Thermally activated Hall creep of flux lines from a columnar defect

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We analyze the thermally activated depinning of an elastic string (line tension ϵ) governed by Hall dynamics from a columnar defect modeled as a cylindrical potential well of depth V_0 for the case of a small external force F. An effective one-dimensional-field Hamiltonian is derived in order to describe the two-dimensional string motion. At high temperatures the decay rate is proportional to $F^{5/2}T^{-1/2}\exp[F_0/F-U(F)/T]$, with F_0 a constant of order of the critical force and $U(F)\sim(\epsilon V_0)^{1/2}V_0/F$ the activation energy. The results are applied to vortices pinned by columnar defects in superclean superconductors. [S0163-1829(98)07105-7]

I. INTRODUCTION

The search for mechanisms of pinning is both scientifically challenging as well as an important problem to be studied in view of technological applications of high-T_c superconductors. Even if the transport current is less than the critical one, energy dissipation takes place due to quantum or thermally activated creep of vortices.[1] Recent measurements of the critical current and of the magnetization relaxation rate in layered high-temperature superconductors show that columnar defects produced by irradiation with heavy ions can strongly suppress the vortex motion.[2] In this paper we present a theoretical study of the thermally activated depinning of a vortex from a columnar defect in the presence of a small transport current (classical creep).

The inverse lifetime Γ of a metastable state can be written in the form $\Gamma=Ae^{-S/\hbar}$, where S is the Euclidean action along the extremal trajectory and A is the prefactor determined by the fluctuations around the saddle-point solution (see Ref. 3 for a general review concerning the decay of metastable states). At low temperatures the saddle-point solution is time dependent and, consequently, S depends on the dynamics of the system (quantum creep), whereas at high temperatures the calculation of S alone does not involve the dynamics (classical creep). Here, we concentrate on high temperatures but go beyond the usual exponential accuracy by calculating the prefactor A, a task which does involve the dynamics of the flux lines as well.

In high-T_c superconductors the dynamics of vortices may be dominated by either the dissipative or Hall term in the equation of motion. Microscopic calculations of the dynamic constants[4] show that the ratio of Hall and dissipative coefficients α/η is approximately equal to $\omega_0\tau$, with ω_0 and τ the level spacing between localized Caroli–de Gennes–Matricon states in the core and the relaxation time, respectively. Recent experimental studies on 90 K crystals of Y-Ba-Cu-O (see Ref. 5) have been interpreted as providing evidence for the superclean limit, with $\omega_0\tau\sim15$ below 15 K, in which case the contribution of dissipative forces can be neglected in this regime.

Quantum depinning of flux lines governed by Hall dynamics from a columnar defect has been considered by Sonin and Horovitz[6] for the case of a small external force. For pancake vortices this problem has been studied by Bulaevskii, Larkin, Maley, and Vinokur.[7] The case $j_c-j\ll j_c$ has been investigated by Chudnovsky, Ferrera, and Vilenkin[8] and by Morais-Smith, Caldeira, and Blatter.[9] On the other hand, the problem of the depinning of a *massive* string from a linear object has been solved by Skvortsov in the whole temperature range[10] and the thermal depinning of a flux line governed by dissipative dynamics has been considered by Krämer and Kulić.[11]

The present paper is organized as follows: In Sec. II we reduce the two-dimensional problem of the string motion to an effective one-dimensional (1D) problem. In Sec. III the decay rate of a trapped string is calculated. In Sec. IV the results are applied to vortices in superclean superconductors.

II. 1D EFFECTIVE HAMILTONIAN

Let us consider a string which is pinned by a columnar defect in the presence of an external force. Both the cylindrical defect as well as the vortex are directed along the c axis of the anisotropic superconductor and the external magnetic field is supposed to be sufficiently small, such that the interaction between the vortices can be neglected. Furthermore, we consider the situation where each vortex is pinned by an individual defect, i.e., the concentration of defects is assumed to be larger than that of vortices. The free-energy density of the string describing this situation is

$$G(\mathbf{u})=\frac{\epsilon}{2}\left(\frac{\partial\mathbf{u}}{\partial z}\right)^2+U(\mathbf{u}),\tag{1}$$

where

$$U(\mathbf{u})=V_{\rm cyl}(\sqrt{u_x^2+u_y^2})+V_{\rm ext}(u_x).\tag{2}$$

Here, ϵ is the elasticity of the string and u_x and u_y describe its displacement along the x and y directions, with the z axis chosen parallel to the defect. In Eq. (2), $V_{\rm cyl}(\sqrt{{u_x}^2+{u_y}^2})$ denotes the cylindrical pinning potential and $V_{\rm ext}=-Fu_x$ is the forcing potential. The function $V_{\rm cyl}(r)$ is supposed to be monotonously increasing and restricted from below and above. We assume boundary conditions $\mathbf{u}(\pm L/2)=0$.

If $F\neq0$, the state of the vortex becomes metastable. Our main goal is to investigate the decay rate as a function of temperature T and force F, where the external force is assumed to be small. The Lagrangian of the vortex (in real time dynamics) can be written as

$$L[u_x,u_y]=\int_{-L/2}^{+L/2}dz\left(\alpha u_y\frac{\partial u_x}{\partial t}-G[u_x,u_y]\right), \tag{3}$$

where α is the Hall coefficient. The corresponding equations of motion take the form

$$\alpha\frac{\partial u_x}{\partial t}=\frac{\partial U}{\partial u_y}-\epsilon\frac{\partial^2 u_y}{\partial z^2}, \tag{4}$$

$$\alpha\frac{\partial u_y}{\partial t}=-\frac{\partial U}{\partial u_x}+\epsilon\frac{\partial^2 u_x}{\partial z^2}. \tag{5}$$

Equations (4) and (5) can be formulated as the equations of motion of the 1D Hamiltonian density

$$H=\left[U\left(x,\frac{p}{\alpha}\right)+\frac{\epsilon}{2}\left(\frac{\partial x}{\partial z}\right)^2+\frac{\epsilon}{2\alpha^2}\left(\frac{\partial p}{\partial z}\right)^2\right], \tag{6}$$

where we have used the definitions $x\equiv u_x$ and $p\equiv\alpha u_y$. Using the variational procedure for the Hamiltonian density H we obtain

$$\dot{x}=\frac{\partial H}{\partial p}-\frac{\partial}{\partial z}\frac{\partial H}{\partial p_z}=\frac{\partial U(x,p/\alpha)}{\partial p}-\frac{\epsilon}{\alpha^2}\frac{\partial^2 p}{\partial z^2}, \tag{7}$$

$$\dot{p}=-\frac{\partial H}{\partial x}+\frac{\partial}{\partial z}\frac{\partial H}{\partial x_z}=-\frac{\partial U(x,p/\alpha)}{\partial x}+\epsilon\frac{\partial^2 x}{\partial z^2}, \tag{8}$$

and one can easily see that Eqs. (7) and (8) are equivalent to Eqs. (4) and (5) with $x=u_x$ and $p=\alpha u_y$. Thus we have reduced the 2D problem of the motion of a vortex governed by Hall dynamics with an action given by Eq. (3) to the 1D problem of the motion of a vortex described by the effective action $\int dz\,dt(p\dot{x}-H)$, with H given by Eq. (6). The above idea has been introduced by Volovik[12] for the tunneling of 2D vortices in a liquid-helium film, the motion of which is equivalent to that of a massless particle in a magnetic field. Later, this idea has been used by Feigel'man, Geshkenbein, Larkin, and Levit[13] within the context of vortex tunneling in superclean high-T_c superconductors. Chudnovsky, Ferrera, and Vilenkin[8] have generalized this idea for the description of the depinning of a flux line governed by Hall dynamics from a columnar defect near criticality. In their case $U(u_x,u_y)=au_x^2/2-bu_x^3/3+cu_y^2/2$ and the elasticity along the y direction can be neglected if $j_c-j\ll j_c$, i.e., the problem can be reduced to the 1D massive string problem. Above we have generalized the problem for the case of an arbitrary potential and nonzero elasticity in the y direction.

