Quantum depinning of a pancake vortex from a columnar defect

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We consider the problem of the depinning of a weakly driven ($F \ll F_c$) pancake vortex from a columnar defect in a Josephson-coupled superconductor, where F denotes the force acting on the vortex (F_c is the critical force). The dynamics of the vortex is supposed to be of the Hall type. The Euclidean action $S_{\text{Eucl}}(T)$ is calculated in the entire temperature range; the result is universal and does not depend on the detailed form of the pinning potential. We show that the transition from quantum-to-classical behavior is second-order like with the temperature T_c of the transition scaling like $F^{4/3}$. Special attention is paid to the regime of applicability of our results, in particular, the influence of the large vortex mass appearing in the superclean limit is discussed. [S0163-1829(98)06905-7]

I. INTRODUCTION

In recent years, quantum creep of vortices in high-T_c superconductors has attracted considerable interest as, on the one hand, this phenomenon is responsible for the dissipation of energy and thus is relevant from a technological point of view, while, on the other hand, it represents an interesting example of a macroscopic quantum phenomenon, the theoretical study of which is challenging. A particularly well defined and technologically relevant configuration are vortices trapped by columnar defects introduced into the sample by heavy-ion irradiation. Measurements of the critical current density and the magnetic relaxation rate show the strong influence these pinning centers have on the vortex dynamics.[1] In this paper we consider the problem of the depinning of a pancake vortex governed by Hall dynamics from a columnar defect in a layered superconductor in the presence of a small ($j \ll j_c$) transport current. The external magnetic field is chosen parallel to the c axis of the superconductor. In the limit $j \rightarrow 0$ the problem is semiclassical and the Euclidean action can be calculated in the whole temperature range. In the limit $T = 0$ the problem discussed above has been considered in Ref. 2, however, it appears that the approximations made are too rough, leading to an inexact result for the decay rate. The main goal of the present work is to improve on the analysis of Ref. 2 and to extend the calculations to the entire temperature range. We adopt the semiclassical approach instead of the lowest-Landau-level (LLL) approximation used in Ref. 2. For a general review concerning the decay of metastable states see the review of Hänggi, Talkner, and Borkovec; see Ref. 3. The problem of quantum and classical Hall creep of vortices in various geometries and for different driving forces $F \lesssim F_c$ and $F \ll F_c$ has been studied by various authors; see Refs. 4–10.

The process of quantum tunneling is described by a time-dependent saddle-point solution. Consequently, the calculations of the decay rate require the specification of the vortex dynamics. The different contributions to the dynamics considered traditionally are of the massive, the dissipative, and the Hall type. The (low-frequency) equation of motion of a single vortex can be written in the form

$$\frac{\Phi_0}{c}\mathbf{j}\wedge\mathbf{n} - \nabla U_{\text{pin}} = -\eta\mathbf{v} + m\dot{\mathbf{v}} + \alpha\mathbf{v}\wedge\mathbf{n}, \quad (1)$$

where the dynamical forces are balanced by the Lorentz and pinning forces. In conventional superconductors one can neglect the contribution of the mass and Hall terms, whereas in high-T_c superconductors the Hall force may become relevant. In particular, it is widely believed[11] that at low temperatures the superclean limit can be reached where the Hall term is large. In this case $\omega_0\tau \gtrsim 1$, where ω_0 is the level spacing inside the vortex core and τ is the quasiparticle relaxation time. Indeed, recent Hall angle measurements[12,13] demonstrate that the limit $\alpha \gtrsim \eta$ can be realized. On the other hand, the large parameter $\omega_0\tau$ also gives rise to a large vortex mass. Microscopic calculations show that the vortex mass is enhanced by a factor $(\epsilon_F/\Delta)^2 \sim 100$ in comparison to the dirty limit.[14–16] Still, it can be shown that for frequencies $\omega < \omega_0$ the Hall force wins over the inertial one, whereas at high temperatures $\omega > \omega_0$ the vortex equation of motion cannot be cast into the simple form (1) [an accurate description produces dispersive transport coefficients $\eta(\omega)$, and $\alpha(\omega)$]. In this situation a good starting point is to ignore the contribution of the vortex mass and solve the remaining Hall tunneling problem. The advantage of this treatment lies in the fact that in the limit $j \ll j_c$ the problem allows for an analytical solution in the whole temperature range with a universal answer: the Euclidean action depends only on the depth of the pinning potential, the detailed shape of the potential being irrelevant. In a second step we establish the consistency of this approximation in the physically relevant regime of parameters. The outline of the paper is as follows: In Sec. II we discuss the model and the qualitative picture of the tunneling process. In Sec. III we calculate the Euclidean action in the whole temperature range and show that the problem always exhibits a second-order-like transition from quantum to classical behavior. Furthermore, we provide estimates for the preexponential factors in the various regimes. Finally, in Sec. IV we discuss the conditions of applicability of our results.

