Parametrically forced sine-Gordon equation and domain wall dynamics in ferromagnets

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A parametrically forced sine-Gordon equation with a fast periodic *mean-zero* forcing is considered. It is shown that π kinks represent a class of solitary-wave solutions of the equation. This result is applied to quasi-one-dimensional ferromagnets with an easy-plane anisotropy, in a rapidly oscillating magnetic field. In this case the π-kink solution we have introduced corresponds to the uniform ''true'' domain-wall motion, since the magnetization directions on opposite sides of the wall are antiparallel. In contrast to previous work, no additional anisotropy is required to obtain a true domain wall. Numerical simulations showed good qualitative agreement with the theory. [S0163-1829(98)01610-5]

The sine-Gordon equation (SGE) arises in various physical applications including Josephson junction transmission lines,[1,2] dislocations in crystals,[3] charge density waves,[4] and waves in quasi-one-dimensional ferromagnetic materials.[5–8] The only stable traveling wave solutions of the ordinary SGE for a scalar field ϕ are localized solutions with identical boundary conditions $\phi=0$ and $\phi=2\pi$, called 2π kinks. On the other hand, nonlocalized kinks separating regions with different values of the field (e.g., π kinks) are also important since they are easier to follow experimentally. This is especially relevant for domain-wall dynamics in quasi-one-dimensional ferromagnets with a strong anisotropy of a hard magnetization axis.[5–7,9,10] In the presence of an external magnetic field (weak compared to the anisotropy) applied within the easy plane, the spin dynamics can be described by the SGE (see, e.g., Ref. 6) with 2π kinks as the only traveling-wave solutions. These localized solutions, preserving the average magnetization, were observed experimentally.[11,12] An interesting effect of the evolution of the average magnetization takes place if adjacent domains have antiparallel magnetizations which correspond to π kinks or ''true'' domain walls. Such π kinks were obtained by introducing additional anisotropy within the easy plane (see, e.g., Refs. 6 and 13, Sec. 8.3).

In this paper we obtain π kinks in a modified SGE with a fast mean-zero parametric excitation. For quasi-one-dimensional ferromagnets with an easy-plane anisotropy, this implies the existence of true domain walls generated by a rapidly oscillating external magnetic field. As opposed to Refs. 6 and 13, no additional anisotropy of the ferromagnet is required for the existence of π kinks.

Various cases of a parametrically forced SGE have arisen in numerous models of physical systems.[2,10,14,15] However, the mean-zero case has been left out of consideration, while it is significant both for theory and applications. Indeed, a zero-average periodic forcing implies that the states $\phi=0$ and $\phi=\pi$ are symmetric. This leads to the existence of π kinks with such a time-dependent macroscopic quantity as the spatial average of the field.

We start with the Hamiltonian[6,16]

$$\mathcal{H}=\sum_{i=1}^{N}\left[-J\mathbf{S}_i\cdot\mathbf{S}_{i+1}+D(S_i^z)^2-g\mu_B HS_i^x\right], \tag{1}$$

where $\mathbf{S}_i$ are dimensionless classical spin vectors, N is the number of spins, and J is a constant of the exchange interaction. The external magnetic field $\mathbf{H}$ is directed along the X-axis. The anisotropy constant D provides the existence of an easy plane, XY, at each site of the chain.

The dynamics of a spin $\mathbf{S}_i$ in the effective magnetic field is governed by[13]

$$\hbar\dot{\mathbf{S}}_i=-\mathbf{S}_i\times\frac{\partial\mathcal{H}}{\partial\mathbf{S}_i}. \tag{2}$$

Representing spins in spherical coordinates $\mathbf{S}_i=S(\cos\theta_i\cos\phi_i,\cos\theta_i\sin\phi_i,\sin\theta_i)$ and taking the continuum limit yields a system of two first-order (in time) partial differential equations for θ and ϕ.[6,16] When the condition $DS\gg g\mu_B H$ holds,[6] the system reduces to the SGE for the polar angle in the easy plane, ϕ,

$$\phi_{xx}-\frac{\ddot{\phi}}{C^2}=\nu H\sin\phi, \tag{3}$$

where $C=2a_0S^2\sqrt{JD}/\hbar$, $\nu=g\mu_B/2JSa_0^2$, and a_0 is the lattice constant. We take the external field $H=H_0a(t/\epsilon)$, where a is a mean-zero periodic function with a unit amplitude, and H_0 is the amplitude of the field. Rescaling the time and coordinate to dimensionless variables, $\tilde{t}=tC\sqrt{\nu H_0}$, $\tilde{x}=x\sqrt{\nu H_0}$, we obtain (after dropping the tildes) a parametrically forced SGE (PSGE)

$$\phi_{tt}-\phi_{xx}+a(t/\epsilon)\sin\ \phi=0. \tag{4}$$

After averaging Eq. (4) directly, over the fast time scale ϵ, we are left with a linear wave equation with wave propagation velocity $c=\pm 1$. In what follows, we use more subtle averaging to obtain π-kink solutions moving with any prescribed velocity.