III. DECAY RATE

The decay rate Γ of an arbitrary metastable Hamiltonian system at high temperatures is given by the expression[14]

$$\hbar\Gamma=\frac{\hbar\omega_0}{\pi}\frac{\mathrm{Im}\,Z}{Z}, \tag{9}$$

where ω_0 is the unstable mode growth rate and Z is the partition function of the system under consideration. Equation (9) is applicable if the temperature T satisfies the condition $T\gg\hbar\omega_0$, otherwise quantum effects become relevant. If the transition from quantum to classical behavior in the decay of the metastable state is of second order, the parameter $T_0=\hbar\omega_0/2\pi$ determines the temperature of the crossover. However, in general the transition can be of first order, in which case the crossover temperature differs from $\hbar\omega_0/2\pi$ (see Ref. 15). Note that at low temperatures the equation for the decay rate can be written as $\hbar\Gamma=2T\,\mathrm{Im}Z/Z$. We emphasize that Eq. (9) can be applied only if the system is properly equilibrized, i.e., if its characteristic relaxation time is much smaller than Γ^{-1}.

We assume that the quasiclassical approximation is applicable. With the partition function written as a path integral,

$$Z=\int\{\mathcal{D}\mathbf{u}\}\exp\left(-\frac{S_{\mathrm{Eucl}}[\mathbf{u}]}{\hbar}\right), \tag{10}$$

we can use the steepest descent method for its calculation. In this approximation the partition function is determined by its stationary points. The most significant contribution to Z arises from the trajectory $\mathbf{u}(z,t)=\mathbf{0}$—the position of the minimum of the potential, whereas the imaginary part of the partition function is determined by the neighborhood of the saddle-point solution $\mathbf{u}(z,t)=\mathbf{u}_0(z)$ which is time independent at high temperatures. Hence, the problem is reduced to the calculation of the unstable mode growth rate ω_0, and of the real and imaginary parts of the statistical sum. Below we shall calculate all these quantities.

A. Unstable mode growth rate

At high temperatures the saddle-point solution does not depend on time. The stationary extremal trajectory $\mathbf{u}_0(z)$ for the Euclidean action

$$S_{\mathrm{Eucl}}[u_x,u_y]=\int_{-L/2}^{+L/2}dz\int_{-\hbar/2T}^{+\hbar/2T}d\tau\left(G[u_x,u_y]-i\alpha u_y\frac{\partial u_x}{\partial\tau}\right) \tag{11}$$

corresponding to the real-time Lagrangian (3) satisfies the Euler equation

$$\epsilon\frac{\partial^2\mathbf{u}}{\partial z^2}=\frac{\partial U(\mathbf{u})}{\partial\mathbf{u}},\quad \mathbf{u}=(u_x,u_y). \tag{12}$$

The unstable mode growth rate then is determined by the negative eigenvalue of the operator $\delta^2_{\mathbf{u}}S_{\mathrm{Eucl}}|_{\mathbf{u_0}(z)}$. Near the saddle-point solution $\mathbf{u}_0(z)$ the perturbed solution can be expanded in the form

$$u_x(z,\tau)=u_{x0}(z)+\psi(z)\exp(i2\pi T\tau/\hbar), \tag{13}$$

$$u_y(z,\tau)=\varphi(z)\exp(i2\pi T\tau/\hbar), \tag{14}$$

with small distortion amplitudes $\psi(z)$ and $\varphi(z)$ for $T\lesssim T_0$. Substituting Eqs. (13) and (14) into the equations of motion for the Euclidean action (11) and expanding, we obtain the system of equations determining the crossover temperature T_0.

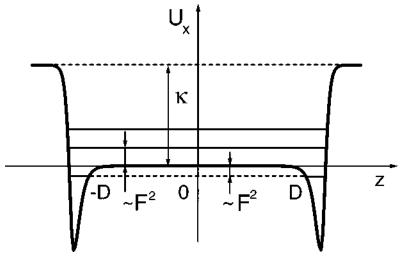

FIG. 1. The ''potential'' $U_x(z)$ for the 1D Schrödinger operator $\hat{H}_x$ [see Eq. (15)]. The characteristic size of the potential is given by $D=\sqrt{2\epsilon V_0}/F$ and its height is equal to κ.

$$-\epsilon\frac{d^2\psi}{dz^2}+\alpha\omega_0\varphi+\frac{d^2V_{\rm cyl}}{dr^2}\Big|_{r=u_{x0}(z)}\equiv\hat{H}_x\psi+\alpha\omega_0\varphi=0, \tag{15}$$

$$-\epsilon\frac{d^2\varphi}{dz^2}-\alpha\omega_0\psi+\frac{1}{r}\frac{dV_{\rm cyl}}{dr}\Big|_{r=u_{x0}(z)}\equiv\hat{H}_y\varphi-\alpha\omega_0\psi=0, \tag{16}$$

subject to the boundary conditions $\psi(\pm L/2),\varphi(\pm L/2)=0$. The (lowest) eigenvalue ω_0 is related to the crossover temperature T_0 through the condition $\omega_0=2\pi T_0/\hbar$.

The ''potentials'' $U_x(z)$ and $U_y(z)$ of the Schrödinger operators defined in Eqs. (15) and (16) are shown in Figs. 1 and 2, respectively.

The operator $\hat{H}_x$ has only one negative eigenvalue: it can be easily seen that in the limit $L\to\infty$ the function du_{x0}/dz is an eigenfunction of the operator $\hat{H}_x$ with zero eigenvalue. Its derivative du_{x0}/dz can be understood as the ''velocity of a particle'' moving in the potential $U(u_{x0},u_{y0})$ with the ''velocity'' changing its sign once. Hence, du_{x0}/dz is an eigenfunction of the 1D Schrödinger operator $\hat{H}_x$ with one node and there must be another function corresponding to the ground-state ''wave function'' with a negative eigenvalue. We conclude that the operator $\hat{H}_x$ has one negative and one zero eigenvalue. The operator $\hat{H}_y$ is positive and has only positive eigenvalues.

Eliminating the function φ from the system of Eqs. (15) and (16) we obtain (we denote eigenvalues of $\hat{H}_x\hat{H}_y$ by $\tilde{\lambda}$ and those of $\hat{H}_x$ by λ)

$$\hat{H}_y\hat{H}_x\psi=\tilde{\lambda}\psi. \tag{17}$$

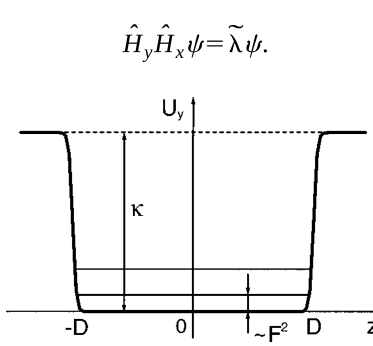

FIG. 2. The ''potential'' $U_y(z)$ for the 1D Schrödinger operator $\hat{H}_y$ [see Eq. (16)]. The characteristic size of the potential is given by $D=\sqrt{2\epsilon V_0}/F$ and its height is equal to κ.