II. MODEL AND QUALITATIVE PICTURE

Consider a pancake vortex with a Hall-force dominated dynamics trapped in a two-dimensional (2D) potential well

$U_0(\sqrt{x^2+y^2})$ and subject to an external force F, i.e., the effective potential $U(x,y)$ takes the form

$$U(x,y)=U_0(\sqrt{x^2+y^2})-Fx. \qquad (2)$$

The function $U_0(r)$ is supposed to be monotonously increasing, $U_0(0)=0$ and $U_0(\infty)=U_0$. At distances r much larger than the characteristic radius a of the pinning potential but still smaller than the magnetic-field penetration length λ_{ab} in the ab plane, $U_0(r)$ behaves as U_0-B/r^2 with B a constant of order U_0a^2 (see Ref. 17).

   The present problem is equivalent to that of the motion of a charged particle in a strong magnetic field. In this case, the appropriate Lagrangian takes the form

$$L=\alpha\dot{x}y-U(x,y). \qquad (3)$$

The precise conditions allowing us to neglect the mass of the particle will be discussed in Sec. IV. The particle whose motion is described by the Lagrangian (3) moves along the equipotential lines of the potential landscape $U(x,y)$. The Euclidean action of the particle as obtained through the substitution $S=\int L\,dt\to-iS$ and $t\to-i\tau$ can be written in the form

$$S_{\mathrm{Eucl}}=\int_{-\hbar/2T}^{+\hbar/2T}[-i\alpha\dot{x}y+U(x,y)]d\tau. \qquad (4)$$

Obviously, the imaginary unit appears in the Euclidean action, i.e., the saddle-point solution is in general complex. However, in the case studied here it is possible to reduce the complex problem to a real one[4,11] via performing the additional transformation $y\to iy$. If the potential $U(x,y)$ satisfies the condition $\mathrm{Im}\{U(x,iy)\}=0$, we obtain a real-time problem. In addition, if after the $y\to iy$-transformation the Lagrangian (4) exhibits a saddle-point solution, we can find it and calculate the decay rate. One can easily see from Eq. (2) that the condition $\mathrm{Im}\{U(x,iy)\}=0$ is indeed satisfied for our potential,[18] i.e., we can study the effective problem of the tunneling of a particle whose dynamics is described by the Euclidean action

$$S_{\mathrm{Eucl}}[x(\tau),y(\tau)]=\int_{-\hbar/2T}^{+\hbar/2T}[\alpha\dot{x}y+U_0(\sqrt{x^2-y^2})-Fx]d\tau. \qquad (5)$$

In Fig. 1 we show the inverted potential after the transformations $t\to-i\tau$ and $y\to iy$ together with the equipotential lines. Obviously, this construction produces a new potential shape exhibiting a bounce solution (saddle point). Note that the inversion affects the potential only in the unstable direction along the force.

   As the potential energy $U(x,iy)$ is preserved during motion, the imaginary time trajectories satisfy the equation $U_0(\sqrt{x^2-y^2})-Fx=\mathrm{const}$. In Fig. 2 we plot these trajectories for the problem at hand. At zero temperature and in the limit $F\to0$ we obtain $U_0(\sqrt{x^2-y^2})-Fx=0$, i.e., $S_{\mathrm{Eucl}}=\alpha\oint y\,dx=\alpha A$, where A is the area encircled during the periodic motion. One can easily see from Fig. 2 that in the limit $F\to0$ the encircled area is equal to that of a triangle CAA'. With $CB\simeq U_0/F$, $AA'\simeq2U_0/F$, we obtain the area $A=(U_0/F)^2$ and the Euclidean action is given by the expression $S(0)=\alpha(U_0/F)^2$ [in Ref. 2 the result $S(0)$
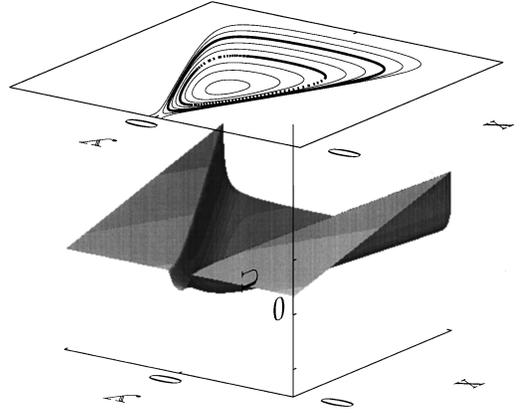
FIG. 1. Inverted potential U after the transformation $t\to-i\tau$, $y\to iy$. Note that the inversion affects only the unstable direction along the x axis. The equipotential lines define the quasiclassical trajectories. The thick solid line corresponds to the zero-temperature instanton, whereas the thick dotted line is a finite-temperature bounce trajectory.

$=(23/5)\alpha(U_0/F)^2$ has been obtained, in contradiction with the above analysis]. In the next section we shall generalize the zero-temperature result to the case of arbitrary temperatures.

