq. (17) is still incomplete. Therefore, the origin of Φ_0 has to be discussed first.

Rewriting Eq. (5) in terms of the vector potential $\mathbf{A}$ and the current density $\mathbf{J}$, we have

$$\nabla\times\mathbf{A}+\mu_0\lambda^2\nabla\times\mathbf{J}=\mathbf{\Phi}_0\delta(\mathbf{r}). \tag{18}$$

Replacing λ^2 and $\mathbf{J}$ in Eq. (18) by Eq. (2) and the quantum-mechanical $\mathbf{J}$ considering the Cooper pairs,

$$\mathbf{J}=\frac{\hbar en}{2m}\left(\nabla\gamma-\frac{2e}{\hbar}\mathbf{A}\right), \tag{19}$$

where γ is the phase of superconducting order parameter, and considering the single valuedness of the order parameter, we make an integration over the interior surface of the circle C as done for Eq. (6) and obtain the vorticity of the phase variable γ,

$$\frac{1}{2\pi}\oint\nabla\gamma\cdot d\mathbf{l}=1, \tag{20}$$

if

$$\Phi_0=\pi\hbar/e=2.07\times10^{-15}\ \text{Wb}. \tag{21}$$

Thus, Φ_0 is a quantum-mechanical constant defined by Eq. (21). Its meaning can be understood in different simple physical situations. For our case, it can correspond in the first instance to the total flux carried by a complete AV. In fact, if the integration made for Eq. (5) is over the interior surface of a circle C of large radius $r\gg\lambda$, the line integral in Eq. (6) becomes negligible owing to the high-degree small boundary current, and we have the total flux of the AV, $\mu_0\int\mathbf{H}\cdot d\mathbf{s}=\Phi_0$. For this reason, Φ_0 is commonly referred to as the flux quantum.[11] As a result, the driving force in Eq. (17) seems to be electromagnetic.

However, if the radius of the circle C is very small, Eq. (6) leads to Eq. (7). This means that Φ_0 is also the circulation of the current density around C multiplied by $\mu_0\lambda^2$. Given this meaning to Φ_0, the driving force in Eq. (17) is exerted between two currents. This can be displayed clearer as follows.

Using Eqs. (12) and (13) and the Ampère law, we rewrite Eq. (16) as

$$\begin{aligned}F_{2x}&=-\mu_0\lambda^2 2\pi\xi|\nabla\times\mathbf{H}_2|\partial H_1/\partial x_2\\&=\mu_0\lambda^2 2\pi\xi J_2(|\mathbf{r}-\mathbf{r}_2|=\xi)J_1(|\mathbf{r}-\mathbf{r}_2|=\xi).\end{aligned} \tag{22}$$

From Eq. (22) we see that the driving force F_{2x} on the second AV does not explicitly depend on the total flux Φ_0 it carries. It is proportional to the circulation of its own current density along the border of the core, $2\pi\xi J_2(|\mathbf{r}-\mathbf{r}_2|=\xi)$, and the current density of the first AV on the same border, $J_1(|\mathbf{r}-\mathbf{r}_2|=\xi)$.

Since the current density $\mathbf{J}$ is related to the electron velocity $\mathbf{v}$ by Eq. (2), the force between two currents should have a kinetic origin.

In order to decide the actual nature of the driving force, we calculate the field energy of an AV from Eq. (8). The result is

$$E_h=\frac{\Phi_0^2}{4\pi\mu_0\lambda^2}\int_{\xi/\lambda}^{\infty}xK_0^2(x)dx=\frac{\Phi_0^2}{8\pi\mu_0\lambda^2}, \tag{23}$$

when $\xi/\lambda\ll1$. In comparison with the total energy E expressed by Eq. (9), this E_h is negligibly small in the low-ξ/λ limit. Thus, the total energy is dominated by the kinetic energy E_k. The same situation occurs for the interaction energy E_{12}. Because this energy is now dominated by the kinetic energy but not the magnetic one, the nature of the driving force should also be mainly kinetic. Accordingly, the relevant meaning of the $\mathbf{\Phi}_0$ appearing in Eq. (17) should be the second, which is kinetic.

