Single-particle Green functions in exactly solvable models of Bose and Fermi liquids

Girish S. Setlur* and Yia-Chung Chang
Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

(Received 12 January 1998)

Based on a class of exactly solvable models of interacting Bose and Fermi liquids, we compute the single-particle propagators of these systems exactly for all wavelengths and energies and in any number of spatial dimensions. The field operators are expressed in terms of Bose fields that correspond to displacements of the condensate in the Bose case and displacements of the Fermi sea in the Fermi case. Unlike some of the previous attempts, the present attempt reduces the answer for the spectral function in any dimension in both Fermi and Bose systems to quadratures. It is shown that when only the lowest-order sea-displacement terms are included, is the random-phase approximation in its many guises recovered in the Fermi case, and Bogoliubov's theory in the Bose case. The momentum distribution is evaluated using two different approaches, exact diagonalization and the equation of motion approach; the novelty being, of course, the exact computation of single-particle properties including short-wavelength behavior. [S0163-1829(98)00624-9]

I. INTRODUCTION

Recent years have seen remarkable developments in many-body theory in the form of an assortment of techniques that may be loosely termed bosonization. The beginnings of these types of techniques may be traced back to the work of Tomonaga[1] and later on by Luttinger[2] and by Lieb and Mattis.[3] The work of Sawada[4] and Arponen and Pajanne[5] in recasting the Fermi-gas problem in a Bose language has to be mentioned. Arponen and Pajanne recover corrections to the random-phase approximation (RPA) of Bohm and Pines [6] in a systematic manner. In nuclear physics, bosonization is widely used to study collective properties, for an introduction see the book by Iachello and Arima.[7] In the 1970s an attempt was made by Luther[8] at generalizing these ideas to higher dimensions. Closely related to this is the work by Sharp *et al.*[9] in current algebra. More progress was made by Haldane[10] which culminated in the explicit computation of the single-particle propagator by Castro-Neto and Fradkin[11] and by Houghton *et al.*[12] and also by Kopietz *et al.*[13] Rigorous work by Frohlich *et al.*[14] is also along similar lines. Also the work of Frau *et al.*[15] on algebraic bosonization is relevant to the present article as the authors have considered effects beyond the linear dispersion in that article. The exactly solvable models of Calogero and Sutherland are of relevance here as well, the exact propagators of these models have been computed by various authors.[16] Recently, these types of models have been generalized to more than one dimension by Ghosh.[17]

The attempt made here is to generalize the concepts of Haldane[10] to accomodate short-wavelength fluctuations where the concept of a linearized bare fermion energy dispersion is no longer valid. To motivate progress in this direction, we find that it is necessary to introduce two different concepts, one is the canonical conjugate of the Fermi/Bose density distribution, the other is the concept of sea/condensate displacements.

Histrorically speaking, the idea that the velocity operator could serve as the canonical conjugate of the density has been around for a long time, and this has been exploited in the study of He-II by Sunakawa *et al.*[18] However, the authors are not aware of a rigorous study of the meaning of this object, in particular, an explicit formula for the canonical conjugate of the density operator has to the best of the authors' knowledge never been written down in terms of the field operators. The work by Sharp *et al.*[9] comes close to what we are attempting here.

The concept of a sea displacement is a generalization of the traditional approach used for bosonizing one-dimensional (1D) systems such as the Tomonaga-Luttinger[1,2] models. There, one introduces Bose fields that correspond physically to displacement of the Fermi surface (in 1D, Fermi points). These Bose fields have simple forms relating them to number-conserving products of Fermi fields. The field operator is obtained by exponentiating the commutation rule between the surface-displacement operator and the field operator. By analogy, we generalize these ideas, so that one is no longer restricted to be close to the Fermi surface. The way this is done is to postulate the existence of Bose fields that correspond to displacements of the Fermi sea rather than just the Fermi surface. From this it is possible to write down formulas for the number-conserving product of two Fermi fields in terms of the Bose fields. A similar construction is possible when the parent fields are bosons, but here, we find that instead of sea displacements, we have to introduce operators that correspond physically to displacements of the condensate. Actually, the Bose case is much simpler and a mathematically rigorous formulation of this correspondence is possible. This is a boon, since we use this fact and make plausible the analogous correspondence in the Fermi case. The assertions in the Fermi case are not proved "rigorously," rather are made exceedingly plausible by analogy. This is the main drawback of this article.

This article is organized as follows. In the next section, we present some formulas that relate the number conserving product of two Fermi/Bose fields to the relevant sea/condensate-displacement operators that are postulated to be canonical bosons. The sea/condensate-displacement operators in turn may be related to the parent Fermi/Bose fields, as it happens, this formula is simple in the case when the parent fields are bosons but is difficult in the case when the parent fields are fermions.

Following this, we write down a generic formula for the Fermi/Bose field operator in terms of the density operator (operator-valued distribution, to be precise) and its canonical conjugate. The new ingredient in this section is the canonical conjugate of the density operator. This quantity may in turn be related to currents and densities. We find that these formulas are ambiguous unless a proper choice is made for a certain phase functional. For bosons, we find that this choice is the zero functional but for fermions it has to be determined by making contact with the free theory (done in Sec. IV).

Combining the two previous sections, we write down in Sec. IV, formulas for currents and densities in terms of the sea/condensate displacements, the field operator has a formula in terms of the sea/condensate-displacements as well. Contact is made with the propagator of the free theory and the undetermined phase functional of the previous sections is determined for the Fermi case. In Sec. V, interaction terms are introduced that correspond to two-body repulsive interactions. It is argued and demonstrated that selectively retaining parts of the interaction that are quadratic in the sea/condensate displacements amounts to using Bogoliubov/RPA theory. Corrections to this quadratic Hamiltonian are easy to write down but are not used to compute corrections to RPA/Bogoliubov theory as this requires significantly more effort. It is found that the diagonalization of the RPA Hamiltonian is rather tricky if one wants to recover both the particle-hole modes and the collective mode. In the end, closed formulas are written down for the Fermi propagator in all three spatial dimensions and their various qualitative features are examined. This completes the solution of the many-body problem in the RPA/Bogoliubov limit.

The Appendixes are as follows: Appendix A contains a detailed proof of the correspondence between the number-conserving product of two Bose fields and the corresponding condensate displacements. Appendix B involves writing down similar ideas for Fermi systems. However here, the various assertions are only made plausible unlike in the Bose case where a rigorous solution is possible. Appendix C is devoted to proving the assertion that retaining only terms linear in the sea displacements in the definition of the density recovers the RPA. Appendix D involves a derivation of the formula for the momentum distribution of the 1D system using the equation of motion approach. Appendix E contains some technical statements regarding the proof of uniqueness of the formula relating the Fermi field with the corresponding currents and densities.

II. EXPRESSING PRODUCTS OF PARENT FIELDS IN TERMS OF SEA DISPLACEMENTS

In this section we introduce canonical Bose fields called sea displacements in the Fermi case and condensate displacements in the Bose case. First, we write down a formula for the number-conserving product of two Bose fields in terms of the condensate-displacement operators. A rigourous proof of this is relegated to Appendix A. The correspondence is made plausible by making several observations about these formulas. Let us first focus on the Bose case. Let $b_{\mathbf{q}}$ and $b^{\dagger}_{\mathbf{q}}$ be canonical Bose operators. From these, we may construct other Bose operators defined as follows($\mathbf{q}\neq\mathbf{0}$):

$$d_{\mathbf{q}/2}(\mathbf{q})=\left(\frac{1}{\sqrt{N_0}}\right)b^{\dagger}_{\mathbf{0}}b_{\mathbf{q}} \tag{1}$$

and

$$d_{\mathbf{0}}(\mathbf{0})=0, \tag{2}$$

where $N_0=b^{\dagger}_{\mathbf{0}}b_{\mathbf{0}}$. This is the condensate-displacement annhilation operator. It is so named for the following reason. The definition suggests that this operator removes a boson from among those that are not in the condensate and returns it to the condensate, thereby displacing the latter. The reason for the redundant momentum label in the notation $d_{\mathbf{q}/2}(\mathbf{q})$ becomes clear if one realizes that a more general object would be a sea-displacement annhilation operator $d_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q})$. Since the condensate corresponds to $\mathbf{k}=0$, we have just the condensate-displacement annhilation operator. In fact, it will be shown subsequently that for the Fermi case we have to deal with this more general object namely, the sea-displacement annhilation operator. It may be shown that (see Appendix A) this object $d_{\mathbf{q}/2}(\mathbf{q})$ satisfies canonical Bose commutation rules. Also a formula is possible for the number-conserving product of two parent bosons in terms of these condensate displacements. The formula is written down below and proved in Appendix A:

$$b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}b_{\mathbf{k}-\mathbf{q}/2}=N_0\,\delta_{\mathbf{k},0}\,\delta_{\mathbf{q},0}+[\,\delta_{\mathbf{k}+\mathbf{q}/2,0}(\sqrt{N_0})\,d_{\mathbf{k}}(-\mathbf{q})+\delta_{\mathbf{k}-\mathbf{q}/2,0}\,d^{\dagger}_{\mathbf{k}}(\mathbf{q})(\sqrt{N_0})]+d^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)}(\mathbf{k}+\mathbf{q}/2)\,d_{(1/2)(\mathbf{k}-\mathbf{q}/2)}(\mathbf{k}-\mathbf{q}/2), \tag{3}$$

where

$$N_0=N-\sum_{\mathbf{q}_1} d^{\dagger}_{\mathbf{q}_1/2}(\mathbf{q}_1)\,d_{\mathbf{q}_1/2}(\mathbf{q}_1) \tag{4}$$

and

$$[\,d_{\mathbf{q}/2}(\mathbf{q}),N]=0, \tag{5}$$

$$N=\sum_{\mathbf{q}} b^{\dagger}_{\mathbf{q}}b_{\mathbf{q}}, \tag{6}$$

also the object $d_{\mathbf{0}}(\mathbf{0})=0$, by definition.

The way the authors initially derived this formula is as follows. One starts off with the observation that the object $b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}b_{\mathbf{k}-\mathbf{q}/2}$ is the only one that enters in the Hamiltonian of number-conserving systems. Furthermore, it satisfies closed commutation rules amongst other members of its kind. One is therefore led to look for formulas for these objects in terms of other bosons with a view to make the full Hamiltonian more easily diagonalizable. In particular, if there were Bose operators $d_{\mathbf{q}/2}(\mathbf{q})$ and $d^{\dagger}_{\mathbf{q}/2}(\mathbf{q})$ such that $b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}b_{\mathbf{k}-\mathbf{q}/2}$ was exactly linear in these bosons, then the full Hamiltonian would indeed be exactly diagonalizable. We find that this is not the case and there are corrections to this linear term and it so happens that introduction of a quadratic term in the condensate displacements in fact makes the correspondence exact. The authors are not aware of a deeper reason behind this simple formula that terminates after including the quadratic term, after all, the formula for the parent annhilation operator $b_{\mathbf{q}}$ in terms of the condensate displacements is formidable as we shall soon see. The Bose case being so simple and exact can be used as a benchmark to write down corresponding formulas in the Fermi case, where rigorous proofs are much harder to come by. The authors also have in mind generalizations to relativistic systems, where one might profit by following the above prescription. In particular, it would be fascinating to see if the ideas above were useful in getting nonperturbative information regarding gauge theories like QED, QCD, etc. But this is far into the future. For now, let us try to write down a similar correspondence for the nonrelativistic Fermi system.