III. DECAY RATE

A. Euclidean action

   As has been shown by Volovik[19] (see also Refs. 4,11), the motion of a massless particle in a magnetic field subject to a potential $U(x,y)$ is equivalent to the 1D dynamics of a particle described by the Hamiltonian $U(x,p/\alpha)$. Consequently, our original 2D problem (3) reduces to the 1D problem with the Hamiltonian given by the expression
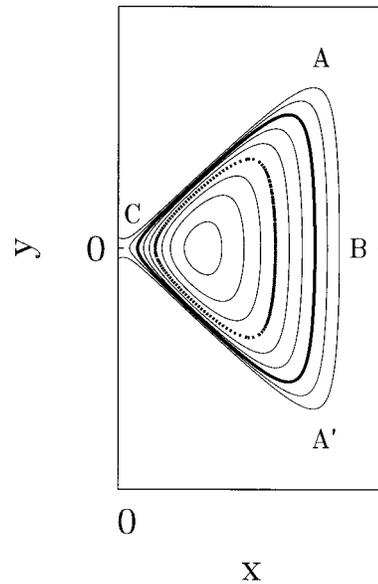
FIG. 2. Quasiclassical (x,y) trajectories corresponding to the tunneling of the vortex. The Euclidean action corresponding to the zero-temperature trajectory (thick line) is equal to the encircled area. The thick dotted line marks a finite-temperature bounce trajectory.

$$H(x,p)=U_0[\sqrt{x^2+(p/\alpha)^2}]-Fx. \qquad (6)$$

Let us show that in the limit $F \to 0$ the problem is semiclassical. The semiclassical wave function can be written in the form[20]

$$\Psi(x)=\frac{C}{\sqrt{\dot{x}}}\exp\left(\frac{i}{\hbar}\int_{x'}^{x}p\,dx\right)$$
$$=C'\exp\left(\int_{x'}^{x}dx\left(\frac{ip}{\hbar}-\frac{1}{2}\frac{\partial^2 H/\partial p^2}{\partial H/\partial p}\frac{dp}{dx}\right)\right) \qquad (7)$$

with x' an integration constant. The semiclassical approximation is applicable if

$$|p|\gg\hbar\left|\frac{\partial^2 H/\partial p^2}{\partial H/\partial p}\frac{dp}{dx}\right|. \qquad (8)$$

Using Eq. (6) we can write for $\partial_p^2 H/\partial_p H$

$$\left|\frac{\partial^2 H/\partial p^2}{\partial H/\partial p}\right|=\frac{1}{|p|}\left|\frac{x^2}{x^2+(p/\alpha)^2}+\frac{(p/\alpha)^2}{\sqrt{x^2+(p/\alpha)^2}}\frac{\partial^2 U_0/\partial u^2}{\partial U_0/\partial u}\right|, \qquad (9)$$

where $u=[x^2+(p/\alpha)^2]^{1/2}$. If $|x^2+(p/\alpha)^2|\ll a^2$, we can use $U_0\cong\kappa u^2/2$, i.e., $|\partial_p^2 H/\partial_p H|\cong 1/|p|$ in the above limit. If $|x^2+(p/\alpha)^2|\gg a^2$,

$$\partial_p^2 U_0/\partial_p U_0\sim(x^2+(p/\alpha)^2)^{-1/2}, \qquad (10)$$

and we obtain $|\partial_p^2 H/\partial_p H|<A/|p|$, with A a constant of order one. Finally, at $|x^2+(p/\alpha)^2|\sim a^2$, we obtain again $|\partial_p^2 H/\partial_p H|\cong A'/|p|$, $A'\sim 1$. Consequently, a sufficient criterion for the applicability of the semiclassical approximation takes the form

$$\left|\frac{\hbar}{p^2}\frac{dp}{dx}\right|\ll 1, \qquad (11)$$

which is the same criterion as for a ''usual'' Hamiltonian of the form $H(x,p)=p^2/2m+U(x)$.

Using the standard technique of binding semiclassical wave functions we obtain the following expression for the imaginary parts of the metastable energy levels:[21]

$$\Gamma_n=\frac{\omega(E_n)}{4\pi}\exp\left(-\frac{2}{\hbar}\int_{c_n}^{b_n}|p|\,dx\right)=\frac{\omega(E_n)}{4\pi}\exp\left(-\frac{S_n}{\hbar}\right), \qquad (12)$$

with E_n the energy levels at zero driving force, $\omega(E_n)$ are the oscillation frequencies, and c_n and b_n denote the turning points. The decay rate Γ can be found by averaging over the Boltzmann distribution (a general discussion concerning finite-temperature decay and the role of dissipation is found in Ref. 3)

$$\Gamma=(2/Z)\sum_n\Gamma_n e^{-E_n/T}, \qquad (13)$$

with Z the partition function for the case $F=0$. As the semiclassical approximation is applicable, we can substitute the sum in Eq. (13) by an integral and make use of the method of steepest descent. The extremal equation then takes the form

$\partial S/\partial E=-\tau(E)$, with $\tau(E)$ the imaginary time oscillation period. Consequently, if one can calculate the function $\tau(E)$ from the solution of the classical equation of motion, the function $S(E)$ can be reconstructed via simple integration, $S(E)=-\int^E\tau(E')dE'$. Let us carry out this program for the present problem.