This fact was overlooked in Anderson and Kim's derivation; they thought that the interaction energy was practically given by ''the naive expression for the magnetic energy,'' so that the driving force was the Lorentz force.[4]

The kinetic nature of the driving force on an AV is supported by recent studies on a planar Josephson-junction array with a centered Josephson vortex (JV). Such an array can be regarded as a discrete version of the type-II superconductor studied above. When the field produced by currents is neglected so that the JV carries a zero flux, the energy in such an array is totally kinetic. A numerical simulation based on the Josephson equation and the current continuity was made for this case when a transport current was fed in the array.[13,14] It was found that the simulated depinning current agreed well with Eq. (1) if the maximum depinning force owing to the discreteness was correctly calculated. This theoretical finding was also indirectly verified experimentally by the resistive transition of large Josephson-junction arrays.[14] As already found by Dang and Gyorffy,[15] it was not immediately obvious how the Lorentz force arose in a model where the magnetic effects of the circulating currents had been neglected. From the present work, we can make another statement about this extreme case: The fact that the driving force can be calculated using a formula containing Φ_0 even though the vortex does not carry any flux just shows that the nature of Φ_0 in Eq. (1) is not magnetic but kinetic.

Replacing $\mathbf{J}_1$ and $\mathbf{\Phi}_0$ in Eq. (17) by Eqs. (2) and (21) results in the driving force

$$\mathbf{F}=-\pi\hbar n\mathbf{v}\times\mathbf{k}, \tag{24}$$

where $\mathbf{k}$ is the unit vector along the z axis. This equation indicates that the driving force is dynamical and proportional to the electron velocity and depends on the electron number density and the Planck constant without an explicit relation to e and m. We notice that, except for a sign difference, this is consistent with Eq. (9) in Ref. 9 for the Magnus force, which was derived in terms of the geometric phase. Thus, we can generally say that the driving force is a dynamical quantum-mechanical force on the AV. It can be expressed electromagnetically as Eq. (17), but as described above, this will mask the actual nature of the driving force and even give rise to a sign confusion. It can also be expressed kinetically (hydrodynamically) as

$$\mathbf{F}=-nm\mathbf{v}\times\mathbf{\Omega}_0, \tag{25}$$

where $\mathbf{\Omega}_0$ is the vectorial quantum of the electron-velocity circulation *closely* around the AV core,

$$\Omega_0=\pi\hbar/m=\Phi_0 e/m=3.637\times10^{-4}\ \text{m}^2/\text{s}. \tag{26}$$

Equation (25) exposes directly the actual kinetic nature of the driving force; Eq. (26) explains precisely the kinetic meaning of Φ_0 mentioned above.

Finally, we should explain that although the driving force is kinetic but not magnetic in the low-ξ/λ limit, the magnetic field does have its own function. Since ξ/λ cannot be zero in any real case and $\xi/\lambda > 0$, the energy will have both the kinetic and magnetic contributions. Although the Ginzburg-Landau equations are more relevant than the London equation in this case, we still use Eqs. (9) and (23) to estimate roughly the magnetic contribution to the total energy. It is about 1%, 10%, and 20%, if $\xi/\lambda = 1/1000$, 1/100, and 1/10, respectively. (We note here that $\xi/\lambda \sim 1/20$ for conventional $A15$ superconductors such as Nb_3Ge and Nb_3Sn and $\sim 1/100$ for most high-T_c superconductors.[10]) Thus, the field also plays its role, though minor, in the total energy and so in the driving force. Actually, different from conventional hydrodynamics, $\mathbf{\Omega}_0$ in Eq. (25) is not a circulation of the electron velocity around the AV core in general but one closely around the core, since this circulation is radius dependent. As a quantum constant, it includes the contribution of the magnetic flux as can be derived from Eq. (18). Equation (25) is reduced to a conventional hydrodynamical force only when the low-ξ/λ limit is considered, since in this case the contribution of the flux is negligible and the circulation is independent of the radius.

Since our results are consistent with Ao and Thouless' theory,[9] experimental tests suggested or made recently to their theory by transport or force measurements should also be basically relevant to our work.[16,17]

In conclusion, the driving force on AV expressed in Eq. (1) has a direction opposite to that predicted by the conventional Lorentz force. It is usually dominated by the kinetic interaction between the transport current and the vortex current and is proportional to the product of both local current densities around the AV core without an explicit relation to the flux carried by the AV. Therefore, it is illogical to call this force the Lorentz force. In order to avoid confusion with the conventional Lorentz force in electrodynamics, a relevant unambiguous name may be the ''London force'' since it can be derived and understood from the London equation in the low-ξ/λ limit with quantum effects being considered. Some simulated and experimental results published in the literature on vortex depinning in Josephson-junction arrays can be interpreted in terms of this concept.

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