As mentioned earlier, for Fermi systems, it is necessary to postulate the existence of a sea-displacement annhilation operator, denoted by $a_{\mathbf{k}}(\mathbf{q})$. A formula for this in terms of the Fermi fields is extremely difficult to deduce. In Appendix B, attempts are made to do exactly this. There it is pointed out that these objects satisfy canonical boson commutation rules. The important issues that enable one to draw practical conclusions, fortunately do not depend very much on the technical details. In Appendix B and in the sections that follow, we show how to extract the necessary physics while cirumventing the technical details. It must be pointed out however that this drawback is regrettable. Let us merely quote the final answers and later on make these formulas plausible. The RPA form of the number conserving product of two Fermi fields in terms of the sea bosons is given by($\mathbf{q}\neq 0$):

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=\left(\frac{N}{\langle N\rangle}\right)^{1/2}[\Lambda_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}}(-\mathbf{q})+a^{\dagger}_{\mathbf{k}}(\mathbf{q})\Lambda_{\mathbf{k}}(-\mathbf{q})]+T_1(\mathbf{k},\mathbf{q})\sum_{\mathbf{q}_1}a^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{k}-\mathbf{q}_1/2}(\mathbf{q}_1-\mathbf{q})$$
$$-T_2(\mathbf{k},\mathbf{q})\sum_{\mathbf{q}_1}a^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)a_{\mathbf{k}+\mathbf{q}_1/2}(\mathbf{q}_1-\mathbf{q}). \tag{7}$$

Here

$$T_1(\mathbf{k},\mathbf{q})=\sqrt{1-\bar{n}_{\mathbf{k}+\mathbf{q}/2}}\sqrt{1-\bar{n}_{\mathbf{k}-\mathbf{q}/2}}, \tag{8}$$

$$T_2(\mathbf{k},\mathbf{q})=\sqrt{\bar{n}_{\mathbf{k}+\mathbf{q}/2}\bar{n}_{\mathbf{k}-\mathbf{q}/2}}, \tag{9}$$

$$\Lambda_{\mathbf{k}}(\mathbf{q})=\sqrt{\bar{n}_{\mathbf{k}+\mathbf{q}/2}(1-\bar{n}_{\mathbf{k}-\mathbf{q}/2})}. \tag{10}$$

Also, the sea-boson commutes with the total number of fermions,

$$[a_{\mathbf{k}}(\mathbf{q}),N]=0 \tag{11}$$

and the operator $a_{\mathbf{k}}(\mathbf{0})=0$. Further,

$$n_{\mathbf{k}}=n^{\beta}(\mathbf{k})\frac{N}{\langle N\rangle}+\sum_{\mathbf{q}}a^{\dagger}_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q})a_{\mathbf{k}-\mathbf{q}/2}(\mathbf{q})$$
$$-\sum_{\mathbf{q}}a^{\dagger}_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q})a_{\mathbf{k}+\mathbf{q}/2}(\mathbf{q}) \tag{12}$$

and

$$n^{\beta}(\mathbf{k})=\frac{1}{\exp(\beta(\epsilon_{\mathbf{k}}-\mu))+1}. \tag{13}$$

Also $\bar{n}_{\mathbf{k}}=\langle n_{\mathbf{k}}\rangle$. The expectation value is with respect to the full interacting ground state. This quantity depends on the interactions that are present in the system and must be evaluated self-consistently. In fact, there is a deeper reason for introducing this. The exact formula for the number conserving product of two Fermi fields and the sea bosons may be expected to involve the number operator itself under the square-root sign. This is made exceedingly likely by analogy with the Bose case, where the square root of the number operator in the zero momentum state appears. In Appendix B the manner in which this exact correspondence may be deduced is hinted at. At this stage, it is pertinent to merely write down a formula for the sea-boson annhilation operator in the RPA limit. The sea boson is defined analogous to the condensate-displacement boson, except the Fermi case is more complicated due to the presence of the Fermi surface. The sea boson may be defined as follows (the rest of the details including a ''proof'' of this fact and how it fits into the Fermi-bilinear-sea-boson correspondence is relegated to Appendix B),

$$a_{\mathbf{k}}(\mathbf{q})=\frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}}c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}\left(\frac{n^{\beta}(\mathbf{k}-\mathbf{q}/2)}{\langle N\rangle}\right)^{1/2}e^{i\theta(\mathbf{k},\mathbf{q})}c_{\mathbf{k}+\mathbf{q}/2}. \tag{14}$$

Here $\theta(\mathbf{k},\mathbf{q})$ is a c number phase that serves to randomly cancel out troublesome terms: this is also related to the ''random phase'' of the random-phase approximation of Bohm and Pines. Thus the above formula for the sea boson is in the ''random-phase'' approximation.

This correspondence recovers the salient features of the finite and zero-temperature aspects of the free theory provided we make the following assumption, the sea bosons do participate in the thermodynamic averaging but come with an infinite negative chemical potential. This means that as far as the free theory is concerned, the average value of the sea-boson occupation is zero in the noninteracting case. The kinetic energy operator in the sea-boson language is given by

$$K=\sum_{\mathbf{k},\mathbf{q}}\frac{\mathbf{k}\cdot\mathbf{q}}{m}a_{\mathbf{k}}^{\dagger}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q})+N\epsilon_0, \tag{15}$$

where ϵ_0 is the kinetic energy per particle. Therefore,

$$\langle a_{\mathbf{k}}^{\dagger}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q})\rangle=\frac{1}{\exp(\beta(\mathbf{k}\cdot\mathbf{q}/m-\mu_B))-1}=0, \tag{16}$$

where $-\mu_B=\infty$. However, when there are interactions in the system, the answer is likely to be different. In particular, it is likely to be a nonanalytic function of the interaction in such a way that it vanishes as the coupling goes to zero (this is demonstrated explicitly in Appendix D). Roughly speaking we may write

$$\langle a_{\mathbf{k}}^{\dagger}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q})\rangle\approx\left(\frac{1}{V}\right)\exp(-1/v), \tag{17}$$

where v is the Coulomb repulsion parameter and V is the volume. All these do come out naturally from the correspondece written down above as we shall soon see. We have thus written down a useful correspondence between Fermi and Bose operators that recovers the salient features of the free theory at zero and finite temperature and it is clear that this correspondence is all that is needed to write down model Hamiltonians with any sort of interaction, such as Coulombic, with phonons, etc., and extract exact nonperturbative (more precisely, nonanalytic in the couplings) solutions. These solutions possess features that are impossible to capture via diagrammatic means let alone mean-field theory. Thus a strong case is to be made for this method as a new paradigm for condensed-matter physics.

III. FIELD OPERATOR IN TERMS OF DENSITY AND ITS CANONICAL CONJUGATE

In this section, we introduce the canonical conjugate of the Fermi/Bose density distribution. The reason for doing this is that we would like to express the field operator itself in terms of the density and its canonical conjugate and consequently in terms of current and densities. None of these ideas are really new. For example, Sunakawa *et al.*[18] use the velocity operator as a canonical conjugate of the density in their investigation of the properties of He-II. The velocity operator is somewhat related to the current operator but is not exactly equal to it. The reason is that the current operator behaves like the conjugate of the density as far as commutation rules with the latter is concerned, but does not commute with members of its own kind (it is difficult to say this in words but will soon become clear). Let us postulate the existence of the object $\Pi(\mathbf{x}\sigma)$ as the canonical conjugate of the density,

$$[\Pi(\mathbf{x}\sigma),\rho(\mathbf{y}\sigma')]=i\,\delta(\mathbf{x}-\mathbf{y})\,\delta_{\sigma,\sigma'}, \tag{18}$$

$$[\Pi(\mathbf{x}\sigma),\Pi(\mathbf{y}\sigma')]=0. \tag{19}$$

It is clear that redefinitions of this object by amounts that involve translations by (more or less arbitrary) functionals of the density are not going to spoil the nature of the commutation rules above. However, we shall take the point of view that Π is defined to be that (almost unique) object that satisfies the relation below (making mathematically rigorous sense out of all this requires the use of functional analysis and will be attempted in Appendix E).

$$\rho(\mathbf{x}\sigma)=-i\frac{\delta}{\delta\Pi(\mathbf{x}\sigma)}. \tag{20}$$

Observe that $\rho(\mathbf{x}\sigma)=\psi^{\dagger}(\mathbf{x}\sigma)\psi(\mathbf{x}\sigma)$ (technical problems involving the multiplication of operator-valued distributions at the same point are alleviated by assuming that we have the whole system in a box, with periodic boundary conditions on the fields, making any infinities only as large as the volume of the box itself, please refer to Appendix E for more details). Observe that (valid for both Bose as well as Fermi systems),

$$[\rho(\mathbf{x}\sigma),\psi(\mathbf{x}'\sigma')]=-\delta^d(\mathbf{x}-\mathbf{x}')\,\delta_{\sigma,\sigma'}\,\psi(\mathbf{x}\sigma). \tag{21}$$

Rewriting this as a differential equation,

$$\left[-i\frac{\delta}{\delta\Pi(\mathbf{x}\sigma)},\psi(\mathbf{x}'\sigma')\right]=-\delta^d(\mathbf{x}-\mathbf{x}')\,\delta_{\sigma,\sigma'}\,\psi(\mathbf{x}\sigma). \tag{22}$$

This may be solved (exponentiation of commutation rules is the more technical term) as

$$\psi(\mathbf{x}\sigma)=\exp(-i\Pi(\mathbf{x}\sigma))F([\rho];\mathbf{x}\sigma). \tag{23}$$

Observe now that $\rho=\psi^{\dagger}\psi$. Therefore,

$$F^{\dagger}([\rho];\mathbf{x}\sigma)F([\rho];\mathbf{x}\sigma)=\rho(\mathbf{x}\sigma). \tag{24}$$

This may in turn be solved and the final density phase variable ansatz (DPVA for short) may be written as

$$\psi(\mathbf{x}\sigma)=e^{-i\Pi(\mathbf{x}\sigma)}e^{i\Phi([\rho];\mathbf{x}\sigma)}(\rho(\mathbf{x}\sigma))^{1/2}. \tag{25}$$

It may be noted above that redefinitions of Π consistent with it being the canonical conjugate to ρ may be absorbed by a suitable redefinition of the phase functional Φ. Therefore, Eq. (25) is in fact quite general. The crucial point of this whole exercise is that the phase functional Φ determines the statistics of the field $\psi(\mathbf{x}\sigma)$. It may be shown (the proof is rather tedious and since this issue is not central to the practical computations, we defer the proof to a future communication) that imposition of Bose/Fermi commutation rules on ψ involves imposing the following restriction on the form of Φ:

$$\begin{aligned}&\Phi([\{\rho(\mathbf{y_1}\sigma_1)-\delta(\mathbf{y_1}-\mathbf{x}')\,\delta_{\sigma_1,\sigma'}\}];\mathbf{x}\sigma)\\&\quad+\Phi([\rho];\mathbf{x}'\sigma')-\Phi([\rho];\mathbf{x}\sigma)\\&\quad-\Phi([\{\rho(\mathbf{y_1}\sigma_1)-\delta(\mathbf{y_1}-\mathbf{x})\,\delta_{\sigma_1,\sigma}\}];\mathbf{x}'\sigma')=m\pi,\end{aligned} \tag{26}$$

where m is an odd integer for fermions and even for bosons. This recursion is to be satisfied for all $(\mathbf{x}\sigma)\neq(\mathbf{x}'\sigma')$. It will be shown later that the restriction is far more severe, brought about by the need to recover the free case properly. It may puzzle the reader that the above statement implies that a random choice of the phase functional that ensures that the recursion is satisfied does not suffice. This is mysterious, but is clarified by a conjecture in Appendix E. This is done by relating the canonical conjugate to the current operator and rewriting the DPVA in terms of current and densities. Again this type of idea has been addressed in the paper by Goldin *et al.*[9] However, many in this field continue to be under the mistaken impression that the formula for the annhilation operator (say the Fermi operator) in terms of the corresponding currents and densities depends on whether the fields in question are free or whether there are interactions in the system. This is shown to be false in the Bose case, by demonstrating that there is a unique Φ namely $\Phi=0$ that reproduces the free theory properly. Interactions just change the form of the Hamiltonian but do not affect the form of the field operator in terms of currents and densities. The same is true but not easily seen in the Fermi case; indeed throughtout this article we find that the Bose case is much simpler and we shall take refuge under this rigorously justifiable edifice when confronted by Fermi systems. Further, the formulas for the field operators suggested by Goldin, Menikoff, and Sharp in their famous paper[9] are according to our results only partially correct, since they have not actually introduced the phase functional Φ and computed it (this will again become clear soon).