The phenomenon of π kinks in the PSGE has a finite-dimensional counterpart: It is the stabilization of the inverted pendulum by a periodic vibration of its suspension point (Kapitza pendulum.[17]) Since this latter phenomenon is responsible for the existence of π kinks, we outline a very simple geometrical explanation of the stability of the inverted pendulum with a vibrating suspension. Full details with other applications can be found in Ref. 20 along with further references to numerous papers on the subject. Consider a pendulum (a bob on a massless rod of length l) in a nearly-upside-down position, with the suspension point undergoing vertical vibration. We assume the latter to have high acceleration and small amplitude. Since the acceleration is large, the force of the rod on the bob is large so that the bob would be expected to follow, in the first approximation, the direction of the rod. This suggests considering an auxiliary system where the velocity of the bob is actually constrained to the line of the rod. In this case the bob will oscillate along an arc of a tractrix (the ''pursuit'' curve; all tangent segments from this curve to a straight line have the same length) and thus will be subject to an average centrifugal force $mk\langle v^2\rangle$, where k is the curvature of the tractrix and v is the speed of the bob which is approximately the speed of the suspension point when the pendulum is near the top. If we now release the constraint, thus releasing the centrifugal force, the bob will behave as if it were subject to a centripetal force $mk\langle v^2\rangle$ which pushes the pendulum towards the top. If this force exceeds the gravitational force, the pendulum is stable; this leads to the simple stability criterion $\langle v^2\rangle > \ell g$ (see Ref. 20).

To average Eq. (4) we apply a series of canonical near-identical transformations via the normal form technique,[18,19] so as to bring the original equation to a better form with rapidly oscillating coefficients moved to higher-order terms. Since the transformations are near identical, the solutions for the reduced Hamiltonian are close to those for the original one.

The Hamiltonian of PSGE (4) is given by

$$H=\int_{-\infty}^{+\infty}\left(\frac{p^2}{2}+\frac{\phi_x^2}{2}-a\ \cos\phi\right)dx, \tag{5}$$

where $p\equiv\phi_t$ (below we omit the limits of integration and dx). Let the first canonical transformation be defined implicitly as follows:

$$p=p_1+\frac{\delta W_1(\phi,p_1)}{\delta\phi},\quad \phi_1=\phi+\frac{\delta W_1(\phi,p_1)}{\delta p_1}. \tag{6}$$

The new Hamiltonian is given by $H_1=H+W_{1t}$ or

$$H_1=\int\left[\frac{1}{2}\left(p_1+\frac{\delta W_1}{\delta\phi}\right)^2+\frac{\phi_x^2}{2}-a\ \cos\phi\right]+W_{1t}. \tag{7}$$

To remove the rapidly oscillating term in H we choose $W_1=\epsilon\int a_{-1}\cos\phi$, where a_{-1} is an antiderivative with zero average. The Hamiltonian takes the form

$$H_1=\int\frac{p_1^2}{2}+\frac{\phi_{1x}^2}{2}-\epsilon a_{-1}p_2\sin\phi_1+\frac{1}{2}\epsilon^2a_{-1}^2\sin^2\phi_1. \tag{8}$$

The last term in the above Hamiltonian cannot be removed by near-identity transformations since it has a nonzero mean with respect to t. However, all other terms with rapidly oscillating coefficients can be removed. Thus, choosing $W_2=\epsilon^2\int a_{-2}p_2\sin\phi_1$ we obtain the Hamiltonian $H_2=\int p_2^2/2+\phi_{2x}^2/2+\frac{1}{2}\epsilon^2\langle a_{-1}^2\rangle\sin^2\phi_2+\epsilon^2R+O(\epsilon^2)$, where $\langle R\rangle=0$. Finally, taking $W_3=\epsilon^3\int R_{-1}$ we obtain the Hamiltonian

$$H_3=\int\left(\frac{p_3^2}{2}+\frac{\phi_{3x}^2}{2}+\frac{1}{2}\epsilon^2\langle a_{-1}^2\rangle\sin^2\phi_3\right)+O(\epsilon^3). \tag{9}$$

After rescaling $X=\epsilon x$, $T=\epsilon t$, $P=2\epsilon^{-1}p_3$, and $\Phi=2\phi_3$ in the equations of motion corresponding to Eq. (9), we obtain

$$\Phi_T=P+O(\epsilon^2),\quad P_T=\Phi_{XX}-\langle a_{-1}^2\rangle\sin\Phi+O(\epsilon). \tag{10}$$

The system (10) is a slightly perturbed SGE with 2π kinks as approximate solutions. After rescaling back to variables (ϕ_3,p_3), we obtain π kinks as approximate solutions $\phi_3\approx U(x,t)$, where

$$U(x,t)=2\arctan\left[\exp\left(\epsilon\sqrt{\langle a_{-1}^2\rangle}\frac{x-ct}{\sqrt{1-c^2}}\right)\right]. \tag{11}$$

Note that by using the normal form technique, our equation (4) has been brought to the form in which it explicitly represents a slightly perturbed SGE (10). Since we have employed only near-identical transformations, the original equation (4) should have solutions close to the solitary waves given by Eq. (11).