The unstable mode growth rate ω_0 is determined by the negative eigenvalue $\tilde{\lambda}_{-1}$ of Eq. (17), $\tilde{\lambda}_{-1}=-(\alpha\omega_0)^2$. In Appendix A we derive a variational principle for the present problem and show that the negative eigenvalue of Eq. (17) indeed exists and is unique. (We exploit that the operator $\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}$ is Hermitian and has the same eigenvalues as $\hat{H}_y\hat{H}_x$). Furthermore, we derive both upper and lower bounds on the eigenvalue $\tilde{\lambda}_{-1}$ of the operator $\hat{H}_x\hat{H}_y$ in Appendix B, such that we finally arrive at the estimate

$$T_0=\frac{\hbar\sqrt{|\tilde{\lambda}_{-1}|}}{2\pi\alpha}=\frac{\xi}{2\pi}\frac{\hbar F^2}{\alpha V_0}, \tag{18}$$

with $\xi\in[1.1530,2.7314]$ a universal constant and V_0 denotes the depth of the potential $V_{\rm cyl}$. The result (18) holds independently of the details of the pinning potential $V_{\rm cyl}$ as long as the driving force F is small, $F\ll F_c$. We note that $T_0\sim F^2$; the same dependence holds for a string with dissipative dynamics. For a massive string depinning from a linear defect $T_0\sim F$, see Ref. 10.

B. Ratio ImZ/Z

The most significant contribution to ReZ arises from the neighborhood of the minimum of the potential $V_{\rm cyl}$. In this region we can write [see Eq. (2)]

$$U(x,p/\alpha)\cong\frac{\kappa}{2}x^2+\frac{\kappa}{2\alpha^2}p^2, \tag{19}$$

where $\kappa=d^2V_{\rm cyl}/du_x^2|_{\mathbf{u}=0}$. The equations of motion (7) and (8) take the form

$$\alpha^2\dot{x}=\left(\kappa-\epsilon\frac{\partial^2}{\partial z^2}\right)p\equiv\hat{H}_0p, \tag{20}$$

$$\dot{p}=-\hat{H}_0x. \tag{21}$$

Hence, $p=\alpha^2\hat{H}_0^{-1}\dot{x}$ and the Euclidean action of the vortex near the equilibrium position can be written in the form [we make use of Eq. (6)]

$$S^0_{\rm Eucl}=\frac{1}{2}\int_{-\hbar/2T}^{+\hbar/2T}d\tau\int_{-L/2}^{+L/2}dz\;x\left(-\alpha^2\frac{\partial^2}{\partial\tau^2}\hat{H}_0^{-1}+\hat{H}_0\right)x. \tag{22}$$

The same procedure as above provides the variation of the Euclidean action near the thermal saddle-point solution

$$S^S_{\rm Eucl}=\frac{1}{2}\int_{-\hbar/2T}^{+\hbar/2T}d\tau\int_{-L/2}^{+L/2}dz\;\delta x\left(-\alpha^2\frac{\partial^2}{\partial\tau^2}\hat{H}_y^{-1}+\hat{H}_x\right)\delta x, \tag{23}$$

where the operators $\hat{H}_x$ and $\hat{H}_y$ are given by Eqs. (15) and (16). Using the usual measures for harmonic oscillators in the path integrals for $S^0_{\rm Eucl}$ and $S^S_{\rm Eucl}$ (see Ref. 16),

$$d\mu^0=\left[\det\left(-\alpha^2\frac{\partial^2}{\partial\tau^2}\hat{H}_0^{-1}\right)\right]^{1/2}\prod_a\frac{dC_a^0}{\sqrt{2\pi\hbar}} \tag{24}$$

and

$$d\mu^S=\left[\det\left(-\alpha^2\frac{\partial^2}{\partial\tau^2}\hat{H}_y^{-1}\right)\right]^{1/2}\prod_b\frac{dC_b^S}{\sqrt{2\pi\hbar}},\qquad(25)$$

(here C_a^0 and C_b^S are expansion coefficients over the systems of normalized functions) and defining the dimensionless operators $\hat{h}_0$, $\hat{h}_x$, and $\hat{h}_y$ as $\hat{H}_0/\kappa$, $\hat{H}_x/\kappa$, and $\hat{H}_y/\kappa$, respectively $[\kappa=V''_{\rm cyl}(0)]$, we obtain for ImZ/Z

$$\frac{{\rm Im}Z}{Z}=\frac{L}{2}\sqrt{\frac{\kappa}{2\pi T}}\left[\int_{-L/2}^{+L/2}dz\left(\frac{\partial u_0}{\partial z}\right)^2\right]^{1/2}\times\exp\left(-\frac{U}{T}\right)\left[\frac{\mu_{x,0}(L)}{\mu_{xy,0}(L)}\right]^{1/2}\times\left|\frac{\det[-(\alpha^2/\kappa^2)(\partial^2/\partial\tau^2)+\hat{h}_0^2]}{\det'[-(\alpha^2/\kappa^2)(\partial^2/\partial\tau^2)+\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}]}\right|^{1/2},\qquad(26)$$

where $(L\to\infty)$

$$U=\epsilon\int_{-L/2}^{+L/2}\left(\frac{\partial u_0}{\partial z}\right)^2dz\simeq\frac{4}{3}\sqrt{2\epsilon V_0}\frac{V_0}{F},\qquad(27)$$

is the activation barrier and L is the length of the string (we performed standard integration over the shift mode, see Refs. 17,18). The prime in Eq. (26) indicates that we exclude the zero eigenvalue of the operator $-\alpha^2\hat{H}_y^{-1}(\partial^2/\partial\tau^2)+\hat{H}_x$. $\mu_{x,0}(L)$ and $\mu_{xy,0}(L)$ are the ‘‘zero’’ eigenvalues of the operators $\hat{h}_x$ and $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$ defined on the interval $[-L/2,+L/2]$, $L\to\infty$.

The eigenvalues of the operators $-(\alpha^2/\kappa^2)(\partial^2/\partial\tau^2)+\hat{h}_0^2$ and $-(\alpha^2/\kappa^2)(\partial^2/\partial\tau^2)+\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$ are equal to

$$\left(\frac{2\pi\alpha T}{\kappa\hbar}n\right)^2+\mu_{0,a}\quad\text{and}\quad\left(\frac{2\pi\alpha T}{\kappa\hbar}n\right)^2+\mu_{xy,b},\qquad n=0,\pm1,\pm2\ \ldots,\qquad(28)$$

with $\mu_{0,a}$ and $\mu_{xy,b}$ the eigenvalues of the operators $\hat{h}_0^2$ and $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$, respectively. Substituting these eigenvalues into Eq. (26) and calculating the product over n using $\Pi_{n=1}^{\infty}(1+x^2/n^2)=\sinh(\pi x)/\pi x$ we obtain

$$\frac{{\rm Im}Z}{Z}=\frac{\alpha L}{4\hbar}\sqrt{\frac{T}{2\pi\kappa}}\exp\left(-\frac{U}{T}\right)\frac{1}{\sin(\pi T_0/T)}\left[\frac{\mu_{x,0}(L)}{\mu_{xy,0}(L)}\right]^{1/2}\times\left[\int_{-L/2}^{+L/2}dz\left(\frac{\partial u_0}{\partial z}\right)^2\right]^{1/2}\times\exp\left[\frac{\hbar\kappa}{2\alpha T}\left(\sum_a\sqrt{\mu_{0,a}}-\sum_b{}''\sqrt{\mu_{xy,b}}\right)\right]\qquad(29)$$

$$\times\frac{\prod_a[1-\exp(-(\hbar\kappa/\alpha T)\sqrt{\mu_{0,a}})]}{\prod_b''[1-\exp(-(\hbar\kappa/\alpha T)\sqrt{\mu_{xy,b}})]}.$$

The double-prime sign reminds us that we excluded the negative and zero eigenvalues of the operator $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$.