Let us now write down a formula for the current operator in terms of the canonical conjugate and density,

$$\mathbf{J}=\left(\frac{1}{2i}\right)[\psi^{\dagger}\nabla\psi-(\nabla\psi)^{\dagger}\psi] \tag{27}$$

using the DPVA Eq. (25),

$$\mathbf{J}(\mathbf{x}\sigma)=\rho(\nabla\Phi)-\rho(\nabla\Pi+[-i\Phi,\nabla\Pi]). \tag{28}$$

From this it possible to deduce a formula for the conjugate in terms of currents and densities,

$$\Pi(\mathbf{x}\sigma)=X_{0\sigma}+\int^{\mathbf{x}}d\mathbf{l}[-1/\rho(\mathbf{y}\sigma)]\mathbf{J}(\mathbf{y}\sigma)+\Phi([\rho];\mathbf{x}\sigma)$$
$$-\int^{\mathbf{x}}d\mathbf{l}[-i\Phi,\nabla\Pi](\mathbf{y}\sigma). \tag{29}$$

The line integral is along an arbitrary path from a remote point where all quantities may be set equal to zero. The field operator may now be rewritten exclusively in terms of currents and densities, like

$$\psi(\mathbf{x}\sigma)=\exp\left\{-iX_{0\sigma}-i\int^{\mathbf{x}}d\mathbf{l}[-1/\rho(\mathbf{y}\sigma)]\mathbf{J}(\mathbf{y}\sigma)\right.$$
$$\left.-i\Phi([\rho];\mathbf{x}\sigma)+i\int^{\mathbf{x}}d\mathbf{l}[-i\Phi,\nabla\Pi](\mathbf{y}\sigma)\right\}$$
$$\times e^{i\Phi([\rho];\mathbf{x}\sigma)}(\rho(\mathbf{x}\sigma))^{1/2}, \tag{30}$$

where $X_{0\sigma}$ is canonically conjugate to the total number of fermions/bosons($[X_{0\sigma},N_{\sigma'}]=i\delta_{\sigma,\sigma'}$) $N_{\sigma}=\Sigma\mathbf{k}c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}$. The need for this is clear. The gradient of Π does not involve the object $X_{0\sigma}$, when in fact it should. To put it differently, the field operator when commuted with N_{σ} should produce itself, whereas if we omit the object $X_{0\sigma}$ then we find that the field operator commutes with the total number, which should not happen. These nuances are not very important for the practical computations as we shall see. It will be shown later that for bosons $\Phi=0$ is the only possible choice and for fermions Φ has to be fixed by making contact with the free theory. Uniqueness is assumed for the Fermi case by making an analogy with the Bose case for which uniqueness may be proved.

IV. MAKING CONTACT WITH THE FREE THEORY

In this section, we write down the kinetic energy operator in terms of the sea displacements and determine the undetermined phase functional Φ in the Fermi case. The reason why the phase functional $\Phi=0$ in the Bose case will also be addressed here. Let us take the Bose case first. It is clear at the outset that the choice $\Phi=0$ satisfies the recursion Eq. (26) for bosons when one assumes that $m=0$, an even integer. That this is the only possible choice is not at all clear. In order to verify this, let us write down the kinetic energy operator in terms of the density and its conjugate and show that an expansion in terms of the density fluctuations recovers the correct form of the dynamical density correlation function of the free theory (just the Bose case).

$$K=\int\frac{d\mathbf{x}}{2m}\left[\rho(\nabla\Pi)^{2}+\frac{(\nabla\rho)^{2}}{4\rho}\right]+c-\text{number}\,. \tag{31}$$

It may now be verified that an expansion in terms of density fluctuations leads to a Hamiltonian that describes free harmonic oscillators, which may be easily diagonalized. It may also be shown that this diagonalized form reproduces the correct dynamical density correlation functions. The expanded form of the operator in Fourier space is reproduced below for convenience:

$$K=\sum_{\mathbf{q}\neq 0}N\epsilon_{\mathbf{q}}X_{\mathbf{q}}X_{-\mathbf{q}}+\sum_{\mathbf{q}\neq 0}\frac{\epsilon_{\mathbf{q}}}{4N}\rho_{\mathbf{q}}\rho_{-\mathbf{q}}\,. \tag{32}$$

A different choice of Φ does not reproduce the free theory correctly. This is attested to by a simple calculation made in 1D. Let us assume a form,

$$\Phi([\rho];x)=2\pi\int_{-\infty}^{+\infty}dy\,\theta(x-y)[\rho(y)-\rho_{0}], \tag{33}$$

where $\theta(x)$ is the Heaviside step function. The above form clearly satisfies the recursion but does not reproduce the free theory as may be easily verified by the reader.

The Fermi case is somewhat more difficult. The difficulty is due to the fact that we must have a choice of $\Phi\neq 0$ that satisifes the recursion at the same time reproducing the free case. We shall take the point of view that the simplest choice for Φ namely linear in ρ should suffice. In any event, for the scheme to have practical significance, it is important for Φ to be a simple functional of the density. We fix the coefficient in this ansatz by making contact with the free theory. Let us focus on the case of spinless fermions. In what follows, we restrict ourselves to zero temperature and a weakly nonideal system, in this case, we are allowed to replace the $\overline{n}_{\mathbf{k}}$ in the definition of $\Lambda_{\mathbf{k}}(\mathbf{q})$ by its noninteracting value at zero temperature. More interesting situations arise when the quantity $\overline{n}_{\mathbf{k}}$ is evaluated self-consistently, but we shall relegate these issues to future publications.[19] From Eq. (30) it is clear that redefinitions of the phase functional by amounts that do not depend on the density, for example, $\Phi([\rho];\mathbf{x}) \rightarrow \Phi([\rho];\mathbf{x}) + f(\mathbf{x})$, do not affect the formula for the field operator in Eq. (30). Therefore, let us try the following ansatz for Φ:

$$\Phi([\rho];\mathbf{x}) = \sum_{\mathbf{q}\neq 0} U_{\mathbf{q}}(\mathbf{x})\rho_{\mathbf{q}}. \tag{34}$$

Let us now write down the kinetic energy operator for fermions using the results of the first section. The kinetic energy operator was written down as

$$K = \sum_{\mathbf{k},\mathbf{q}} \frac{\mathbf{k}\cdot\mathbf{q}}{m} a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) + N\epsilon_0. \tag{35}$$

It has been demonstrated in Appendix C that if one uses the form of the density-fluctuation operator obtained by dropping quadratic terms in the sea displacements (the existence of such quadratic terms are hinted in Appendix B), this reproduces the RPA dielectric function. Since we know from prior experience that the RPA is exact in the ultrahigh density limit, we can use these two pieces of information to deduce a formula for $U_{\mathbf{q}}(\mathbf{x})$ in terms of the properties of the free theory. First let us write down the RPA form of the density-fluctuation operator,

$$\tilde{\rho}_{\mathbf{q}} = \sum_{\mathbf{k}} [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + \Lambda_{\mathbf{k}}(-\mathbf{q}) a^{\dagger}_{\mathbf{k}}(\mathbf{q})], \tag{36}$$

where

$$\Lambda_{\mathbf{k}}(\mathbf{q}) = \sqrt{\overline{n}_{\mathbf{k}+\mathbf{q}/2}(1-\overline{n}_{\mathbf{k}-\mathbf{q}/2})} \tag{37}$$

and the corresponding conjugate variable may be written down (that is, Π in Fourier space),

$$\tilde{X}_{\mathbf{q}} = \left(-\frac{1}{2iN\epsilon_{\mathbf{q}}}\right) \sum_{\mathbf{k}} [\Lambda_{\mathbf{k}}(-\mathbf{q})\omega_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) - \Lambda_{\mathbf{k}}(\mathbf{q})\omega_{\mathbf{k}}(-\mathbf{q}) a^{\dagger}_{\mathbf{k}}(-\mathbf{q})], \tag{38}$$

where the dispersion is given by $\omega_{\mathbf{k}}(\mathbf{q}) = \mathbf{k}\cdot\mathbf{q}/m$. From this the Fermi-field operator may be written down as

$$\psi(\mathbf{x}) = e^{-iU_1(\mathbf{x})} e^{iU_2(\mathbf{x})} \sqrt{\rho_0}, \tag{39}$$

where

$$U_1(\mathbf{x}) = \sum_{\mathbf{q}\neq 0} e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{X}_{\mathbf{q}}, \tag{40}$$

$$U_2(\mathbf{x}) = \sum_{\mathbf{q}\neq 0} U_{\mathbf{q}}(\mathbf{x}) \tilde{\rho}_{\mathbf{q}}. \tag{41}$$

Using these facts, let us compute the equal-time version of the propagator below in the Bose language and in the usual Fermi language and equate the two expressions. In the sea-displacement language it comes out as

$$\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t)\rangle = \rho_0 e^{-\sum_{\mathbf{k},\mathbf{q}\neq 0} g^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) g_{\mathbf{k},\mathbf{q}}(\mathbf{x})} e^{\sum_{\mathbf{k},\mathbf{q}\neq 0} g^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) g_{\mathbf{k},\mathbf{q}}(\mathbf{x}')}, \tag{42}$$

where

$$g_{\mathbf{k},\mathbf{q}}(\mathbf{x}) = -e^{-i\mathbf{q}\cdot\mathbf{x}} \left(\frac{1}{2N\epsilon_{\mathbf{q}}}\right) \Lambda_{\mathbf{k}}(-\mathbf{q})\omega_{\mathbf{k}}(\mathbf{q}) + iU_{\mathbf{q}}(\mathbf{x})\Lambda_{\mathbf{k}}(-\mathbf{q}) = -f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}). \tag{43}$$

In the original Fermi language it is

$$\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t)\rangle = \frac{1}{V}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x})} \theta(k_f - |\mathbf{q}|). \tag{44}$$

Set $U_{\mathbf{q}}(\mathbf{x}) = e^{-i\mathbf{q}\cdot\mathbf{x}} U_0(\mathbf{q})$ and $U_0(\mathbf{q})$ is real. In order to derive a formula for $U_0(\mathbf{q})$, let us equate the logarithm of the two expressions

$$\begin{aligned}
\ln[\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t)\rangle] &= \ln(\rho_0) + \sum_{\mathbf{k},\mathbf{q}\neq 0} \left[\left(\frac{1}{2N\epsilon_{\mathbf{q}}}\right)^2 \left(\frac{\mathbf{k}\cdot\mathbf{q}}{m}\right)^2 + (U_0(\mathbf{q}))^2\right] (\Lambda_{\mathbf{k}}(-\mathbf{q}))^2 (e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} - 1) \\
&= \ln(\rho_0) + \ln\left(1 + \frac{1}{N}\sum_{\mathbf{q}\neq 0} (e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} - 1)\theta(k_f - |\mathbf{q}|)\right) \\
&\approx \ln(\rho_0) + \frac{1}{N}\sum_{\mathbf{q}\neq 0} (e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} - 1)\theta(k_f - |\mathbf{q}|).
\end{aligned} \tag{45}$$

This leads to the following formula for the coefficient:

$$U_0(\mathbf{q}) = \frac{1}{N}\left(\frac{\theta(k_f - |\mathbf{q}|) - w_1(\mathbf{q})}{w_2(\mathbf{q})}\right)^{1/2}, \tag{46}$$

$$w_1(\mathbf{q}) = \left(\frac{1}{4N\epsilon_{\mathbf{q}}^2}\right)\sum_{\mathbf{k}}\left(\frac{\mathbf{k}\cdot\mathbf{q}}{m}\right)^2(\Lambda_{\mathbf{k}}(-\mathbf{q}))^2, \tag{47}$$

$$w_2(\mathbf{q}) = \left(\frac{1}{N}\right)\sum_{\mathbf{k}}(\Lambda_{\mathbf{k}}(-\mathbf{q}))^2. \tag{48}$$

In fact, in principle, we could go all the way back to the expression in Eq. (30) and say that we now have a unique correspondence between the Fermi-field operator and the corresponding currents and densities. In the next section, we write down and diagonalize the Hamiltonian of interacting systems. It is shown that when the lowest-order sea-displacement terms/condensate displacement terms are included, it amounts to using RPA/Bogoliubov theory. This Hamiltonian is diagonalized in the Fermi and Bose cases and the single-particle spectral functions are computed. The Bose case comes out nicely since, it is just the Bogoliubov theory, but in the Fermi case, we have to take extra care in properly diagonalizing the Hamiltonian in order not to lose the particle-hole mode, the collective mode being more obvious.