The semiclassical trajectories can be found as the solution of the equation $H(x,p)=E$, with E the energy, i.e.,

$$p(x)=\pm\alpha\sqrt{f^2(E+Fx)-x^2}, \qquad (14)$$

with $f=U_0^{-1}$ the inverse function of the potential shape U_0. There is a region $c\leq x\leq b$ $(b>c>0)$, where the function $p(x)$ is purely imaginary (the x coordinates c and b are associated with the turning points C and B in Fig. 2). The equation $f^2(E+Fx)-x^2=0$ has two solutions: At small x, $f^2(E+Fx)\cong(2/\kappa)(E+Fx)$, i.e., $c=F/\kappa+\sqrt{F^2/\kappa^2+2E/\kappa}$. As $x\to(U_0-E)/F$, $f^2(E+Fx)\to\infty$, the equation $f^2(E+Fx)-x^2=0$ has another root $b\cong(U_0-E)/F$ (the expressions for c and b are applicable for any energy not too close to U_0). The decay rate of a metastable state with an energy E is proportional to $\exp[-(2/\hbar)\int_{c(E)}^{b(E)}|p|dx]$. Note that everywhere inside the interval $c\leq x\leq b$, except for the vicinity of the points c and b, $|p|\cong\alpha x$ and the condition (11) is fulfilled. At the points b and c, $p=0$. These points play the role of turning points in the ''usual'' semiclassical approximation.[20] Consequently, we have shown that in the limit $F\to 0$ the semiclassical approximation is applicable.

Let us calculate the Euclidean action. Using Eqs. (12) and (14) we can write for S

$$S(E)=2\alpha\int_c^b\sqrt{x^2-f^2(E+Fx)}dx. \qquad (15)$$

The oscillation time $\tau(E)$ satisfies the equation $\tau(E)=-\partial S/\partial E$, i.e.,

$$\tau(E)=\frac{2\alpha}{F}\int_c^b\frac{x\,dx}{\sqrt{x^2-f^2(E+Fx)}}. \qquad (16)$$

In the limit $F\to 0$ almost everywhere inside the interval $[c,b]$ we have $x^2\gg f^2(E+Fx)$ and one can write for the period $\tau(E)$

$$\tau(E)=\frac{2\alpha}{F}(b-c)+C(c)+C(b), \qquad (17)$$

with $C(c)$ and $C(b)$ the contributions of the turning points c and b where the function $x^2-f^2(E+Fx)$ vanishes. It can be shown that $C(c)$ is relevant only if E is close to $-F^2/2\kappa$ (see below) and the contribution of the point b is always negligible. Calculating $C(c)$, substituting the result into Eq. (17), and taking into account that for E not very close to U_0, $b-c\simeq(U_0-E)/F$, we obtain

$$\tau(E)\simeq\frac{2\alpha}{F^2}(U_0-E)+\frac{2\alpha}{\kappa}\ln\frac{2\widetilde{x}}{\sqrt{F^2/\kappa^2+2E/\kappa}}, \qquad (18)$$

where $\widetilde{x}$ is an E-independent cutoff parameter arising from the integration in the vicinity of the point c.

Next, we need to solve the equation $\tau(E)=\hbar/T$ and find the energy E of the saddle-point trajectory at finite tempera-

ture T. The second term in Eq. (18) is relevant only if the solution of the equation $\tau(E) = \hbar/T$ is very close to $-F^2/2\kappa$, i.e., at low temperature where the equation $2\alpha(U_0 - E)/F^2 = \hbar/T$ has no solution (this is the case when $T < \hbar F^2/2\alpha U_0$), and produces merely an exponentially small correction to the zero-temperature result. This behavior is typical for a Hamiltonian problem where the finite-temperature boundary conditions have a vanishingly small effect on the bounce solution at small temperatures.[22] On the other hand, if $T > \hbar F^2/2\alpha U_0$, the solution of Eq. (18) is given by $E = U_0 - \hbar F^2/2\alpha U_0$ (up to exponentially small corrections). After a simple integration $S(E) = -\int^E \tau(E') dE'$ we obtain the Euclidean action $S_{\text{Eucl}} = \alpha(U_0 - E)^2/F^2 + E/T$ [the integration constant is obtained from the condition $S(0) = \alpha U_0^2/F^2$]. In summary, for temperatures $T < \hbar F^2/2\alpha U_0$ the Euclidean action is constant up to exponentially small corrections. At $T_1 = \hbar F^2/2\alpha U_0$, S_{Eucl} begins to decrease, $S_{\text{Eucl}}(T) = \hbar U_0/T - \hbar^2 F^2/4\alpha T^2$. We then can write the following expression for the Euclidean action in the whole temperature interval (about the applicability of this result to the high-temperature regime see below)

$$
S_{\text{Eucl}} = \begin{cases} \alpha \left(\dfrac{U_0}{F} \right)^2, & T < \hbar F^2/2\alpha U_0 \equiv T_1, \\[2ex] \dfrac{\hbar U_0}{T} - \dfrac{\hbar^2 F^2}{4\alpha T^2}, & T > T_1. \end{cases} \tag{19}
$$