C. Result for Γ

The positive part of the spectrum of $\hat{h}_x^{1/2}\hat{h}_y\hat{h}_x^{1/2}$ consists of M discrete eigenvalues $\mu_{xy,b}$ and a continuous part $\mu_{xy}(k)=(1+\epsilon k^2/\kappa)^2$ with a spectral density $\rho^S(k)$. The spectrum of $\hat{h}_0^2$ is continuous, $\mu_0(k)=(1+\epsilon k^2/\kappa)^2$ with the spectral density $\rho^0(k)$. Using

$$\frac{\prod_a f(\mu_{0,a})}{\prod_b f(\mu_{xy,b})}=\exp\left[\int dk\ \delta\rho(k)\ln f(\mu(k))\right],\qquad(30)$$

with $\delta\rho(k)=\rho^0(k)-\rho^S(k)$ we arrive at the final expression for the decay rate.

$$\Gamma=\frac{\alpha T_0}{2\hbar^2}\sqrt{\frac{T}{2\pi\kappa}}\exp\left(-\frac{U^*}{T}\right)\frac{L}{\sin(\pi T_0/T)}\times\left[\int_{-L/2}^{+L/2}dz\left(\frac{\partial u_0}{\partial z}\right)^2\right]^{1/2}\left[\frac{\mu_{x,0}(L)}{\mu_{xy,0}(L)}\right]^{1/2}\times\exp\left\{\int_0^\infty dk\ \delta\rho(k)\ln\left[1-\exp\left(-\frac{\hbar\kappa}{\alpha T}\left(1+\frac{\epsilon}{\kappa}k^2\right)\right)\right]\right\}\times\prod_{b=1}^{M}\left[1-\exp\left(-\frac{\hbar\kappa}{\alpha T}\sqrt{\mu_{xy,b}},\right)\right]^{-1},\qquad(31)$$

with

$$U^*=U-\frac{\hbar\kappa}{2\alpha}\left[\int_0^\infty dk\ \delta\rho(k)\left(1+\frac{\epsilon}{\kappa}k^2\right)-\sum_{b=1}^{M}\sqrt{\mu_{xy,b}}\right]\qquad(32)$$

the quantum renormalized activation energy. Let us investigate the correction to the activation energy (32) arising from large k values. We will see that the large-k modes lead to an ultraviolet divergence which we have to cut off by some physical length scale. It is convenient to rewrite the integral $\int_0^k dk'\,\delta\rho(k')(1+\epsilon k'^2/\kappa)$ as

$$\int_0^k dk'\,\delta\rho(k')\left(1+\frac{\epsilon}{\kappa}k'^2\right)=\sum_n^{N_0(k)}(1+\epsilon k_n^2/\kappa^2)-\sum_n^{N_S(k)}(1+\epsilon q_n^2/\kappa^2),\qquad(33)$$

where k_n and q_n are trivially related to the eigenvalues of the operators $\hat{h}_0^2$ and $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$ and $N_0(k)$ and $N_S(k)$ denote the number of eigenvalues inside the interval $[0,k]$. Note that $N_0(\infty)-N_S(\infty)=M+2$. The two operators $\hat{h}_0^2$ and $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$ define two scattering problems. We define appropriate scattering states $\psi_k(z)$ and $\psi_q(z)$ with asymptotics e^{ikz} and e^{iqz} $(z\to-\infty)$ and $e^{ikz},e^{iqz+\delta(q)}$ $(z\to+\infty)$, with $\delta(q)$ the appropriate scattering phase shift. The general solutions of the ‘‘Schrödinger’’ equations can be written as $\Psi_k(z)$

$=C_1\psi_k(z)+C_2\psi_{-k}(z)$ and $\Psi_q(z)=C_1'\psi_q(z)+C_2'\psi_{-q}(z)$. Since we require that the "wave functions" satisfy the condition $\Psi_p(\pm L/2)=0$, $L\gg\sqrt{\epsilon V_0}/F$, we obtain the discrete levels through the phase equations (see also Ref. 19)

$$k_nL=\pi n, \quad q_nL+\delta(q)=\pi n, \tag{34}$$

and hence

$$k_n^2-q_n^2=\left(\frac{\pi n}{L}\right)^2-\left(\frac{\pi n-\delta(q)}{L}\right)^2 \simeq+\frac{2\pi n}{L^2}\delta(q)\simeq+\frac{2q}{L}\delta(q). \tag{35}$$

For $q\gg\sqrt{\kappa/\epsilon}$ we can make use of the semiclassical approximation for the operator $\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2}$ and calculate the phase shift directly: With the usual ansatz $\Psi(z)\sim e^{i\int q(z)dz}$ the function $q(z)$ can be found from the equation

$$[\epsilon q^2(z)+U_x][\epsilon q^2(z)+U_y]=\kappa^2E_q, \tag{36}$$

where $E_q=(1+\epsilon q^2/\kappa)^2\simeq\epsilon^2q^4/\kappa^2$ is the eigenvalue of the continuous spectrum of $\hat{h}_y^{1/2}\hat{h}_x^{1/2}\hat{h}_y^{1/2}$. At large q we obtain

$$q(z)\simeq\sqrt{\frac{\kappa}{\epsilon}}E_q^{1/4}\left(1-\frac{(U_x+U_y)}{4\kappa\sqrt{E_q}}\right). \tag{37}$$

Similarly, for the operator $\hat{h}_0^2$ we obtain

$$k(z)\simeq\sqrt{\frac{\kappa}{\epsilon}}E_k^{1/4}\left(1-\frac{1}{2\sqrt{E_k}}\right), \tag{38}$$

and we arrive at the difference in the phase shift

$$\delta(q)\simeq\sqrt{\frac{\kappa}{\epsilon}}E_q^{1/4}\int_{-L/2}^{+L/2}dz\frac{(2\kappa-U_x-U_y)}{4\kappa\sqrt{E_q}}. \tag{39}$$

Taking into account that at large q, $E_q\simeq\epsilon^2q^4/\kappa^2$ and that $U_x(z),U_y(z)\ll\kappa$ for $|z|<\sqrt{2\epsilon V_0}/F$, we obtain

$$\delta(q)\simeq\sqrt{\frac{2V_0}{\epsilon}}\frac{\kappa}{qF}, \tag{40}$$

i.e., $k_n^2-q_n^2=2\kappa\sqrt{2V_0/\epsilon}/FL$. At large k the number of states in the interval dk is equal to $L\,dk/2\pi$ and the integral

$$\int_0^k dk'\,\delta\rho(k')\left(1+\frac{\epsilon k'^2}{\kappa}\right)\simeq\frac{\epsilon}{\kappa}\sum_n^{N_0(k)}(k_n^2-q_n^2) \simeq\int_0^k dk'\,\frac{\sqrt{2\epsilon V_0}}{\pi F} \tag{41}$$

diverges linearly at large k (for the case of a massive string the correction to the activation diverges as $\ln k$, see Refs. 10,18). The integral then has to be cut off at some wave vector k^*. For a string the natural cutoff is π/r, with r the radius of the string. For a vortex parallel to the c axis of an anisotropic superconductor $k^*\sim\pi/\max(\xi_c,d)$, with ξ_c the coherence length in the c direction and d being the distance between the superconducting layers.