V. SPECTRAL FUNCTION OF INTERACTING SYSTEMS

Let us make the following observation for future reference:

$$\textbf{RPA/Bogoliubov} \rightarrow \textbf{Leave out the quadratic part in Eq. (B4) and Eq. (3)}. \tag{49}$$

It is pertinent at this stage to remark on the physical meaning of the above relation. In the case of bosons, it is simple to visualize. Bogoliubov's theory is exact provided there are large number bosons in the zero momentum state so that we may legitimately replace the number operator by its c-number expectation value. Also it is important that the system be weakly interacting so that the fluctuations of the number operator in the zero momentum state are small compared with its macroscopic expectation value. In the Fermi case an analogous statement would be that the momentum distribution be sufficiently different from zero or unity for all values of the momenta. Also the fluctuations of the momentum distribution must be small. Thus for the Fermi system, our approach gives good answers even for strong interactions that drive the momentum distribution away from zero or unity for all momenta so long as the fluctuations around these nonideal averages are small. In any event, the philosophy is that we have an exactly solvable class of models that describe correlation effects in many different contexts and this alone merits attention and serious study. In the end experiments may have to be used to ''calibrate'' these models so that they become a true description of the low-energy real world.

A. Bose system

Let us focus on the Bose case first. The Bogoliubov Hamiltonian may be written down by following the prescription of Eq. (49):

$$H_{\rm bog} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} d^{\dagger}_{(1/2)\mathbf{k}}(\mathbf{k}) d_{(1/2)\mathbf{k}}(\mathbf{k}) + \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2}[\sqrt{N_0}d_{-\mathbf{q}/2}(-\mathbf{q}) + d^{\dagger}_{\mathbf{q}/2}(\mathbf{q})\sqrt{N_0}] \times[\sqrt{N_0}d_{\mathbf{q}/2}(\mathbf{q}) + d^{\dagger}_{-\mathbf{q}/2}(-\mathbf{q})\sqrt{N_0}]. \tag{50}$$

In the above equation N_0 is an operator, therefore this is the nonlocal Bogoliubov Hamiltonian. But we shall assume that it is legitimate to replace it with its c-number expectation value. It would be interesting to see what corrections to Bogoliubov theory come about by incorporating this square root of the operator. These correction terms tell us that fluctuations of the number of particles in the condensate are important and lead to correlations beyond the Bogoliubov theory. This is in addition to correlations coming from quadratic terms that the prescription Eq. (49) neglects. When these approximations are implemented, and a further approximation $N_0 \approx N$ is made, it becomes exactly the Bogoliubov theory introduced by Bogoliubov and Bogliubov and Zubarev.[20] It may be diagonalized quite easily,

$$H_{\rm bog} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} f^{\dagger}_{\mathbf{q}} f_{\mathbf{q}} \tag{51}$$

and

$$f_{\mathbf{q}} = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2} d_{\mathbf{q}/2}(\mathbf{q}) + \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2} d^{\dagger}_{-\mathbf{q}/2}(-\mathbf{q}), \tag{52}$$

$$d_{\mathbf{q}/2}(\mathbf{q}) = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2} f_{\mathbf{q}} - \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2} f^{\dagger}_{-\mathbf{q}}. \tag{53}$$

The dispersion is given by

$$\omega_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}^2 + 2\rho_0 v_{\mathbf{q}}\epsilon_{\mathbf{q}}}, \tag{54}$$

where ρ_0 is the density of bosons in the condensate (not the overall density). From this one may deduce the filling fraction and dynamical structure factor, *Filling fraction*:

$$f_0=N_0/N=1-(1/N)\sum_{\mathbf{q}}\ \langle d^{\dagger}_{(1/2)\mathbf{q}}(\mathbf{q})d_{(1/2)\mathbf{q}}(\mathbf{q})\rangle, \quad (55)$$

in other words

$$f_0=N_0/N=1-(1/2\pi^2\rho)\int_0^{\infty}dq\ q^2\left(\frac{-\omega_{\mathbf{q}}+\epsilon_{\mathbf{q}}+\rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right), \quad (56)$$

where ρ is the total density of bosons including those that are not in the condensate.

Dynamical structure factor:

$$\begin{aligned}S^{>}(\mathbf{q}t)&=\langle\rho_{\mathbf{q}}(t)\rho_{-\mathbf{q}}(0)\rangle\\&=N_0\langle[d_{-(1/2)\mathbf{q}}(-\mathbf{q})(t)+d^{\dagger}_{(1/2)\mathbf{q}}(\mathbf{q})(t)]\\&\quad\times[d_{(1/2)\mathbf{q}}(\mathbf{q})(0)+d^{\dagger}_{-(1/2)\mathbf{q}}(-\mathbf{q})(0)]\rangle, \quad (57)\end{aligned}$$

in other words

$$S^{>}(\mathbf{q},t)=N_0\left(\frac{\epsilon_{\mathbf{q}}}{\omega_{\mathbf{q}}}\right)\exp(-i\omega_{\mathbf{q}}t). \quad (58)$$

This method is truly powerful when applied to compute single-particle properties. The single-particle green function is difficult to obtain using conventional digrammatic methods or otherwise (see Kadanoff and Baym, Ref. 21). For this one must first write down the field operator in terms of the condensate displacements:

$$\Pi(\mathbf{x})\approx\left(\frac{i}{2\sqrt{N_0}}\right)\sum_{\mathbf{q}}\ \exp(i\mathbf{q}\cdot\mathbf{x})[d_{\mathbf{q}/2}(\mathbf{q})-d^{\dagger}_{-\mathbf{q}/2}(-\mathbf{q})] \quad (59)$$

and the expression for the field operator is

$$\psi(\mathbf{x})\approx e^{-i\Pi(\mathbf{x})}\sqrt{\rho}. \quad (60)$$

The propagator (all propagators in this article are evaluated at zero temperature, this means we may set the chemical potential equal to zero in the Bose case) may now be computed and shown to be equal to the free propagator at ultrahigh density. The interacting case is more interesting. The time-evolved version is

$$\psi(\mathbf{x},t)\approx e^{-i\Pi(\mathbf{x},t)}\sqrt{\rho} \quad (61)$$

and

$$\begin{aligned}\Pi(\mathbf{x},t)&=\left(\frac{i}{2\sqrt{N_0}}\right)\sum_{\mathbf{q}}\ \exp(i\mathbf{q}\cdot\mathbf{x})(A_{\mathbf{q}}+B_{\mathbf{q}})\\&\quad\times[f_{\mathbf{q}}e^{-i\omega_{\mathbf{q}}t}-f^{\dagger}_{-\mathbf{q}}e^{i\omega_{\mathbf{q}}t}], \quad (62)\end{aligned}$$

$$\langle\psi^{\dagger}(\mathbf{0},0)\psi(\mathbf{x},t)\rangle=\rho\langle e^{i\Pi(\mathbf{0},0)}e^{-i\Pi(\mathbf{x},t)}\rangle. \quad (63)$$

In order to ensure that the free case is properly recovered, we use this somewhat illegal trick, but a trick that should be very palatable to most physicists, namely multiply and divide by the free propagator and in the division use the free propagator predicted by the bosonized theory and in the numerator use the free propagator obtained from elementary considerations.

$$\begin{aligned}\langle\psi^{\dagger}(\mathbf{0},0)\psi(\mathbf{x},t)\rangle&=\exp\left[\left(\frac{1}{4N_0}\right)\sum_{\mathbf{q}}\ f_{\mathbf{q}}(\mathbf{x},t)\right]\\&\quad\times\langle\psi^{\dagger}(\mathbf{0},0)\psi(\mathbf{x},t)\rangle_{\text{free}}, \quad (64)\end{aligned}$$

where

$$A_{\mathbf{q}}=\left(\frac{\omega_{\mathbf{q}}+\epsilon_{\mathbf{q}}+\rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2}, \quad (65)$$

$$B_{\mathbf{q}}=\left(\frac{-\omega_{\mathbf{q}}+\epsilon_{\mathbf{q}}+\rho_0 v_{\mathbf{q}}}{2\omega_{\mathbf{q}}}\right)^{1/2}. \quad (66)$$

Similarly,

$$\begin{aligned}\langle\psi(\mathbf{x},t)\psi^{\dagger}(\mathbf{0},0)\rangle&=\rho\langle e^{-i\Pi(\mathbf{x},t)}e^{i\Pi(\mathbf{0},0)}\rangle\\&=\exp\left[\left(\frac{1}{4N_0}\right)\sum_{\mathbf{q}}\ f_{\mathbf{q}}(-\mathbf{x},-t)\right]\\&\quad\times\langle\psi(\mathbf{x},t)\psi^{\dagger}(\mathbf{0},0)\rangle_{\text{free}}, \quad (67)\end{aligned}$$

where

$$f_{\mathbf{q}}(\mathbf{x},t)=(e^{-i\mathbf{q}\cdot\mathbf{x}}e^{i\omega_{\mathbf{q}}t}-1)(A_{\mathbf{q}}+B_{\mathbf{q}})^2-(e^{-i\mathbf{q}\cdot\mathbf{x}}e^{i\epsilon_{\mathbf{q}}t}-1). \quad (68)$$

From Kadanoff and Baym[21] the spectral function may be deduced as follows.

The spectral function:

$$\begin{aligned}A(\mathbf{p},\omega)=\int d\mathbf{x}\int_{-\infty}^{+\infty}dt\,e^{-i\mathbf{p}\cdot\mathbf{x}+i\omega t}&\left\{\exp\left[\frac{1}{4N_0}\sum_{\mathbf{q}}\ f_{\mathbf{q}}(-\mathbf{x},-t)\right]\right.\\&\left.\times[\rho+u(\mathbf{x},t)]-\exp\left[\frac{1}{4N_0}\sum_{\mathbf{q}}\ f_{\mathbf{q}}(\mathbf{x},t)\right]\rho\right\} \quad (69)\end{aligned}$$

and

$$u(\mathbf{x},t)=\frac{1}{V}\sum_{\mathbf{k}}\ e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\epsilon_{\mathbf{k}}t}. \quad (70)$$

The above answer is the exact answer for the spectral function provided Bogoliubov's theory is adequate. Now we move on to the Fermi case which is far more interesting and important.

B. Fermi system

In order to compute the full propagator for these systems, it is desirable to first ascertain, under what conditions these formulas are going to be valid. The answer is given by the assertion in Eq. (49). Thus these answers for the single-particle properties are valid in the same limit in which RPA/Bogoliubov's theory is exact. The assertion in the Bose case in Eq. (49) has been verified. In order to verify the analogous assertion in the Fermi case, we have to diagonalize the full Hamiltonian given below. [The fact that the RPA dielectric function comes out naturally from the prescription in Eq.