B. Crossover to classical behavior

Next, let us calculate the crossover temperature T_c from the thermally assisted quantum regime to the purely thermal activation. Below we use the perturbative procedure which is applicable only for second-order transitions from quantum to classical behavior. For a first-order transition this approach breaks down. However, we will show that if the potential $U_0(r)$ satisfies the required conditions [$U(r)$ is monotonously increasing and $U(r) \simeq U_0 - B/r^2, r \to \infty$] a second-order transition takes place. The crossover temperature T_c is equal to $\hbar/\tau_0$, where τ_0 is the imaginary time oscillation period of the system in the vicinity of the time-independent thermal saddle-point solution. This solution is given by the equation $x(\tau) = x_{\text{max}}$ with x_{max} the point where the function $U(x,0)$ takes its maximal value. Near this point $U(x,y) \simeq U_0 - B/(x^2 + y^2) - Fx$. For this dependence the function $\tau(E)$ [see Eq. (16)] can be calculated exactly,

$$
\tau(E) = \frac{2\alpha B}{F^2} \frac{1}{(b+d)^{3/2}(b+c)(b-d)^{1/2}}
$$

$$
\times \left[(b+d) F\left(\frac{\pi}{2}, \sqrt{\frac{(b-c)(b+c)}{(b-d)(b+d)}} \right) \right.
$$

$$
\left. + (c-d) \Pi\left(\frac{\pi}{2}, \frac{(b-c)(b+c)}{(b-d)(b+d)}, \sqrt{\frac{(b-c)(b+c)}{(b-d)(b+d)}} \right) \right], \tag{20}
$$

where $b \geq c > d$ are the three roots of the equation

$$
U_0 - \frac{B}{x^2} - Fx = E. \tag{21}
$$

Again b and c are the turning points of the imaginary time trajectory, $\Pi(\pi/2, n, k) = \int_0^{\pi/2} d\phi [(1 - n\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}]^{-1}$ is the complete elliptic integral of third order, and $F(\pi/2, k) = \int_0^{\pi/2} d\phi [1 - k^2\cos^2\phi]^{-1/2}$ is the complete elliptic integral of second order. Here we consider only the physically relevant case $E \lesssim E_{\text{max}} = U_0 - (3/2^{2/3}) B^{1/3} F^{2/3}$. If $E = E_{\text{max}}$, $b = c = x_{\text{max}} = (2B/F)^{1/3}$, and $d = -1/2^{2/3}(B/F)^{1/3}$. Substituting these values into Eq. (20) we obtain

$$
T_c = \frac{\sqrt{3}}{2^{4/3}\pi} \frac{\hbar F^{4/3}}{\alpha B^{1/3}}. \tag{22}
$$

Inserting $T > T_c$ into Eq. (19) in the limit $F \to 0$ we obtain $S_{\text{Eucl}}(T > T_c) \simeq U_0/T$, such that we can use the result (19) at any temperature in this limit. The dependence $S_{\text{Eucl}}(T)$ is plotted in Fig. 3.

Let us show that the problem exhibits a second-order-like transition at $T = T_c$. The 1D Hamiltonian system produces a smooth transition at T_c if its imaginary time oscillation period $\tau(E)$ is a monotonous function of energy.[23–26] A simple criterion to verify the monotonicity of the function $\tau(E)$ is given by the derivative $\partial_E \tau$ evaluated at E_{max} (see also Ref. 26): For $\partial_E \tau|_{E_{\text{max}}} < 0$ the function is monotonous and we have a second-order-like transition. If $b \simeq (U_0 - E)/F \gg c$, (i.e., for small F and E not too close to E_{max}), one can use Eq. (17) to show that the function $\tau(E)$ is monotonous in this interval. On the other hand, one can use Eq. (20) as long as the roots b and c greatly exceed the characteristic radius a of the pinning potential. In the limit $F \to 0$ and for E close to E_{max}, $b,c \sim (B/F)^{1/3}$, and the condition $b,c \gg a$ is well satisfied.

We reparametrize the energy in the form

$$
E = U_0 - \lambda B^{1/3} F^{2/3}, \tag{23}
$$

FIG. 3. Euclidean action as a function of temperature. At $T < T_1 = \hbar F^2/2\alpha U_0$, S_{Eucl} is a constant up to exponentially small corrections. In the regime $T > T_1$ the Euclidean action begins to decrease, see Eq. (19). The temperature $T = T_c$ [see Eq. (22)] marks the second-order-like transition from quantum to classical behavior. In the vicinity of T_c Eq. (27) gives an accurate description of the action.