Cutting off the integral in Eq. (32) at large momentum k^* we obtain

$$U^*\simeq U-\frac{\hbar\kappa k^*}{\sqrt{2}\pi\alpha F}\sqrt{V_0\epsilon}. \tag{42}$$

The theory is self-consistent if the correction to the activation energy is small as compared to U (here we have assumed the cutoff k^* to be large such that we can neglect the contribution from the bound states).

In case of a continuous transition from quantum to classical behavior the result Eq. (31) with Eq. (32) is applicable for any $T>T_0$ except for a narrow temperature interval $\sim\hbar^{3/2}$ around T_0 (see Ref. 20). An explicit expression for the decay rate can be obtained only for the case $\hbar\to0$, i.e., in the classical limit. In this limit we can expand the exponents in Eq. (31), $1-\exp(-\hbar\kappa/\alpha T)\simeq\hbar\kappa/\alpha T$ and the quantum correction to the activation energy (32) tends to zero. Performing these steps we obtain the result

$$\Gamma=\frac{\kappa L}{2\pi\alpha}\sqrt{\frac{\kappa}{2\pi T}}\exp\left(-\frac{U}{T}\right)\left[\int_{-L/2}^{+L/2}dz\left(\frac{\partial u_0}{\partial z}\right)^2\right]^{1/2} \times\left[\frac{\mu_{x,0}(L)}{\mu_{xy,0}(L)}\right]^{1/2}\left[\frac{\det(\hat{h}_0^2)}{\det''(\hat{h}_y^{1/2}\hat{h}_x\hat{h}_y^{1/2})}\right]^{1/2}. \tag{43}$$

Using the same technique as before, see Eq. (26), we arrive at the more suitable form

$$\Gamma=\frac{\kappa L}{2\pi\alpha}\sqrt{\frac{\kappa}{2\pi T}}\exp\left(-\frac{U}{T}\right)\left[\int_{-L/2}^{+L/2}dz\left(\frac{\partial u_0}{\partial z}\right)^2\right]^{1/2} \times\sqrt{|\mu_{xy,-1}|}\lim_{L\to\infty}\sqrt{\mu_{x,0}(L)}\left|\frac{\det(\hat{h}_0)}{\det(\hat{h}_x(L))}\right|^{1/2}\left[\frac{\det(\hat{h}_0)}{\det(\hat{h}_y)}\right]^{1/2}, \tag{44}$$

where $\mu_{x,0}(L)$ is the zero eigenvalue of the operator $\hat{h}_x(L)$.

The calculation of the instanton determinant ratios (44) can be elegantly performed using the Gelfand-Yaglom formula.[21] The renormalized[21] determinant $\det\hat{H}$ of the Schrödinger operator $\hat{H}=-\partial^2/\partial z^2+p(z)$ is the solution of the differential equation $\hat{H}f(z)=0$ with the initial conditions $f|_{z=-L/2}=0$ and $df/dz|_{z=-L/2}=1$. The value of the function $f(z=L/2)$ provides the renormalized determinant. As has been shown in Ref. 11, in the limit $F\to0$, $\mu_{x,0}(L)\sim(Fa/V_0)\exp[-\sqrt{\kappa/\epsilon}(L-2\sqrt{2\epsilon V_0}/F)]$, $\det(\hat{h}_0)/|\det(\hat{h}_x(L))|\sim\exp(\sqrt{\kappa/\epsilon}L)$, and $\det(\hat{h}_0)/\det(\hat{h}_y)\sim(Fa/V_0)\exp(2\sqrt{2\kappa V_0}/F)$, where the coefficients of proportionality depend on the detailed form of the pinning potential and a is the characteristic radius of the pinning well. The eigenvalue $\mu_{xy,-1}=-\xi^2F^4/\kappa^2V_0^2$, see Eq. (18), and we make use of Eq. (27). Substituting these dependences into Eq. (44) we obtain

$$\Gamma=g\frac{V_0L}{\alpha a^3}\sqrt{\frac{V_0}{\epsilon}}\left(\frac{F}{F_c}\right)^{5/2}\left(\frac{\tilde{T}}{T}\right)^{1/2}\exp\left(-\frac{U}{T}+\frac{2\sqrt{2\kappa V_0}}{F}\right), \tag{45}$$

where g is a dimensionless proportionality coefficient which depends on the detailed form of the pinning potential, $\widetilde{T} = a\sqrt{\epsilon V_0}$, F_c is the critical force, and the activation energy U is given by Eq. (27). As an illustration we have explicitly solved the differential equations for a pinning potential given by the equation

$$V_{\rm cyl} = V_0\left[1 - \cos^2\left(\frac{\pi}{2a}\sqrt{u_x^2+u_y^2}\right)\theta(a^2-u_x^2-u_y^2)\right], \quad (46)$$

where $\theta(t)$ is the step-function [$\theta(t)=1$ if $t>0$ and $\theta(t)=0$ otherwise]. The external force has the form $F(u_x) = F\theta(|u_x|-a)$.

We find $\mu_{x,0}(L) = (24Fa/\pi V_0)\exp[-\sqrt{\kappa/\epsilon}(L-2\sqrt{2\epsilon V_0}/F)]$, $\det(\hat{h}_0)/|\det(\hat{h}_x)| = (1/2)\exp(\sqrt{\kappa/\epsilon}L)$, and $\det(\hat{h}_0)/\det(\hat{h}_y) = 4Fa/\pi V_0 \exp(2\sqrt{2\kappa V_0}/F)$, and substituting these expressions into Eq. (44) we obtain $g=\pi\xi/2^{5/4} \simeq 1.32\xi$.

A real physical system will involve a finite cutoff k^*, which we can account for in Eq. (31) by a restricted integration over the modes $k<k^*$. However, in making use of the Gelfand-Yaglom formula we actually account for unphysical fluctuations with $k>k^*$. In order for the classical result (45) to be valid we then must require the cutoff k^* to be sufficiently large such as to validate the Gelfand-Yaglom approach. On the other hand, k^* has to be small enough in order to arrive at a small quantum correction in Eq. (32).

IV. CONCLUSION

Let us first specify the regime of applicability of our results. The zero-temperature Euclidean action takes the value $S_{\rm Eucl}(T=0) = \alpha\Omega$, where Ω is the volume encircled by the string in the course of the tunneling motion. In the x and y directions the characteristic size of the loop is V_0/F and the length of the string segment along the z direction is of order $\sqrt{\epsilon V_0}/F$, hence $S_{\rm Eucl}(T=0) \sim \alpha\sqrt{\epsilon/V_0}(V_0/F)^3$. On the other hand, $S_{\rm Eucl}(T=T_0) = \hbar U/T_0 \sim S_{\rm Eucl}(T=0)$, i.e., even if the transition from quantum to classical behavior at $T_c > T_0$ is first-order like, its temperature will still be of order $T_c \sim T_0$ and our results are applicable in the regime $T>T_0$.