(49) will be demsonstrated in Appendix C.] In the Fermi case, we have to diagonalize the full Hamiltonian given below:

$$H=\sum_{\mathbf{k},\mathbf{q}} \omega_{\mathbf{k}}(\mathbf{q})a^{\dagger}_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q}) + \sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k},\mathbf{k}'}[\Lambda_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}}(-\mathbf{q})+\Lambda_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}}(\mathbf{q})] \times[\Lambda_{\mathbf{k}'}(-\mathbf{q})a_{\mathbf{k}'}(\mathbf{q})+\Lambda_{\mathbf{k}'}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(-\mathbf{q})], \tag{71}$$

where $\omega_{\mathbf{k}}(\mathbf{q})=(\mathbf{k}\cdot\mathbf{q}/m)\Lambda_{\mathbf{k}}(-\mathbf{q})$. The zero-temperature case is somewhat special, here we may assume that the sea boson annhilates the noninteracting Fermi sea, which means that we have to introduce a factor of $\Lambda_{\mathbf{k}}(-\mathbf{q})$ in the dispersion that makes the kinetic energy operator positive definite. In order to diagonalize this we proceed as follows. Assume that the diagonalized form is

$$H=\sum_{i,\mathbf{q}} \tilde{\omega}_i(\mathbf{q})b^{\dagger}_i(\mathbf{q})b_i(\mathbf{q}), \tag{72}$$

where $b_i(\mathbf{q})$ and $b^{\dagger}_i(\mathbf{q})$ are ''dressed-sea-displacement'' operators. The objects i take on values from an index set. The size of this set is the big issue here. Is it finite or does it have the same size as the number of points in k space, or is it equal to the number of points on the Fermi surface? We shall find that answers to these questions are hard, and may be addressed only after coming to an agreement as to what sort of physics we hope to capture. Indeed, in many cases in physics one is forced to bend the rules or reinterpret mathematical formulas in order to capture what one is looking for. We find that we have to resort to such methods here as well. In particular, we find the following general feature. $\tilde{\omega}_i(\mathbf{q})$ are the roots of the RPA-dielectric function. Now the RPA-dielectric function is a complex quantity, as it is usually introduced in the textbooks. Therefore finding roots cannot mean finding the zeros of both the real and imaginary parts at the same time for this gives no root, and both the real and imaginary part cannot be zero simultaneously. This leaves us with the following options, reinterpret the zeros of the RPA-dielectric function to be the maxima of the dynamical structure factor, in which case one gets both the particle-hole mode as well as the collective mode. The better option is to delay taking the thermodynamic limit until after all the summation over momenta have been performed. Then assume that the density is high enough and at the very end go to the thermodynamic limit, this ensures that both the particle-hole mode and the collective mode are properly recovered. These are admitedly difficult issues to grapple with, and the authors have attempted a different approach to deal with them. However, the traditional viewpoint on this matter is presented in the paper by Castro-Neto and Fradkin.[11] The diagonalization proceeds as follows:

$$b_i(\mathbf{q})=\sum_{\mathbf{k}}[b_i(\mathbf{q}),a^{\dagger}_{\mathbf{k}}(\mathbf{q})]a_{\mathbf{k}}(\mathbf{q}) - \sum_{\mathbf{k}}[b_i(\mathbf{q}),a_{\mathbf{k}}(-\mathbf{q})]a^{\dagger}_{\mathbf{k}}(-\mathbf{q}) \tag{73}$$

the corresponding inverted formula reads

$$a_{\mathbf{k}}(\mathbf{q})=\sum_i [a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_i(\mathbf{q})]b_i(\mathbf{q}) - \sum_i [a_{\mathbf{k}}(\mathbf{q}),b_i(-\mathbf{q})]b^{\dagger}_i(-\mathbf{q}). \tag{74}$$

The quantities $[b_i(\mathbf{q}),a_{\mathbf{k}}(-\mathbf{q})]$ and $[a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_i(\mathbf{q})]$ are c numbers and real. The i here could span a continuum (particle-hole mode) or be finite (actually there is just one collective mode). The diagonalization continues unabated,

$$[b_i(\mathbf{q}),a^{\dagger}_{\mathbf{k}}(\mathbf{q})]=\left(\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}_i(\mathbf{q})-\mathbf{k}\cdot\mathbf{q}/m}\right)g_i(\mathbf{q})=[a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_i(\mathbf{q})], \tag{75}$$

$$[b_i(\mathbf{q}),a_{\mathbf{k}}(-\mathbf{q})]=-\left(\frac{\Lambda_{\mathbf{k}}(\mathbf{q})}{\tilde{\omega}_i(\mathbf{q})-\mathbf{k}\cdot\mathbf{q}/m}\right)g_i(\mathbf{q}), \tag{76}$$

$$[a_{\mathbf{k}}(\mathbf{q}),b_i(-\mathbf{q})]=\left(\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m}\right)g_i(-\mathbf{q}), \tag{77}$$

$$g_i(\mathbf{q})=\left[\sum_{\mathbf{k}}\frac{n_F(\mathbf{k}-\mathbf{q}/2)-n_F(\mathbf{k}+\mathbf{q}/2)}{[\tilde{\omega}_i(\mathbf{q})-\mathbf{k}\cdot\mathbf{q}/m]^2}\right]^{-1/2}. \tag{78}$$

The eigenvalues $\tilde{\omega}_i(\mathbf{q})$ are zeros of the real part of the RPA dielectric function. The RPA dielectric function is written down below:

$$\epsilon_{\text{RPA}}(\mathbf{q},\tilde{\omega})=1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{n_F(\mathbf{k}+\mathbf{q}/2)-n_F(\mathbf{k}-\mathbf{q}/2)}{\tilde{\omega}-\mathbf{k}\cdot\mathbf{q}/m}. \tag{79}$$

As it stands, the above sum is ill-defined. In particular, if one takes the thermodynamic limit at the outset, and treats the above sum as the principal part, then one gets the real part of the RPA dielectric function. On the other hand, if one defers the taking of the thermodynamic limit until the very end, and instead takes the high density limit first, then one obtains the particle-hole mode as the argument below will attest. Let us rewrite the sum in the RPA dielectric function as

$$\epsilon_{\text{RPA}}(\mathbf{q},\tilde{\omega})=1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\Lambda^2_{\mathbf{k}}(\mathbf{q})-\Lambda^2_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}-\mathbf{k}\cdot\mathbf{q}/m} =1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\Lambda^2_{\mathbf{k}}(\mathbf{q})}{\tilde{\omega}+\omega_{\mathbf{k}}(-\mathbf{q})}-\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\neq\mathbf{k}_i}\frac{\Lambda^2_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}-\omega_{\mathbf{k}}(\mathbf{q})} -\frac{v_{\mathbf{q}}}{V}\frac{\Lambda^2_{\mathbf{k}_i}(-\mathbf{q})}{\tilde{\omega}-\omega_{\mathbf{k}_i}(\mathbf{q})}. \tag{80}$$

Let us now assume that the volume V is fixed and we now go to the high density limit ($k_F\rightarrow\infty$, or equivalently when , $|\mathbf{q}|\ll k_f$), then we find, due to the fact below

$$\Lambda_{\mathbf{k}}(-\mathbf{q})=0;\ \textit{unless}\ |\mathbf{k}|\approx k_f\ \textit{and}\ \mathbf{k}\cdot\mathbf{q}>0. \tag{81}$$

The total number of terms in the above two sums is a small fraction of the total volume and as k_F keeps increasing, the fraction gets smaller and smaller until it becomes small compared to unity and may be neglected. This means

$$1-\left(\frac{v_{\mathbf{q}}}{V}\right)\frac{\Lambda^2_{\mathbf{k}_i}(-\mathbf{q})}{\tilde{\omega}_i(\mathbf{q})-\omega_{\mathbf{k}_i}(\mathbf{q})}=0. \tag{82}$$

From this we may deduce the particle-hole mode as

$$\tilde{\omega}_i(\mathbf{q})=\omega_{\mathbf{k}_i}(\mathbf{q})+\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda^2_{\mathbf{k}_i}(-\mathbf{q}). \tag{83}$$

As is clear from the above derivation, two points must be borne in mind, one is, we have to defer the taking of the thermodynamic limit until the very end, the other is to exploit the property of the object $\Lambda_{\mathbf{k}}(\mathbf{q})$, namely, if $|\mathbf{q}|\ll k_f$, and $\Lambda_{\mathbf{k}}(-\mathbf{q})=1$ ($\Lambda_{\mathbf{k}}(-\mathbf{q})=0,1$ *always*) then $|\mathbf{k}|\approx k_f$. Alternatively, we can solve for $\tilde{\omega}_i(\mathbf{q})$ as shown below:

$$\tilde{\omega}_i(\mathbf{q})=\omega_{\mathbf{k}_i}(\mathbf{q})+\left(\frac{v_{\mathbf{q}}}{V}\right)\frac{\Lambda^2_{\mathbf{k}_i}(-\mathbf{q})}{1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\Lambda^2_{\mathbf{k}}(\mathbf{q})}{\omega_{\mathbf{k}_i}(\mathbf{q})+\omega_{\mathbf{k}}(-\mathbf{q})}-\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\neq\mathbf{k}_i}\frac{\Lambda^2_{\mathbf{k}}(-\mathbf{q})}{\omega_{\mathbf{k}_i}(\mathbf{q})-\omega_{\mathbf{k}}(\mathbf{q})}}. \tag{84}$$

We shall find these formulas useful later on when we try to write down the propagator. The collective mode in 1D and 3D may be written down as shown below:

$$\omega_{c-1D}(q)=\left(\frac{|q|}{m}\right)\sqrt{\frac{(k_f+q/2)^2-(k_f-q/2)^2\exp(-\lambda(q))}{1-\exp(-\lambda(q))}}, \tag{85}$$

$$\lambda(q)=\left(\frac{2\pi q}{m}\right)\left(\frac{1}{v_q}\right). \tag{86}$$

We may also write

$$\omega^2_{c-1\mathrm{D}}(q)=\left(\frac{k_f q}{m}\right)^2+\epsilon_q^2+2\epsilon_q\left(\frac{k_f q}{m}\right)\coth\left(\frac{\lambda(q)}{2}\right). \tag{87}$$

In 3D it is more familiar[22] (only for Coulomb repulsion),

$$\omega_{c-3\mathrm{D}}(\mathbf{q})=\omega_p\left[1+\frac{3}{10}\frac{(qv_f)^2}{\omega_p^2}\right]. \tag{88}$$

For more general forms of interaction in 3D the answer may be obtained by computing the roots of the equation below:

$$1-\frac{n_0(v_{\mathbf{q}}q^2)/m}{\omega^2}\left\{1+\frac{1}{\omega^2}\left[\frac{3}{5}(qv_F)^2-\epsilon^2_{\mathbf{q}}\right]\right\}=0. \tag{89}$$

In 2D, the answer is not available in the books and may be deduced after some algebra as ($\omega\gg k_f|q|m$)

$$\omega_{c-2D}(\mathbf{q})=\frac{(k_f|\mathbf{q}|/m)(1+2\pi/m/v_{\mathbf{q}})}{\sqrt{4\pi/m/v_{\mathbf{q}}+(2\pi/m/v_{\mathbf{q}})^2}}. \tag{90}$$

After all this, it is relatively simple to deduce the full propagator. For reference the free propagator is

$$\langle\psi^\dagger(\mathbf{x},t)\,\psi(\mathbf{x}',t')\rangle=\rho_0 e^{-\sum_{\mathbf{k},\mathbf{q}\neq0}g^*_{\mathbf{k},\mathbf{q}}(\mathbf{x})g_{\mathbf{k},\mathbf{q}}(\mathbf{x})}\times e^{\sum_{\mathbf{k},\mathbf{q}\neq0}g^*_{\mathbf{k},\mathbf{q}}(\mathbf{x})g_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\omega_{\mathbf{k}}(\mathbf{q})(t'-t)}}, \tag{91}$$

$$\langle\psi(\mathbf{x}',t')\,\psi^\dagger(\mathbf{x},t)\rangle=\rho_0 e^{-\sum_{\mathbf{k},\mathbf{q}\neq0}f^*_{\mathbf{k},\mathbf{q}}(\mathbf{x}')f_{\mathbf{k},\mathbf{q}}(\mathbf{x}')}\times e^{\sum_{\mathbf{k},\mathbf{q}\neq0}f^*_{\mathbf{k},\mathbf{q}}(\mathbf{x})f_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\omega_{\mathbf{k}}(\mathbf{q})(t-t')}}, \tag{92}$$

$$f_{\mathbf{k},\mathbf{q}}(\mathbf{x})=e^{i\mathbf{q}\cdot\mathbf{x}}\left(\frac{1}{2N\epsilon_{\mathbf{q}}}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\,\omega_{\mathbf{k}}(\mathbf{q})+iU_{-\mathbf{q}}(\mathbf{x})\Lambda_{\mathbf{k}}(-\mathbf{q}), \tag{93}$$

$$g_{\mathbf{k},\mathbf{q}}(\mathbf{x})=-e^{-i\mathbf{q}\cdot\mathbf{x}}\left(\frac{1}{2N\epsilon_{\mathbf{q}}}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\,\omega_{\mathbf{k}}(\mathbf{q})+iU_{\mathbf{q}}(\mathbf{x})\Lambda_{\mathbf{k}}(-\mathbf{q}) =-f^*_{\mathbf{k},\mathbf{q}}(\mathbf{x}), \tag{94}$$

and

$$\mathcal{Z}_0=e^{i\sum_{\mathbf{k},\mathbf{q}\neq0}U_0(\mathbf{q})(1/2N\epsilon_{\mathbf{q}})(\Lambda_{\mathbf{k}}(-\mathbf{q}))^2\omega_{\mathbf{k}}(\mathbf{q})}\times e^{1/2\sum_{\mathbf{k},\mathbf{q}\neq0}(1/2N\epsilon_{\mathbf{q}})^2(\Lambda_{\mathbf{k}}(-\mathbf{q}))^2(\omega_{\mathbf{k}}(\mathbf{q}))^2}\times e^{1/2\ \sum_{\mathbf{k},\mathbf{q}\neq0}(U_0(\mathbf{q}))^2(\Lambda_{\mathbf{k}}(-\mathbf{q}))^2}. \tag{95}$$