FIG. 4. The function $\widetilde{\tau}$ [solid line, see Eq. (24)], which is directly proportional to the imaginary time oscillation period $\tau(E)$, as a function of λ/λ_{min} [see Eq. (23)], $\lambda_{min}=3/2^{2/3}$. Surprisingly, the slope of $\widetilde{\tau}$ near λ_{min} is very close to its asymptotic value at large λ. The dotted line shows the function $g(\lambda)=(2^{1/3}/\sqrt{3})\pi + (4\sqrt{3}/27)\pi(\lambda-3/2^{2/3})$, see Eq. (25). The dashed line illustrates the slope of the function $f(\lambda)=\lambda+\text{const}$ corresponding to the asymptotic expression for $\tau(E)$, $\tau(E)\simeq 2\alpha(U_0-E)/F^2$. Obviously, the slope of $f(\lambda)$ is very close to those of $\widetilde{\tau}(\lambda)$ and $g(\lambda)$, indicating that one can use Eq. (18) in almost the whole energy range.

where $\lambda=\lambda(E)$ is a dimensionless parameter. Substituting this expression into Eq. (21) we obtain for large λ (E away from E_{max}), $c\simeq\lambda^{-1/2}(B/F)^{1/3}$, and $b\simeq\lambda(B/F)^{1/3}$, i.e., $b\gg c$ for $\lambda\gtrsim 5$ and Eq. (18) is valid in this region. Consequently, if we show that the oscillation period given by Eq. (20) is a monotonous function of λ for $\lambda<5$, a second-order transition from quantum to classical behavior takes place. The function $[\widetilde{\tau}=(\tau F/2\alpha)(F/B)^{1/3}]$

$$
\widetilde{\tau}(\lambda)=\frac{1}{(\widetilde{b}+\widetilde{d})^{3/2}(\widetilde{b}+\widetilde{c})(\widetilde{b}-\widetilde{d})^{1/2}}
$$
$$
\times\left[(\widetilde{b}+\widetilde{d})F\left(\frac{\pi}{2},\sqrt{\frac{(\widetilde{b}-\widetilde{c})(\widetilde{b}+\widetilde{c})}{(\widetilde{b}-\widetilde{d})(\widetilde{b}+\widetilde{d})}}\right)\right.
$$
$$
\left.+(\widetilde{c}-\widetilde{d})\Pi\left(\frac{\pi}{2},\frac{(\widetilde{b}-\widetilde{c})(\widetilde{b}+\widetilde{c})}{(\widetilde{b}-\widetilde{d})(\widetilde{b}+\widetilde{d})},\sqrt{\frac{(\widetilde{b}-\widetilde{c})(\widetilde{b}+\widetilde{c})}{(\widetilde{b}-\widetilde{d})(\widetilde{b}+\widetilde{d})}}\right)\right],
$$
(24)

is plotted in Fig. 4 (solid line); here $\widetilde{b}\geqslant\widetilde{c}>\widetilde{d}$ are the roots of the equation $1/x^2+x=\lambda$. We can see that the function $\widetilde{\tau}(\lambda)$ is monotonously increasing. Surprisingly, even for $\lambda=\lambda_{min}\simeq 3/2^{2/3}$, the slope of $\widetilde{\tau}$ is close to its asymptotic value $\widetilde{\tau}(\lambda)\sim\lambda+\text{const},\lambda\to\infty$. Close to the point $\lambda_{min}=3/2^{2/3}$, we find

$$
\widetilde{\tau}(\lambda)\simeq\frac{2^{1/3}}{\sqrt{3}}\pi+\frac{4\sqrt{3}}{27}\pi\left(\lambda-\frac{3}{2^{2/3}}\right),
$$
(25)

and the derivative $\partial_E\tau|_{E_{max}}$ is indeed negative

$$
\partial\tau/\partial E|_{E_{max}}=-\frac{8\sqrt{3}\,\pi}{27}\frac{\alpha B^{1/3}}{F^{4/3}}.
$$
(26)

After solving the equation $\tau(E)=\hbar/T$ in the vicinity of the point $E=E_{max}$ we obtain the expression for the action

$$
S_{Eucl}(T)\simeq\frac{\hbar}{T}\left(U_0-\frac{3}{2^{2/3}}B^{1/3}F^{2/3}\right)
$$
$$
-2^{4/3}\sqrt{3}\,\pi^3\frac{\alpha^3 B}{\hbar^2 F^4}(T-T_c)^2\theta(T_c-T), \quad (27)
$$

which improves on the result (19) in the vicinity of T_c.

C. Preexponential factor

Finally, let us estimate the preexponential factor at high and at low temperatures. At $T>T_c$ the saddle-point solution is time independent and is given by the equation $x=x_{max}$ with x_{max} the maximum of the function $H(x,0)$. In the vicinity of the point $(x_{max},0)$,

$$
H(x,p)\simeq U_0-\frac{3B}{x_{max}^4}(x-x_{max})^2+\frac{B}{\alpha^2 x_{max}^4}p^2.
$$
(28)

The imaginary time Hamiltonian corresponding to Eq. (28) has the form of a harmonic oscillator with the mass $\alpha^2 x_{max}^4/2B$ and the stiffness $6B/x_{max}^4$. Consequently, at high temperatures one can use the result for the decay rate of a massive particle (see Refs. 3,27)

$$
\Gamma=\frac{\sqrt{3}F^{4/3}}{2^{4/3}\pi\alpha B^{4/3}}\frac{\sinh(\hbar\kappa/2\alpha T)}{\sin[(\sqrt{3}/2^{4/3})(\hbar F^{4/3}/\alpha B^{4/3}T)]}\exp\left(-\frac{U_0}{T}\right).
$$
(29)

This expression is applicable at any temperature higher than T_c except for a narrow temperature interval $\sim\hbar^{3/2}$ around T_c, see Ref. 27. At $T\gg T_c,\hbar\kappa/\alpha$ we obtain the simple result $\Gamma=(\hbar\kappa/2\pi\alpha)\exp(-U_0/T)$.