Next, we estimate the temperature T_0 using parameters for the moderately anisotropic superonductor $YBa_2Cu_3O_{7-\delta}$. We choose our columnar defect in the form of a cylindrical cavity of radius $a \simeq \xi_{ab}(0)$, where ξ_{ab} is the coherence length in the ab plane. The expression for the pinning potential then can be written in the form[22]

$$U(r) = \nu\frac{\epsilon_0}{4}\frac{r^2}{r^2+2\xi_{ab}^2}, \quad (47)$$

where $\epsilon_0 = (\Phi_0/4\pi\lambda_{ab})^2$ is the relevant energy scale and the factor ν $(0<\nu<1)$ describes the pinning efficiency factor,[23] Φ_0 and λ_{ab} are the flux quantum and the penetration length in the ab plane, respectively. For the model potential (47) the temperature T_0, the critical force F_c, and the temperature $\hbar\kappa/\alpha$ can be easily calculated (the Hall coefficient α is related to the electron density n as $\alpha = \pi\hbar n$ at $T=0$ in the superclean limit $\omega_0\tau \gg 1$): $T_0 = (\nu\xi/64\pi^2)(\epsilon_0/n\xi_{ab}^2)(F/F_c)^2$, $F_c = (\sqrt{2}\nu/16)\epsilon_0/\xi_{ab}$, $\hbar\kappa/\alpha = \nu\hbar\epsilon_0/4\alpha\xi_{ab}^2$. Substituting $\xi_{ab}=16$ Å, $\lambda_{ab}=1400$ Å, $\lambda_c/\lambda_{ab}=5$, $\alpha=\pi\hbar n$, $n=2\times10^{21}$ cm^{-3}, $\epsilon=\epsilon_0\lambda_{ab}^2/\lambda_c^2$, and $\xi\simeq2$, we obtain $T_0 = 0.6$ K$\cdot\nu(F/F_c)^2$, and $\hbar\kappa/\alpha = 15$ K$\cdot\nu$. Taking into account that ν might be ~0.1, see Ref. 23, we see that at $T\sim10-15$ K when the Hall force appears to be large,[5] the conditions $T \gg T_0, \hbar\kappa/\alpha$ are well satisfied.

The ratio $\alpha/\eta \simeq 15$ for $T \lesssim 15$ K (see Ref. 5) was obtained from indirect measurements. In Ref. 24 direct measurements of the Hall angle in 60 K $YBa_2Cu_3O_{7-\delta}$ are reported to yield a ratio $\alpha/\eta \simeq 1$. We wish to point out that even if $\alpha/\eta \lesssim 1$ it is still possible to use Eq. (45): We have shown that the inverse lifetime of a vortex pinned by a columnar defect behaves at high temperatures as

$$\Gamma \sim F^{5/2}T^{-1/2}\exp\left(-\frac{U}{T} + \frac{2\sqrt{2\kappa V_0}}{F}\right). \quad (48)$$

It is interesting to note that the same dependence of the decay rate on the external force and temperature has been obtained in Refs. 11 for a flux line governed by dissipative dynamics. Moreover, the equation for the decay rate from Refs. 11 can be obtained up to a numerical prefactor from Eq. (45) by the substitution $\alpha\to\eta$, indicating that in the regimes $\alpha \lesssim \eta$ or $\alpha \ll \eta$ it is possible to use Eq. (45) with $\alpha \to \max(\alpha,\eta)$.

The scaling law (48) for the decay rate leads then to the resistivity scaling

$$\rho(F) \sim \left[\sqrt{\frac{F}{T}}F\exp(2\sqrt{2\kappa V_0}/F)\right]\exp\left(-\frac{U}{T}\right), \quad (49)$$

where the factor in square brackets represents the correction to the standard result, which arises from the inclusion of classical fluctuations around the saddle point (prefactor).

We note that Eq. (48) is not valid for $T > FU/2\sqrt{2\kappa V_0}$ as the quasiclassical approximation is not applicable at these large temperatures. It is well known that the problem of a vortex pinned by a cylindrical potential is equivalent to that of a 2D quantum particle trapped in a radially symmetric 2D potential. The latter always has a bound state in 2D, and thus the vortex is pinned by the defect at any temperature. However, the thermal fluctuations lead to a large downward renormalization of the pinning energy at sufficiently high temperatures.[22,23]

Finally, let us make an estimate of the cutoff momentum k^*. The theory developed above works well if $k^* \gg \sqrt{\kappa/\epsilon}$; with $k^* = \pi/d$, where $d=12$ Å is the distance between superconducting layers, we obtain $k^*/\sqrt{\kappa/\epsilon} \simeq 5$, such that this condition is marginally satisfied.

Briefly summarizing, we have considered the problem of the thermally activated depinning of a flux line governed by Hall dynamics from a columnar defect in the presence of a small $(F \ll F_c)$ external force. We have shown how to reduce the 2D problem of the string motion to a 1D effective problem. The expression for the decay rate has been obtained for the whole temperature region where the saddle-point solution is time independent [see Eqs. (31), (44), and (45) for high temperatures]. An analytical expression for the classical asymptotics of Γ has been calculated for a model potential as given by Eq. (46), and we have discussed possible applications of these results to high-T_c superconductors.

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APPENDIX A: EXISTENCE AND UNIQUENESS

We prove that a solution of Eq. (17) with a negative eigenvalue exists. Second, we show the uniqueness of this solution. We mention that in the limit $L\to\infty$ the boundary condition $\psi\to 0$ for $|z|\to\infty$ is asymptotically satisfied.

Existence. As we already know, the operator $\hat{H}_x$ has one negative eigenvalue. We show that the operator $\hat{H}_y\hat{H}_x$ has a negative eigenvalue, too. It is possible to rewrite the equation $\hat{H}_y\hat{H}_x\psi=\tilde{\lambda}\psi$ in the form

$$\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}(\hat{H}_y^{-1/2}\psi)=\tilde{\lambda}(\hat{H}_y^{-1/2}\psi) \tag{A1}$$

(the operator $\hat{H}_y$ has only positive eigenvalues, hence, operators like $\hat{H}_y^{+1/2}$ or $\hat{H}_y^{-1/2}$ are well defined and Hermitian). The operator $\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}$ is also Hermitian and its lowest eigenvalue can be obtained by minimization of the functional

$$F[\chi]=\frac{\langle\chi|\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}|\chi\rangle}{\langle\chi|\chi\rangle} \tag{A2}$$

in the space of functions with integrable modulus squared. Using the fact that $H_y^{1/2}$ is self-conjugate and defining $\phi=\hat{H}_y^{1/2}\chi$, the functional can be rewritten as

$$F[\phi]=\frac{\langle\phi|\hat{H}_x|\phi\rangle}{\langle\hat{H}_y^{-1/2}\phi|\hat{H}_y^{-1/2}\phi\rangle}=\frac{\langle\phi|\hat{H}_x|\phi\rangle}{\langle\phi|\hat{H}_y^{-1}|\phi\rangle}. \tag{A3}$$

The operator $\hat{H}_x$ has a negative eigenvalue, i.e., there is a function $|\phi\rangle$ such that $\langle\phi|\hat{H}_x|\phi\rangle<0$. The operator $\hat{H}_y$ has only positive eigenvalues, i.e., the form $\langle\phi|\hat{H}_y^{-1}|\phi\rangle$ is always positive. Consequently we constructed the Rayleigh-Ritz principle for the equation $\hat{H}_y\hat{H}_x\psi=\tilde{\lambda}\psi$ and proved that there is a function with a negative average value. The eigenvalue $\tilde{\lambda}_{-1}$ is even lower, thus we have demonstrated the existence of a negative eigenvalue for the problem (17).