The time-evolved field operator in the interacting case is

$$\psi^\dagger(\mathbf{x},t)=\exp\left(\sum_{\mathbf{k},\mathbf{q}\neq0,i}U^i_{\mathbf{k},\mathbf{q}}(\mathbf{x})b^\dagger_i(\mathbf{q})e^{i\tilde{\omega}_i(\mathbf{q})t}\right)\times\exp\left(-\sum_{\mathbf{k},\mathbf{q}\neq0,i}U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})b_i(\mathbf{q})e^{-i\tilde{\omega}_i(\mathbf{q})t}\right)\mathcal{R}_0\mathcal{Z}^*_0\sqrt{\rho_0}, \tag{96}$$

where

$$U^{i}_{\mathbf{k},\mathbf{q}} = f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_{i}(\mathbf{q})] + f_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}),b_{i}(\mathbf{q})],$$

$$\mathcal{R}_0 = \exp\left(-\sum_{\mathbf{k},\mathbf{q},i} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) f_{\mathbf{k},\mathbf{q}}(\mathbf{x})[b_i(\mathbf{q}),a^{\dagger}_{\mathbf{k}}(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_{i}(\mathbf{q})]\right) \times \exp\left(-\frac{1}{2}\sum_{\mathbf{k},\mathbf{q},i} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) f^{*}_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}),b_i(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_{i}(\mathbf{q})]\right) \tag{97}$$

$$\times \exp\left(-\frac{1}{2}\sum_{\mathbf{k},\mathbf{q},i} f_{\mathbf{k},\mathbf{q}}(\mathbf{x}) f_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}),b_i(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}),b^{\dagger}_{i}(\mathbf{q})]\right). \tag{98}$$

The two full Fermi propagators may be written down as

$$\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t')\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^2\rho_0 e^{\sum_{\mathbf{k},\mathbf{q},i}U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\tilde{\omega}_i(\mathbf{q})(t'-t)}}, \tag{99}$$

$$\langle \psi(\mathbf{x}',t')\psi^{\dagger}(\mathbf{x},t)\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^2\rho_0 e^{\sum_{\mathbf{k},\mathbf{q},i}U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}')U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})e^{i\tilde{\omega}_i(\mathbf{q})(t-t')}}. \tag{100}$$

Again, it is desirable to use the trick we used in the Bose case, namely multiply and divide by the free propagator and in the division use the form predicted by the bosonized theory and in the multiplication, use the form predicted by elementary considerations. This procedure also ensures that in spite of the fact we have not verified that the Fermi fields written down in terms of the Bose fields anticommute, the anticommutation rules are forced on the propagators by the free propagators which we know anticommute in the right fashion. This leads to the following forms for the propagators:

$$\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t')\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^4 e^{\sum_{\mathbf{k},\mathbf{q},i}U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\tilde{\omega}_i(\mathbf{q})(t'-t)}} \times e^{-\sum_{\mathbf{k},\mathbf{q}} g^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})g_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\omega_{\mathbf{k}}(\mathbf{q})(t'-t)}}\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}',t')\rangle, \tag{101}$$

$$\langle \psi(\mathbf{x}',t')\psi^{\dagger}(\mathbf{x},t)\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^4 e^{\sum_{\mathbf{k},\mathbf{q},i}U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}')U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})e^{i\tilde{\omega}_i(\mathbf{q})(t-t')}} \times e^{-\sum_{\mathbf{k},\mathbf{q}} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})f_{\mathbf{k},\mathbf{q}}(\mathbf{x}')e^{i\omega_{\mathbf{k}}(\mathbf{q})(t-t')}}\langle \psi(\mathbf{x}',t')\psi^{\dagger}(\mathbf{x},t)\rangle. \tag{102}$$

In the above formula, the index i runs over both the collective mode as well as the particle-hole mode ($i=c,\mathbf{k}_i$) The momentum distribution may be evaluated in a different way by computing the expectation value of the number operator in Eq. (12). This leads to the following answer. It includes the contribution from both the particle-hole mode and the collective mode. In Appendix D, we show how to derive the same momentum distribution using the equation of motion approach (actually just the collective part, for purposes of illustration). The full momentum distribution including the particle-hole mode is given below.

$$\langle c^{\dagger}_{\mathbf{k}}c_{\mathbf{k}}\rangle = \theta(k_f-|\mathbf{k}|)F_1(\mathbf{k}) + [1-\theta(k_f-|\mathbf{k}|)]F_2(\mathbf{k}), \tag{103}$$

$$F_1(\mathbf{k}) = 1 - \sum_{i,\mathbf{q}} \frac{1-n_F(\mathbf{k}+\mathbf{q})}{[\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m+\epsilon_{\mathbf{q}}]^2} g^2_i(-\mathbf{q}), \tag{104}$$

$$F_2(\mathbf{k}) = \sum_{i,\mathbf{q}} \frac{n_F(\mathbf{k}-\mathbf{q})}{[\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m-\epsilon_{\mathbf{q}}]^2} g^2_i(-\mathbf{q}). \tag{105}$$

In the above sum over i, one must include both the collective mode and the particle-hole mode ($i=c,\mathbf{k}_i$). A more general result is possible for systems that are significantly more nonideal. This comes about when one does not use the zero-temperature noninteracting values in the Fermi-bilinear sea-boson correspondence. The form of the momentum distribution suggested by this is given in Appendix D. It is now very easy to write down a criterion for the breakdown of Fermi-liquid behavior. It is given by equating the step at the Fermi surface to zero (the quasiparticle residue):

$$Z_f = F_1(k_f) - F_2(k_f) = 0. \tag{106}$$

In the end, it is pertinent to address the claim made in the abstract namely that we are able to capture short-wavelength behavior. The real issue here is that we have two length scales, one is the inverse of the Fermi momentum the other is the Bohr radius. When one speaks of short wavelengths, one means wavelengths comparable to the Bohr radius. In the ultrahigh density limit, where all the answers we have been deriving are valid, the inverse of the Fermi momentum is much too small (compared to the Bohr radius) for the wavelength of any external field to be comparable to it. In other words, even if you have an external field that varies so rapidly in space that it changes sign over a Bohr radius, the effective field induced by such an external field is still described by the RPA. To put it yet another way, the RPA is exact in the ultrahigh density limit. Some have argued that this limit is uninteresting since in this limit, the Coulomb interaction is completely screened out and therefore in this regime we just have a Fermi liquid. We find that this argument is not entirely true. In fact, we have shown[19] that when the inverse of the Fermi momentum is small compared to the Bohr radius, it is still possible to increase the value of the dimensionless coupling strength (for a delta-function interaction) sufficiently so that Fermi-liquid behavior is destroyed. We find that Fermi-liquid behavior persists in 1D for sufficiently weak-coupling strengths (when we assume the interactions are hard-core δ-function interactions), in contrast to the Lieb-Mattis solution of the Tomonaga-Luttinger model. We also find that Fermi-liquid behavior breaks down in more than one dimension for sufficiently strong values of the coupling strength in contrast to the answers obtained by Castro-Neto and Fradkin.[11] In fact we find that Fermi-liquid behavior persists in all three dimensions for sufficiently small values of the coupling strength and is destroyed in all three dimensions for sufficiently large values of the coupling strength. It may be argued by the reader that our results are not foolproof either, for one, we have neglected several terms in the Hamiltonian and those terms are small only in the limit when RPA is exact. The other points are the technical shortcomings, such as the fact that we have not proved the Fermi case as rigorously as the Bose case, like the Fermi commutation rules are not explictly verified, etc. Notwithstanding all these shortcomings, a case is to be made for the revision of entrenched dogma about Fermi and Luttinger liquids.

VI. CONCLUSIONS

Let us summarize the results obtained so far. We have succeeded in reducing to quadratures the propagators of both Bose and Fermi systems. We have also computed the momentum distribution of interacting Fermi systems and written down a formula for the quasiparticle residue in terms of the electron-electron repulsion. From this we obtain a criterion for the breakdown of Fermi-liquid behavior. The results we obtain contradict some widely held views about 1D systems, in particular the Lieb-Mattis solution[3] of the Tomonaga-Luttinger model suggests that the momentum distribution of a 1D system with δ-function interactions exhibits no discontinuity at the Fermi momentum. This is in contrast with the results obtained above that does in fact exhibit such a discontinuity for sufficiently weak values of the coupling strength and is destroyed only for larger values of the coupling strength. We attribute this discrepency to assumptions used in the linearized dispersion model (i.e., Tomonaga-Luttinger model). Luttinger-liquid theory is based on the assumption that the low-energy behavior of the homogeneous interacting Fermi system in one dimension is correctly described by the exactly solvable Tomonaga-Luttinger model. Our results show that the important qualitative features of the homogeneous interacting Fermi system namely the presence or absence of a Fermi surface cannot be surmised by examining the properties of the Tomonaga-Luttinger model, especially when the interactions are weak.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor A. H. Castro-Neto and Professor D. K. Campbell for providing important references and encouragement and the former for useful discussions as well, and Professor A. J. Leggett for giving his valuable time and advice on matters related to the pursuit of this work and also for providing important references and for useful discussions. Thanks are also due to Professor Ilias E. Perakis for providing the authors with an important reference. We also acknowledge Professor P. W. Anderson for critically evaluating an early version of this article. We thank Dr. S. Chitanvis for correcting the authors' misreading of the Lieb-Mattis solution. This work was supported in part by ONR Grant No. N00014-90-J-1267 and the University of Illinois, Materials Research Laboratory, under Grant No. NSF/DMR-89-20539, and in part by the Department of Physics at the University of Illinois at Urbana-Champaign.

APPENDIX A

In this appendix we prove some assertions made earlier. First the definition of the condensate-displacement annhilation operator:

$$d_{\mathbf{q}/2}(\mathbf{q})=\left(\frac{1}{\sqrt{N_0}}\right)b_{\mathbf{0}}^{\dagger}b_{\mathbf{q}}. \tag{A1}$$

In order to define the quantity $\mathcal{O}=(1/\sqrt{N_0})$ in a manner acceptable to most physicists, we proceed as follows. $\mathcal{O}$ is defined to be that operator that commutes with the number operator N_0

$$[\mathcal{O},N_0]=0 \tag{A2}$$

in the basis in which N_0 is diagonal and possesses nonzero eigenvalues (not an unreasonable assumption considering the fact that even in the most strongly interacting systems N_0 is macroscopic, call them $\{N_0^r\}$), then the matrix elements of $\mathcal{O}$ in the same basis are going to be $1/\sqrt{N_0^r}$. Having thus provided all the matrix elements, the definition of $\mathcal{O}$ is complete. We have to now show that $d_{\mathbf{q}/2}(\mathbf{q})$ satisfies canonical Bose commutation rules. The simplest way of doing this is to use the polar decomposition of $b_{\mathbf{0}}$

$$b_{\mathbf{0}}=\exp(-iX_0)\sqrt{N_0}, \tag{A3}$$

where X_0 is the Hermitian operator canonically conjugate to $N_0=b_{\mathbf{0}}^{\dagger}b_{\mathbf{0}}$, that is, $[X_0,N_0]=i$. This decomposition correctly reproduces the Bose commutation rules of $b_{\mathbf{0}}$ and $b_{\mathbf{0}}^{\dagger}$. For example,

$$[b_{\mathbf{0}},b_{\mathbf{0}}^{\dagger}]=b_{\mathbf{0}}b_{\mathbf{0}}^{\dagger}-b_{\mathbf{0}}^{\dagger}b_{\mathbf{0}}=\exp(-iX_0)N_0\exp(iX_0)-N_0=1. \tag{A4}$$