At zero temperature the decay involves only tunneling out of the ground state, i.e., $\Gamma\simeq[\omega(E_0)/2\pi]\times\exp[-(\alpha/\hbar)(U_0/F)^2]$, see Ref. 23. Near the metastable minimum the vortex oscillates with the frequency $\omega=\kappa/\alpha$, and we arrive at the result

$$
\Gamma\simeq\frac{\kappa}{2\pi\alpha}\exp\left[-\frac{\alpha}{\hbar}\left(\frac{U_0}{F}\right)^2\right].
$$
(30)

However, one should point out that the preexponential factor in Eq. (30) is only an order of magnitude estimate as the quasiclassical approximation is not properly applicable to the ground state.

IV. APPLICABILITY

Let us discuss the conditions for the applicability of the above results. First of all, we have to account for a nonzero vortex mass. In this case, the system is described by the Hamiltonian

$$\hat{H} = \frac{\hat{P}_x^2 + (\hat{P}_y^2 + \alpha x)^2}{2m} + U_0(\sqrt{x^2 + y^2}) - Fx, \qquad (31)$$

which is equivalent to the Hamiltonian of a massive 2D charged particle in a magnetic field oriented perpendicular to the plane of motion. Let us discuss the conditions which should be satisfied in order to use the Hamiltonian (6) instead of that given by Eq. (31). It is useful to introduce the four new operators (see also Ref. 28) $\hat{\xi}_y = -l^2\hat{P}_x/\hbar$, $\hat{X} = -l^2\hat{P}_y/\hbar$, $\hat{\xi}_x = l^2\hat{P}_y/\hbar + x$, $\hat{Y} = l^2\hat{P}_x/\hbar + y$, where $l = \sqrt{\hbar/\alpha}$ is the magnetic length.[20] We assume that the condition $l \ll a$ is satisfied (below we will verify this condition using parameters for high-T_c superconductors). These new operators satisfy the commutation relations $[\hat{\xi}_y, \hat{\xi}_x] = [\hat{X}, \hat{Y}] = il^2$, $[\hat{\xi}_x, \hat{Y}] = [\hat{\xi}_x, \hat{X}] = [\hat{\xi}_y, \hat{X}] = [\hat{\xi}_y, \hat{Y}] = 0$. Going over to the X, ξ_y representation the Hamiltonian (31) can be written in the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi_y^2} + \frac{m\omega_c^2\xi_y^2}{2} + \int U(\mathbf{k})e^{ik_x(\hat{\xi}_x + \hat{X}) + ik_y(\hat{\xi}_y + \hat{Y})}d^2k. \qquad (32)$$

In a classical language the problem we study involves two types of motion: A fast rotation of the particle with the cyclotron frequency $\omega_c = \alpha/m$ superimposed on the slow guiding center motion. The term

$$\hat{h} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi_y^2} + \frac{m\omega_c^2\xi_y^2}{2} \qquad (33)$$

describes the "fast" part of the Hamiltonian. The lowest eigenvalue of $\hat{h}$ is equal to $\hbar\omega_c/2$. If the characteristic variation of the potential on the length l is much smaller than $\hbar\omega_c/2$, we can average the Hamiltonian (32) over the ground-state eigenfunction $\chi(\xi_y)$ of the operator $\hat{h}$, see Ref. 29. This process corresponds to the averaging over the fast rotation with the frequency ω_c. After averaging we obtain

$$\hat{H} \rightarrow \langle\hat{H}\rangle_{\text{fast}} = \int U(\mathbf{k})\exp\left[-\frac{\mathbf{k}^2l^2}{4} + i(k_x\hat{X} + k_y\hat{Y})\right]d^2k. \qquad (34)$$

Note that in the X representation $\hat{Y} = -il^2\partial/\partial X$. If $l \ll a$, we can neglect the exponent $\exp(-\mathbf{k}^2l^2/4)$ in Eq. (34). On the other hand, if l is small, $[\hat{X}, \hat{Y}] \rightarrow 0$ and we can easily perform the integration in Eq. (34) to arrive at the new Hamiltonian

$$\hat{H}_{\text{eff}} = U_0(\sqrt{\hat{X}^2 + \hat{Y}^2}) - F\hat{X} \qquad (35)$$

describing the guiding center motion. As the operators $\hat{X}$ and $\hat{Y}$ obey the commutation relation $[\hat{X}, \hat{Y}] = il^2$, we can identify $\hat{X}$ with the coordinate and $\hat{Y}$ with the momentum. This Hamiltonian is identical to that in Eq. (6).