Uniqueness. Let us rewrite the operator $\hat{H}_x$ in the following form:

$$\hat{H}_x=\sum_\alpha \lambda_\alpha|\alpha\rangle\langle\alpha|=\sum_{\alpha\neq-1}\lambda_\alpha|\alpha\rangle\langle\alpha|+\lambda_{-1}|-1\rangle\langle-1|, \tag{A4}$$

where λ_α are the eigenvalues of $\hat{H}_x$, λ_{-1} is the lowest eigenvalue of the operator $\hat{H}_x$, the vector $|-1\rangle$ denoting the ''ground state''of the operator $\hat{H}_x$. The operator $\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}$ can be rewritten in the form

$$\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}=\hat{H}_y^{1/2}\sum_{\alpha\neq-1}\lambda_\alpha|\alpha\rangle\langle\alpha|\hat{H}_y^{1/2} +\hat{H}_y^{1/2}\lambda_{-1}|-1\rangle\langle-1|\hat{H}_y^{1/2}\equiv\hat{A}+\hat{B}. \tag{A5}$$

The image of the operator $\hat{B}=\hat{H}_y^{1/2}\lambda_{-1}|-1\rangle\langle-1|\hat{H}_y^{1/2}$ is one-dimensional, i.e., $\hat{B}$ is a self-adjoint projector of rank 1. The operator

$$\hat{A}=\hat{H}_y^{1/2}\left(\sum_{\alpha\neq-1}\lambda_\alpha|\alpha\rangle\langle\alpha|+0\cdot|-1\rangle\langle-1|\right)\hat{H}_y^{1/2} \tag{A6}$$

is a non-negative Hermitian operator with two zero eigenvalues. We perturb the operator $\hat{A}$ with a projector of rank 1 and make use of the following theorem.[25]

Theorem. *Let $\hat{A}$ be a self-adjoint operator and $\hat{B}$ a self-adjoint operator of rank 1. Take an interval D on the real axis and denote by $n(D)$ the number of eigenvalues of $\hat{A}$ in this interval. Then the number $m(D)$ of eigenvalues of $\hat{A}+\hat{B}$ within D satisfies the estimate:*

$$n(D)-1\leqslant m(D)\leqslant n(D)+1. \tag{A7}$$

Applying this theorem to the interval $(-\infty,0)$ (we exclude zero) one can easily see that the operator $\hat{H}_y^{1/2}\hat{H}_x\hat{H}_y^{1/2}$ has not more than one negative eigenvalue. But the eigenvalues of this operator are the same as those of the operator $\hat{H}_y\hat{H}_x$ so that our problem has only one negative eigenvalue.

APPENDIX B: NEGATIVE EIGENVALUE PROBLEM

We construct upper and lower bounds for the negative eigenvalue $\tilde{\lambda}_{-1}$ of the operator $\hat{H}_x\hat{H}_y$. In the limit $F\to 0$ the lowest eigenvalues of the operators $\hat{H}_x$ and $\hat{H}_y$ do not depend on the detailed form of the pinning potential.[11] This is due to the rapid decay of the eigenfunctions in the region $|z|>D$, where the form of the potential is irrelevant. The lowest eigenvalue of the operator $\hat{H}_x$ is equal to $\lambda_{-1}=-\mu^2F^2/2V_0$, with μ the root of $\mu\tanh\mu=1$ (see Refs. 11). In the region $|z|<D$ the normalized ''ground-state'' wave function of the operator $\hat{H}_x$ is

$$\phi_{-1}(z)=\frac{\cosh(\sqrt{|\lambda_{-1}|/\epsilon}z)}{\sqrt{D}\cosh(\sqrt{|\lambda_{-1}|/\epsilon}D)}. \tag{B1}$$

As the characteristic depth κ of the ''potential well'' $U_y(z)$ is much larger than the ''size quantization energy'' ϵ/D^2 (see Fig. 2), we can approximate the eigenvalues and eigenfunctions of the operator $\hat{H}_y$ by those of the infinitely deep quantum well

$$\lambda_n'=\frac{\pi^2F^2}{8V_0}n^2,\quad \phi_n'(z)=\frac{1}{\sqrt{D}}\cos\left(\frac{\pi n}{2D}z+\frac{\pi}{2}(n-1)\right), \quad n=1,2\ldots. \tag{B2}$$

Using the variational principle (A3) we can write

$$\tilde{\lambda}_{-1}\leqslant\frac{\langle\phi_{-1}|\hat{H}_x|\phi_{-1}\rangle}{\langle\phi_{-1}|\hat{H}_y^{-1}|\phi_{-1}\rangle}=-\frac{\mu^2F^2/2V_0}{\sum_m(|\langle\phi_{-1}|\phi_m'\rangle|^2/\lambda_m')}. \tag{B3}$$

Substituting the eigenvalues and eigenfunctions (B2) into Eq. (B3) we obtain

$$\tilde{\lambda}_{-1}\leqslant -\frac{\mu^2F^4/2V_0^2}{\sum_{m=0}^{\infty}\{8/[\mu^2+\pi^2(2m+1)^2/4]\}^2}=-1.3292\frac{F^4}{V_0^2}. \tag{B4}$$

In order to obtain a lower estimate $\tilde{\lambda}_{-1}\geqslant B>-\infty$ [see Eq. (A3)], we use the following inequality:

$$\tilde{\lambda}_{-1}=\min_{\phi\in\mathcal{L}^2}F[\phi]\geqslant\min_{\Psi\in\mathcal{D}}\frac{\lambda_{-1}}{\langle\phi|\hat{H}_y^{-1}|\phi\rangle}, \tag{B5}$$

where the functions belonging to $\mathcal{D}$ satisfy the conditions (i) $\langle\phi|\phi\rangle=1$, (ii) $\langle\phi|\hat{H}_x|\phi\rangle<0$, and (iii) $\phi(z)=\phi(-z)$. The last condition is a consequence of the parity symmetry of the operator $\hat{H}_x\hat{H}_y$. Following the discussion in Sec. III A, the operator $\hat{H}_x$ has one negative eigenvalue with an even eigenfunction. For any odd function ϕ, $\langle\phi|\hat{H}_x|\phi\rangle\geqslant 0$, thus an odd function cannot minimize the functional (A3), hence the eigenfunction corresponding to the lowest eigenvalue of the operator $\hat{H}_x\hat{H}_y$ has to be even. Our goal is to find some (large) constant b such that $\langle\Psi|\hat{H}_y^{-1}|\Psi\rangle>b>0$ for any $\Psi\in\mathcal{D}$. We rewrite

$$\langle\phi|\hat{H}_y^{-1}|\phi\rangle=\sum_m\frac{|\langle\phi|\phi_m'\rangle|^2}{\lambda_m'}\geqslant\frac{|\langle\phi|\phi_1'\rangle|^2}{\lambda_1'} \tag{B6}$$

and expanding ϕ in the eigenfunctions of the operator $\hat{H}_x$, $\phi=\Sigma_{n=-1}c_n\phi_n\equiv c_{-1}\phi_{-1}+\tilde{\phi}$, we can write

$$\begin{aligned}|\langle\phi|\phi_1'\rangle|^2 &=c_{-1}^2|\langle\phi_{-1}'|\phi_1'\rangle|^2+|\langle\tilde{\phi}|\phi_1'\rangle|^2+2c_{-1}\langle\phi_{-1}|\phi_1'\rangle\langle\tilde{\phi}|\phi_1'\rangle\\ &\geqslant c_{-1}^2|\langle\phi_{-1}|\phi_1'\rangle|^2+|\langle\tilde{\phi}|\phi_1'\rangle|^2-2|c_{-1}||\langle\phi_{-1}|\phi_1'\rangle||\langle\tilde{\phi}|\phi_1'\rangle|.\end{aligned} \tag{B7}$$