This means that $d_{\mathbf{q}/2}(\mathbf{q})=z_0^{*}b_{\mathbf{q}}$, where $z_0^{*}=\exp(iX_0)$. Since, $[z_0,b_{\mathbf{q}}]=0$ and $[z_0,b_{\mathbf{q}}^{\dagger}]=0$, and $[z_0,z_0^{*}]=0$, it follows that $d_{\mathbf{q}/2}(\mathbf{q})$ and $b_{\mathbf{q}}$ both satisfy the same commutation rules since z_0^{*} now behaves effectively as a c number (as regards commutation rules with $b_{\mathbf{q}}$, $b_{\mathbf{q}}^{\dagger}$, and z_0. It is worthwhile pointing out

$$[d_{\mathbf{q}/2}(\mathbf{q}),N_0]\neq 0,$$

rather

$$[d_{\mathbf{q}/2}(\mathbf{q}),N]=0, \tag{A5}$$

though not obviously so. In order to prove this

$$[d_{\mathbf{q}/2}(\mathbf{q}),N]=[d_{\mathbf{q}/2}(\mathbf{q}),N_0]+\left[d_{\mathbf{q}/2}(\mathbf{q}),\sum_{\mathbf{q}'\neq 0} b^{\dagger}_{\mathbf{q}'}b_{\mathbf{q}'}\right]$$
$$=[\exp(iX_0),N_0]b_{\mathbf{q}}+\exp(iX_0)\sum_{\mathbf{q}'\neq 0}[b_{\mathbf{q}},b^{\dagger}_{\mathbf{q}'}b_{\mathbf{q}'}]$$
$$=[\exp(iX_0)N_0-N_0\exp(iX_0)]b_{\mathbf{q}}+\exp(iX_0)b_{\mathbf{q}}$$
$$=[iX_0,N_0]\exp(iX_0)b_{\mathbf{q}}+\exp(iX_0)b_{\mathbf{q}}=-\exp(iX_0)b_{\mathbf{q}}+\exp(iX_0)b_{\mathbf{q}}=0. \quad \text{(A6)}$$

Next, one would like to prove Eq. (3). For this we simply plug in definition Eq. (A1) into Eq. (3) and verify that is reduces to an identity. The details are as follows:

$$L_{\mathbf{k},\mathbf{q}}=N_0\delta_{\mathbf{k},0}\delta_{\mathbf{q},0}$$
$$+[\delta_{\mathbf{k+q/2},0}(\sqrt{N_0})d_{\mathbf{k}}(-\mathbf{q})+\delta_{\mathbf{k-q/2},0}d^{\dagger}_{\mathbf{k}}(\mathbf{q})(\sqrt{N_0})]$$
$$+d^{\dagger}_{(1/2)(\mathbf{k+q/2})}(\mathbf{k+q/2})d_{(1/2)(\mathbf{k-q/2})}(\mathbf{k-q/2}). \quad \text{(A7)}$$

The proof involves these cases:

(i) $\mathbf{k}=0$ and $\mathbf{q}=0$. In this case,

$$L_{\mathbf{0},\mathbf{0}}=N_0=b^{\dagger}_0 b_0. \quad \text{(A8)}$$

(ii) $\mathbf{k}+\mathbf{q}/2=0$ but $\mathbf{k}-\mathbf{q}/2\neq 0$

$$L_{\mathbf{k}=-\mathbf{q}/2,\mathbf{q}}=(\sqrt{N_0})d_{-\mathbf{q}/2}(-\mathbf{q})=b^{\dagger}_0 b_{-\mathbf{q}}. \quad \text{(A9)}$$

(iii) $\mathbf{k}-\mathbf{q}/2=0$ but $\mathbf{k}+\mathbf{q}/2\neq 0$

$$L_{\mathbf{k}=\mathbf{q}/2,\mathbf{q}}=d^{\dagger}_{\mathbf{q}/2}(\mathbf{q})(\sqrt{N_0})=b^{\dagger}_{\mathbf{q}}b_{\mathbf{0}}. \quad \text{(A10)}$$

(iii) $\mathbf{k}-\mathbf{q}/2\neq 0$ and $\mathbf{k}+\mathbf{q}/2\neq 0$

$$L_{\mathbf{k},\mathbf{q}}=d^{\dagger}_{(1/2)(\mathbf{k}+\mathbf{q}/2)}(\mathbf{k}+\mathbf{q}/2)d_{(1/2)(\mathbf{k}-\mathbf{q}/2)}(\mathbf{k}-\mathbf{q}/2)$$
$$=b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}\exp(-iX_0)\exp(iX_0)b_{\mathbf{k}-\mathbf{q}/2}$$
$$=b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}b_{\mathbf{k}-\mathbf{q}/2}. \quad \text{(A11)}$$

Therefore in all cases,

$$L_{\mathbf{k},\mathbf{q}}=b^{\dagger}_{\mathbf{k}+\mathbf{q}/2}b_{\mathbf{k}-\mathbf{q}/2}, \quad \text{(A12)}$$

and thus Eq. (3) follows. Finally, we would like to clarify the finite-temperature case. In particular, what is the chemical potential of the condensate-displacement bosons? Is it zero or is it the same as that of the parent bosons? The answer may be found by computing the thermodynamic expectation value of the number of bosons in the condensate N_0,

$$\langle N_0\rangle=N-\sum_{\mathbf{q}\neq 0}\langle d^{\dagger}_{(1/2)\mathbf{q}}(\mathbf{q})d_{(1/2)\mathbf{q}}(\mathbf{q})\rangle. \quad \text{(A13)}$$

We also know the answer from elementary considerations, it is

$$\langle N_0\rangle=N-\sum_{\mathbf{q}\neq 0}\frac{1}{\exp(\beta(\epsilon_{\mathbf{q}}-\mu))-1}, \quad \text{(A14)}$$

where μ is the chemical potential of the parent bosons. Then it follows that

$$\langle d^{\dagger}_{(1/2)\mathbf{q}}(\mathbf{q})d_{(1/2)\mathbf{q}}(\mathbf{q})\rangle=\frac{1}{\exp(\beta(\epsilon_{\mathbf{q}}-\mu))-1}. \quad \text{(A15)}$$

In other words, the chemical potential of the condensate displacement bosons is the same as that of the parent bosons

$$\mu_{\text{parent}}=\mu_{\text{cond/displ}}. \quad \text{(A16)}$$

APPENDIX B

In this appendix, we try to to make plausible the correspondence between the number-conserving product of two Fermi fields and the sea bosons. Let us rewrite the Bose case [Eq. (3)] more suggestively,

$$b^{\dagger}_{\mathbf{k+q/2}}b_{\mathbf{k-q/2}}=O(\mathbf{k})\delta_{\mathbf{q},\mathbf{0}}+[\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})+A^{\dagger}_{\mathbf{k}}(\mathbf{q})\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}]$$
$$+\sum_{\mathbf{q}_1}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$
$$-\sum_{\mathbf{q}_1}A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1). \quad \text{(B1)}$$

In the Bose case

$$A_{\mathbf{k}}(\mathbf{q})=\delta_{\mathbf{k}-\mathbf{q}/2,0}d_{\mathbf{q}/2}(\mathbf{q}) \quad \text{(B2)}$$

and

$$O(\mathbf{k})=N\delta_{\mathbf{k},0}. \quad \text{(B3)}$$

Observe that the suggestively extravagant notation in Eq. (B1) is meant to imply that a very similar relation holds in the Fermi case which we reproduce below:

$$c^{\dagger}_{\mathbf{k+q/2}}c_{\mathbf{k-q/2}}=O(\mathbf{k})\delta_{\mathbf{q},\mathbf{0}}+[\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}A_{\mathbf{k}}(-\mathbf{q})+A^{\dagger}_{\mathbf{k}}(\mathbf{q})\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}]$$
$$+\sum_{\mathbf{q}_1}A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)$$
$$-\sum_{\mathbf{q}_1}A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1). \quad \text{(B4)}$$

Here $A_{\mathbf{k}}(\mathbf{q})$ depends on two momentum labels unlike in the Bose case. This has to do with the fact the now $O(\mathbf{k})$ no longer has the simple structure we saw in the Bose case. We must now invert this relation and obtain a formula for the operator $A_{\mathbf{k}}(\mathbf{q})$. It is not at all clear that this object will behave like an exact boson annhilation operator. The alternative is to write down an ansatz for an exact boson in analogy with the Bose case and determine the unknown in the formula by imposing canonical Bose commutation rules:

$$a_{\mathbf{k}}(\mathbf{q})=\frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}}c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}M(\mathbf{k},\mathbf{q})c_{\mathbf{k}+\mathbf{q}/2}. \tag{B5}$$

The unknown operator $M(\mathbf{k},\mathbf{q})$ has to be related to some number-conserving Fermi bilinear by demanding that the operator $a_{\mathbf{k}}(\mathbf{q})$ obey canonical Bose commutation rules

$$[a_{\mathbf{k}}(\mathbf{q}),a_{\mathbf{k}'}(\mathbf{q}')]=0, \tag{B6}$$

$$[a_{\mathbf{k}}(\mathbf{q}),a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')]=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}. \tag{B7}$$

It is at present beyond the authors to arrive at a formula for $M(\mathbf{k},\mathbf{q})$. Notwithstanding this, it is still useful to capture some sort of an approximate correspondence like the one introduced in Sec. II. The relations written down there have the following positive features

(i) They recover the RPA dielectric function at zero and finite temperatures.

(ii) They capture the correct four-point and six-point functions at zero and finite temperatures.

(iii) The formula for the sea boson in Eq. (14) when plugged into the correspondence for the number operator in Eq. (12) gives an identity.

The only negative aspect of this correspondence is that the mutual commutation rules between the off-diagonal Fermi bilinears is recovered correctly only up to terms linear in the sea bosons. That is, somehow the operators on the right side of these commutations rules should not be too different from their approximations obtained by dropping terms higher than the linear order. This is no doubt a strong assumption. This is in fact equivalent to RPA (perhaps even better than RPA).

The definition of the sea boson is incomplete without a prescription for the phase $\theta(\mathbf{k},\mathbf{q})$. In order to derive an expression for this, we again make heavy use of the Bose case which we have proved rigorously in Appendix A. There we found that plugging in the expression for the condensate-displacement boson into the correspondence resulted in an identity when $\mathbf{q}\neq 0$ (the $\mathbf{q}=0$ case being special). This identity comes about in a very specific fashion. In the general form of the correspondence outlined in Eq. (B1), we find that the sum on the right that comes with a negative sign is identically zero (for $\mathbf{q}\neq 0$) and the sum on the right that comes with a positive sign is equal to the left-hand side, except in ''rare'' cases when either $\mathbf{k}+\mathbf{q}/2=0$ or $\mathbf{k}-\mathbf{q}/2=0$. We shall adopt the same approach in the Fermi case and try to fix the phase $\theta(\mathbf{k},\mathbf{q})$ such that the identity is satisfied in the manner just described. Let us now write down the potential identity,

$$\begin{aligned}
c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}=&\Lambda_{\mathbf{k}}(\mathbf{q})\frac{1}{\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}}c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}\left(\frac{n^{\beta}(\mathbf{k}+\mathbf{q}/2)}{\langle N\rangle}\right)^{1/2}e^{i\theta(\mathbf{k},-\mathbf{q})}c_{\mathbf{k}-\mathbf{q}/2}\\
&+\Lambda_{\mathbf{k}}(-\mathbf{q})c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}e^{-i}\theta(\mathbf{k},\mathbf{q})\left(\frac{n^{\beta}(\mathbf{k}-\mathbf{q}/2)}{\langle N\rangle}\right)^{1/2}c_{\mathbf{k}-\mathbf{q}/2}\frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}}\\
&+T_1(\mathbf{k},\mathbf{q})c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}\left(\sum_{\mathbf{q}_1\neq\mathbf{q},\mathbf{0}}\frac{n^{\beta}(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1)}{\langle N\rangle}e^{i\theta(\mathbf{k}-\mathbf{q}_1/2,-\mathbf{q}+\mathbf{q}_1)}e^{-i\theta(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2,\mathbf{q}_1)}\right)c_{\mathbf{k}-\mathbf{q}/2}\\
&-T_2(\mathbf{k},\mathbf{q})c_{\mathbf{k}-\mathbf{q}/2}\frac{1}{\sqrt{n_{\mathbf{k}-\mathbf{q}/2}}}\frac{1}{\sqrt{n_{\mathbf{k}+\mathbf{q}/2}}}c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}\left(\frac{n^{\beta}(\mathbf{k}+\mathbf{q}/2)}{\langle N\rangle}\right)^{1/2}\left(\frac{n^{\beta}(\mathbf{k}-\mathbf{q}/2)}{\langle N\rangle}\right)^{1/2}\\
&\times\sum_{\mathbf{q}_1\neq\mathbf{q},\mathbf{0}}n_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1}e^{i\theta(\mathbf{k}+\mathbf{q}_1/2,-\mathbf{q}+\mathbf{q}_1)}e^{-i\theta(\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2,\mathbf{q}_1)}.
\end{aligned} \tag{B8}$$