In order to carry out the above procedure the characteristic variation of the potential $U(x,y)$ on the scale l has to be much smaller than $\hbar\omega_c$, i.e., $\kappa l^2 \ll \hbar\omega_c$, and with $\kappa \sim U_0/a^2$ we obtain the condition $\alpha^2 a^2/mU_0 \gg 1$. Consequently, we arrive at the following two conditions for the applicability of the approach we have used:

$$l \ll a \quad \text{and} \quad \frac{\alpha^2 a^2}{mU_0} \gg 1. \qquad (36)$$

Let us estimate the parameters in Eq. (36) appropriate for high-T_c superconductors: $\alpha = \pi\hbar ns$, where $n = 2 \times 10^{21}$ cm^{-3} is the electron density and $s = 15$ Å is the interlayer spacing in Bi-2:2:1:2. For the magnetic length we then obtain $l = \sqrt{\hbar/\alpha} \simeq 4$ Å $\ll a \simeq 30$ Å. For the second condition in Eq. (36) we use the expression for the vortex mass predicted for the superclean limit[15] $m = m_e(\epsilon_F/\Delta)^2 \simeq 100m_e$ and the depth of the potential as given by the equation $U_0 = (\Phi_0/4\pi\lambda_{ab})^2 s$, with Φ_0 the magnetic-flux quantum and $\lambda_{ab} = 2000$ Å being the ab-magnetic penetration length. Finally, we obtain $\alpha^2 a^2/mU_0 \simeq 10 \gg 1$, i.e., we find that both conditions in Eq. (36) are well satisfied.

The above results should be compared to those in Ref. 2 where the condition $\hbar\alpha/mU_0 \gg 1$ for the applicability of the results has been used. Estimating this parameter we obtain $\hbar\alpha/mU_0 \simeq 0.1 \ll 1$. In Ref. 2 an estimate $\hbar\alpha/mU_0 \simeq 200$ has been obtained using the vortex mass from the dirty limit,[30–32] which is much smaller than the mass in the superclean limit. We see that the LLL approach of Ref. 2 breaks down in the superclean limit if one does not take explicitly the condition $l \ll a$ into account.

Finally, let us show that the second condition in Eq. (36) can be easily obtained from a simple analysis of the semiclassical bounce trajectories: If the Euclidean action is much larger than unity, we can use the instanton method for the calculation of the imaginary part of the partition function which determines the decay rate $\Gamma \propto \text{Im}Z/Z$. In order to determine $\text{Im}Z$ within exponential accuracy we have to solve the classical equation of motion describing the bounce solution. If it turns out that the correction of the bounce trajectory due to the mass term is small, we arrive at an effective Hall tunneling problem. It is possible to neglect the mass if everywhere along the trajectory $m|\dot{v}| \ll \alpha|v|$, where $\dot{v}$ is the characteristic acceleration of the vortex during the imaginary time motion and v its characteristic velocity. If a vortex is moving along the sides CA, AA', or $A'C$ of the triangle CAA' (see Fig. 2), we can make the estimate $|\dot{v}| \sim v^2/(U_0/F)$, whereas for v the estimate $v \sim U_0/\alpha a$ holds, i.e., the following condition should be satisfied:

$$\frac{F}{F_c} \ll \frac{\alpha^2 a^2}{mU_0}, \qquad (37)$$

with $F_c \sim U_0/a$ the critical force. On the other hand, in the vicinity of the point C, we can write $v \sim U_0/\alpha a$ and $\dot{u} \sim v^2/a$, i.e., the condition $\alpha^2 a^2/mU_0 \gg 1$ has to be satisfied, which is identical to the second condition in Eq. (36). Comparing this result with Eq. (37) we see that the second condition in Eq. (36) is stronger since $F < F_c$.

V. SUMMARY

In Sec. IV we have described how to reduce the dynamics of a massive particle in a magnetic field to the guiding center motion along equipotential lines of a smooth potential. Carrying out this procedure for vortices is not entirely unproblematic. In fact, we have seen that the averaging process over the fast component introduces a frequency $\omega_c = \alpha/m$

$\simeq \omega_0$ which is at the limit of the applicability of the low-frequency vortex dynamics as described by Eq. (1). This problem has been ignored in previous works as the inertial term has been dropped on the classical level of Eq. (1). As shown above, both methods produce the same condition $\alpha^2 a^2/mU_0 \gg 1$ for the irrelevance of the mass term, however, starting from the description (31) one obtains that the frequency $\omega_c = \alpha/m$ naturally shows up in the quantum description.

Briefly summarizing, we have considered the decay of the metastable state of a massless particle in a magnetic field trapped in a cylindrical attractive potential and subject to a small external force. The above problem is an appropriate model for the depinning of pancake vortices in superclean Josephson-coupled superconductors for the case of small ($j \ll j_c$) transport currents. The Euclidean action as a function of temperature is given by Eq. (19); for $T < T_1$ the action is constant up to exponentially small corrections and decreases smoothly above T_1. At $T = T_c$, the crossover to the classical regime takes place. Close to and above T_c the decay rate can be accurately described by Eq. (27) which reduces to the exact classical result above T_c. For the potential satisfying the conditions (i) $U(r)$ is a monotonous function of r and (ii) $U(r) \cong U_0 - B/r^2, r \to \infty$, the transition from quantum to classical behavior is second-order like and the crossover temperature T_c is proportional to $F^{4/3}$; see Eq. (22).

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