As $\langle\phi|\hat{H}_x|\phi\rangle<0$, $-c_{-1}^2|\lambda_{-1}|+c_1^2\lambda_1+c_2^2\lambda_2+\cdots<0$, and taking into account that $\Sigma_n c_n^2=1$ we obtain

$$1>|c_{-1}|>\sqrt{\frac{\lambda_1}{|\lambda_{-1}|+\lambda_1}}=0.9191, \tag{B8}$$

where we used the fact that $\lambda_{-1}=-1.4392\epsilon/D^2$ and $\lambda_1=7.8309\epsilon/D^2$ (see Refs. 10,11). As $|\langle\tilde{\phi}|\phi_1'\rangle|\leqslant||\tilde{\phi}||||\phi_1'||\leqslant\sqrt{1-c_{-1}^2}=0.3941$, we obtain $|\langle\phi|\phi_1'\rangle|^2\geqslant f(|c_{-1}|,K)$, where

$$f(|c_{-1}|,K)=c_{-1}^2|\langle\phi_{-1}|\phi_1'\rangle|^2+K^2-2|c_{-1}||\langle\phi_{-1}|\phi_1'\rangle|K. \tag{B9}$$

We note that $|c_{-1}|\in[0.9191,1]$ and $K\in[0,0.3941]$. A simple analysis shows that the minimal value of $f(|c_{-1}|,K)$ is equal to $f(0.9191,0.3941)=0.119$ and substituting this value into Eqs. (B5) and (B6) we obtain

$$\tilde{\lambda}_{-1}\geqslant-7.4603\frac{F^4}{V_0^2}. \tag{B10}$$

Hence, we have shown that in the limit $F\to 0$ the lowest eigenvalue $\tilde{\lambda}_{-1}$ of the operator $\hat{H}_x\hat{H}_y$, $\tilde{\lambda}_{-1}$ behaves as $-\gamma F^4/V_0^2$ with $\gamma\in[1.3292,7.4603]$. The temperature T_0 and the eigenvalue $\tilde{\lambda}_{-1}$ are connected by the equation $T_0=\hbar\sqrt{|\tilde{\lambda}_{-1}|}/2\pi\alpha$ and we obtain the estimate

$$T_0=\frac{\xi}{2\pi}\frac{\hbar F^2}{\alpha V_0}, \tag{B11}$$

with $\xi\in[1.1530,2.7314]$.

[1] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[2] L. Civale, A. D. Marwick, T. K. Worthington, M. A. Kirk, J. R. Thompson, L. Krusin-Elbaum, Y. Sun, J. R. Clem, and F. Holtzberg, Phys. Rev. Lett. **67**, 648 (1991); V. Hardy, D. Groult, J. Provost, M. Hervieu, and R. Raveau, Physica C **178**, 255 (1991); W. Gerhäuser, G. Ries, H. W. Neumüller, W. Schmidt, O. Eibl, G. Saemann-Ischenko, and S. Klaumünzer, Phys. Rev. Lett. **68**, 879 (1992).
[3] P. Hängii, P. Talkner, and B. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[4] N. B. Kopnin and V. E. Kravtsov, Pis'ma Zh. Éksp. Teor. Fiz. **23**, 631 (1976) [JETP Lett. **23**, 578 (1976)]; N. B. Kopnin and V. E. Kravtsov, Zh. Éksp. Teor. Fiz. **71**, 1644 (1976) [Sov. Phys. JETP **44**, 861 (1976)].
[5] Y. Matsuda, N. P. Ong, Y. F. Yan, J. M. Harris, and J. B. Peterson, Phys. Rev. B **49**, 4380 (1994).
[6] E. B. Sonin and B. Horovitz, Phys. Rev. B **51**, 6526 (1995).
[7] L. N. Bulaevskii, A. I. Larkin, M. P. Maley, and V. M. Vinokur, Phys. Rev. B **52**, 9205 (1995). The approximations made in this paper lead to an inexact result; see the discussion in D. A. Gorokhov and G. Blatter, following paper, Phys. Rev. B **57**, 3586 (1998).
[8] E. M. Chudnovsky, A. Ferrera, and A. Vilenkin, Phys. Rev. B **51**, 1181 (1995).
[9] C. Morais-Smith, A. O. Caldeira, and G. Blatter, Phys. Rev. B **54**, 784 (1996).
[10] M. A. Skvortsov, Phys. Rev. B **55**, 515 (1997).
[11] A. Krämer and M. L. Kulić, Phys. Rev. B **48**, 9673 (1993); **50**, 9484 (1994).

[12] G. E. Volovik, Pis'ma Zh. Éksp. Teor. Fiz. **15**, 116 (1972) [JETP Lett. **15**, 81 (1972)].
[13] M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin, and S. Levit, Pis'ma Zh. Éksp. Teor. Fiz. **57**, 699 (1993) [JETP Lett. **57**, 711 (1993)].
[14] J. S. Langer, Ann. Phys. (N.Y.) **54**, 258 (1969).
[15] I. M. Lifshitz and Yu. Kagan, Zh. Éksp. Teor. Fiz. **62**, 1 (1972) [Sov. Phys. JETP **35**, 206 (1972)]; S. V. Meshkov, *ibid.* **89**, 1734 (1985) [*ibid.* **62**, 1000 (1985)]; E. Chudnovsky, Phys. Rev. A **46**, 8011 (1992); D. A. Gorokhov and G. Blatter, Phys. Rev. B **56**, 3130 (1997).
[16] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1990).
[17] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, New York, Oxford, 1982).
[18] B. I. Ivlev and V. I. Mel'nikov, Phys. Rev. B **36**, 6889 (1987).
[19] A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Usp. Fiz. Nauk **136**, 553 (1982) [Sov. Phys. Usp. **25**, 195 (1982)].
[20] I. Affleck, Phys. Rev. Lett. **46**, 388 (1981).
[21] I. M. Gelfand and A. M. Yaglom, J. Math. Phys. **1**, 48 (1960); H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (World Scientific, Singapore, 1990).
[22] D. R. Nelson and V. M. Vinokur, Phys. Rev. B **48**, 13 060 (1993).
[23] L. Krusin-Elbaum, L. Civale, J. R. Thompson, and C. Feild, Phys. Rev. B **53**, 11 744 (1996).
[24] J. M. Harris, Y. F. Yan, O. K. C. Tsui, Y. Matsuda, and N. P. Ong, Phys. Rev. Lett. **73**, 1711 (1994).
[25] M. S. Birman and M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space* (Reidel, Dordrecht, 1987).