We have verified our results by the numerical simulations of PSGE (4), using the second-order leap frog method. To obtain initial conditions for the original variables $\phi(x,0)$ and $p(x,0)$, we start with the initial conditions for the transformed variables ϕ_3 and p_3, generated from Eq. (11) as $\phi_3(x,0)=U(x,0)$, and $p_3(x,0)=U_t(x,0)$. Retracing our canonical transformations and keeping the lowest-order terms only we arrive at

$$\phi(x,0)=U(x,0),\quad p(x,0)=U_t(x,0)-\epsilon a_{-1}(0)\sin U(x,0). \tag{12}$$

In Fig. 1 we compare the results of the simulations (dashed line) with the analytical solution (11) (thin solid line). One can see from the figure a good quantitative agreement between the theory and numerical simulations (the two curves in the figure are almost indistinguishable). We have also simulated PSGE (4), starting with initial conditions of shapes different from Eq. (12). We have found that these solutions split into two linear wave packets moving in opposite directions with velocities $c=\pm 1$. This can be explained by the closeness of the PSGE with mean-zero excitation to the linear wave equation, as was mentioned above.

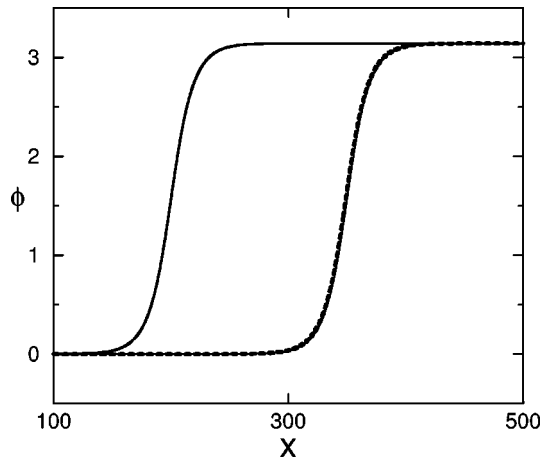

FIG. 1. The behavior of the π-kink solution in PSGE (4). The thick solid line corresponds to the initial profile given by Eq. (12) with $c=1/2$, $a=\sin(t/\epsilon)$. The thin solid and dashed lines correspond to the approximate solution (11) and the result of simulation, respectively, at $t=300$. The analytical and numerical curves almost coincide. The parameters of the simulations are $\epsilon=0.1$, time step $dt=0.01$, mesh size $dx=0.05$, and system size $L=500$.

In terms of the original physical system described by Eq. (3), the π-kink solutions represent moving true domain walls generated by a rapidly oscillating external magnetic field. Now we return from the dimensionless variables to the original variables $t=\tilde{t}/C\sqrt{\nu H_0}$, and $x=\tilde{x}/\sqrt{\nu H_0}$ to estimate the values of physical parameters, which provide the formation of a realistic domain wall. The lattice constant a_0 and energy constants J and D are taken, according to Ref. 11, as $a_0 \sim 5$ Å, and $J \sim D \sim 10$ K. In dimensionless units the frequency $\tilde{\omega}$ of the field and the width of the domain wall $\Delta\tilde{x}$ are of order ϵ^{-1}. Then, for the field amplitude $H_0 \sim 1$G and $\epsilon \sim 0.1$–0.01, we obtain $\omega \sim 10^9$ Hz and $\Delta x \sim 10$ μm. A magnetic field $H_0 \sin\omega t$, with such values of H_0 and ω, can be created in an experiment, to observe the predicted effect.

In summary, we have found π-kink solutions to the PSGE with a fast periodic mean-zero forcing. As applied to quasi-one-dimensional ferromagnets with only the anisotropy of an easy plane, these solutions imply moving true domain walls, generated by a rapidly oscillating magnetic field. Our theoretical results are in good quantitative agreement with numerical simulations of the PSGE. These results are also applicable to essentially two- and three-dimensional easy-plane ferromagnets, when plane front solutions are stable with respect to small deformations. This problem, as well as the stability of π kinks in the PSGE, will be addressed in future investigations.

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