Here, since we are not involved in proving the rigorous correspondence, but just the salient features, we are entitled to some leeway. In particular, we shall turn a blind eye to the fact that there exist these objects $T_1(\mathbf{k},\mathbf{q})$ and $T_2(\mathbf{k},\mathbf{q})$, in fact set them both equal to unity, just for the moment. The exact correspondence in terms of the $A_{\mathbf{k}}(\mathbf{q})$ seems to suggest exactly this. Then we find that, if we choose our $\theta(\mathbf{k},\mathbf{q})$ to be such that

$$\theta(\mathbf{k}-\mathbf{q}_1/2,-\mathbf{q}+\mathbf{q}_1)=\theta(\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2,\mathbf{q}_1) \tag{B9}$$

and

$$\sum_{\mathbf{q}_1\neq\mathbf{0},\mathbf{q}}\bar{n}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1}e^{i\theta(\mathbf{k}+\mathbf{q}_1/2,-\mathbf{q}+\mathbf{q}_1)}e^{-i\theta(\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2,\mathbf{q}_1)}=0, \tag{B10}$$

then all is well. Terms that were linear in the sea bosons are vanishingly small in the thermodynamic limit, and are important only when both the sums on the right side are identically zero for some reason, that is, it is ''rarely'' important just like in the Bose case. It is not really important to write down an explicit formula for the phase function $\theta(\mathbf{k},\mathbf{q})$, it is merely sufficient to show that it does what is required of it, namely, it provides the ''random phase'' that cancels terms that enable the whole machinery to run smoothly. Lastly, we have not yet verified that this sea boson obeys canonical commutation rules. This is again a tricky problem, it is likely to be resolved by the exact approach which is beyond the scope of this article. It is merely sufficient to point out that this is likely to come about due to the strong likelihood that the phase $\theta(\mathbf{k},\mathbf{q})$ is actually a functional of the number operator.

The correspondence that we have just defended is nothing but a more elegant version of the correspondence introduced by the pioneers like Castro-Neto and Fradkin.[11] Any criticism that may be leveled against our approach may equally well be leveled against theirs. The only difference between our approach and theirs is that the single-particle properties which they are so fervently seeking are far more elegantly recovered by our approach since we do not linearize the bare fermion dispersion or use the clumsy Luther construction.[8] Indeed, we have even shown that the answers for the 1D case are different from the Tomonaga-Luttinger model that linearize the bare fermion dispersion.

The other issue worth addressing at this stage is the validity of the prescription in Eq. (49). It can be seen from the exact correspondence in Eq. (B4) that as $\mathbf{q}\rightarrow\mathbf{0}$ terms that correspond to corrections to the RPA form of the full Hamiltonian vanish at least as fast as $|\mathbf{q}|/k_f$. The RPA terms themselves do not vanish and tend toward $[\lim_{\mathbf{q}\rightarrow\mathbf{0}}A_{\mathbf{k}}(-\mathbf{q})\neq 0$ as in the Bose case]

$$\sum_{\mathbf{k}}\ \sqrt{n_{\mathbf{k}}}A_{\mathbf{k}}(-\mathbf{q})+\sum_{\mathbf{k}}\ A^{\dagger}_{\mathbf{k}}(\mathbf{q})\sqrt{n_{\mathbf{k}}}. \tag{B11}$$

In order for the prescription in Eq. (49) to be accurate, it is important for the interaction $v_{\mathbf{q}}$ to possess these properties, but first it must vanish for large enough $\mathbf{q}$ (or small interparticle separation)

$$\lim_{|\mathbf{q}|\rightarrow c_0k_f}v_{\mathbf{q}}\rightarrow 0, \tag{B12}$$

where c_0 is small compared to unity and positive. This ensures that the only possible contributions come from small $\mathbf{q}$ where corrections to the RPA form themselves are small. In addition, if we also make sure that the interaction vanishes fast enough for large interparticle separations so that

$$\lim_{|\mathbf{q}|\rightarrow 0}v_{\mathbf{q}}\rightarrow|\mathbf{q}|^{D}, \tag{B13}$$

where $D=0,1,2,\ldots$ (larger the better), then our formalism is in fact exact as $k_f\rightarrow\infty$ (or sufficiently large). It may be argued that this state of affairs is most likely uninteresting since it may not be realizable in practice, when it is, it merely leads to a Fermi liquid. This is a valid point. But it is worth pointing out that non-Fermi-liquid behavior can still emerge in such systems when the interaction strength (with the same functional form) becomes strong enough. These considerations also tell us that for an interaction of the δ function type in 1D, provided the strength is weak enough, we have a Fermi liquid in contrast to the Lieb-Mattis solution of the Tomonaga-Luttinger model.

In any event, the philosophy is that having introduced sea bosons, we more or less forget about the fact that it was fermions that motivated their introduction in the first place, and instead try to write down a whole new set of models in terms of the sea bosons and calibrate them appropriately so that they capture the salient features of the real world. It is not a tautology to remark that we have in our hands a whole class of exactly solvable models of correlated fermions that is easier to use than mean-field theory itself but with capture effects significantly beyond diagrammatic perturbation theory, like the nonanalytic dependence of the momentum distribution on the coupling strength (written down in Appendix D).

APPENDIX C

In this appendix we demonstrate that the RPA dielectric function is recovered exactly by selectively retaining parts of the Coulomb interaction that lead to RPA. We know that the kinetic energy in the Bose language is given by

$$H_{\rm kin}=\sum_{\mathbf{k},\mathbf{q}}\left(\frac{\mathbf{k}\cdot\mathbf{q}}{m}\right)a^{\dagger}_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q}). \tag{C1}$$

For this let us choose

$$H_I=\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\tilde{\rho}_{\mathbf{q}}\tilde{\rho}_{-\mathbf{q}}, \tag{C2}$$

where

$$\tilde{\rho}_{\mathbf{q}}=\sum_{\mathbf{k}}\ [\Lambda_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}}(-\mathbf{q})+\Lambda_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}}(\mathbf{q})]. \tag{C3}$$

From this it may be shown that the RPA dielectric function is recovered as the following demonstration shows. Assume that a weak time-varying external perturbation is applied as shown below

$$H_{\rm ext}=\sum_{\mathbf{q}\neq 0}[U_{\rm ext}(\mathbf{q},t)+U^{*}_{\rm ext}(-\mathbf{q},t)]\tilde{\rho}_{\mathbf{q}}, \tag{C4}$$

where

$$U_{\rm ext}(\vec{r},t)=U_0e^{i\mathbf{q}\cdot\vec{r}-i\omega t}. \tag{C5}$$

Let us now write down the equations of motion for the various Bose fields

$$i\frac{\partial}{\partial t}\langle a^{t}_{\mathbf{k}}(\mathbf{q})\rangle=\omega_{\mathbf{k}}(\mathbf{q})\langle a^{t}_{\mathbf{k}}(\mathbf{q})\rangle+\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\sum_{\mathbf{k}'}\ [\Lambda_{\mathbf{k}'}(-\mathbf{q})\langle a^{t}_{\mathbf{k}'}(\mathbf{q})\rangle+\Lambda_{\mathbf{k}'}(\mathbf{q})\langle a^{t\dagger}_{\mathbf{k}'}(-\mathbf{q})\rangle]$$
$$+[U_{\rm ext}(\mathbf{q},t)+U^{*}_{\rm ext}(-\mathbf{q},t)]\Lambda_{\mathbf{k}}(-\mathbf{q}), \tag{C6}$$

$$-i\frac{\partial}{\partial t}\langle a^{t\dagger}_{\mathbf{k}}(-\mathbf{q})\rangle=\omega_{\mathbf{k}}(-\mathbf{q})\langle a^{t\dagger}_{\mathbf{k}}(-\mathbf{q})\rangle+\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(\mathbf{q})\sum_{\mathbf{k}'}\ [\Lambda_{\mathbf{k}'}(-\mathbf{q})\langle a^{t}_{\mathbf{k}'}(\mathbf{q})\rangle+\Lambda_{\mathbf{k}'}(\mathbf{q})\langle a^{t\dagger}_{\mathbf{k}'}(-\mathbf{q})\rangle]$$
$$+[U_{\rm ext}(\mathbf{q},t)+U^{*}_{\rm ext}(-\mathbf{q},t)]\Lambda_{\mathbf{k}}(\mathbf{q}). \tag{C7}$$

Now, let us decompose the expectation values as follows:

$$\langle a^t_{\mathbf{k}}(\mathbf{q})\rangle = U_{\rm ext}(\mathbf{q},t)C_{\mathbf{k}}(\mathbf{q}) + U^*_{\rm ext}(-\mathbf{q},t)D_{\mathbf{k}}(\mathbf{q}), \tag{C8}$$

$$\langle a^{t\dagger}_{\mathbf{k}}(-\mathbf{q})\rangle = U^*_{\rm ext}(-\mathbf{q},t)C^*_{\mathbf{k}}(-\mathbf{q}) + U_{\rm ext}(\mathbf{q},t)D^*_{\mathbf{k}}(-\mathbf{q}). \tag{C9}$$

The coefficients $C_{\mathbf{k}}(\mathbf{q})$ and $D^*_{\mathbf{k}}(-\mathbf{q})$ satisfy

$$\omega C_{\mathbf{k}}(\mathbf{q}) = \omega_{\mathbf{k}}(\mathbf{q})C_{\mathbf{k}}(\mathbf{q}) + \left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\sum_{\mathbf{k}'}\,[\Lambda_{\mathbf{k}'}(-\mathbf{q})C_{\mathbf{k}'}(\mathbf{q}) + \Lambda_{\mathbf{k}'}(\mathbf{q})D^*_{\mathbf{k}'}(-\mathbf{q})] + \Lambda_{\mathbf{k}}(-\mathbf{q}), \tag{C10}$$

$$\omega D^*_{\mathbf{k}}(-\mathbf{q}) = \omega_{\mathbf{k}}(-\mathbf{q})D^*_{\mathbf{k}}(-\mathbf{q}) + \left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(\mathbf{q})\sum_{\mathbf{k}'}\,[\Lambda_{\mathbf{k}'}(\mathbf{q})D^*_{\mathbf{k}'}(-\mathbf{q}) + \Lambda_{\mathbf{k}'}(-\mathbf{q})C_{\mathbf{k}'}(\mathbf{q})] + \Lambda_{\mathbf{k}}(\mathbf{q}). \tag{C11}$$

Now, the effective potential may be written as

$$U_{\rm eff}(\mathbf{q},t) = U_{\rm ext}(\mathbf{q},t) + \left(\frac{v_{\mathbf{q}}}{V}\right)\langle\rho_{-\mathbf{q}}\rangle' U_{\rm ext}(\mathbf{q},t), \tag{C12}$$

where

$$\langle\rho_{-\mathbf{q}}\rangle = U_{\rm ext}(\mathbf{q},t)\langle\rho_{-\mathbf{q}}\rangle' + U^*_{\rm ext}(-\mathbf{q},t)\langle\rho_{-\mathbf{q}}\rangle'', \tag{C13}$$

using the fact that

$$\langle\rho_{-\mathbf{q}}\rangle' = \sum_{\mathbf{k}}\,\Lambda_{\mathbf{k}}(-\mathbf{q})C_{\mathbf{k}}(\mathbf{q}) + \sum_{\mathbf{k}}\,\Lambda_{\mathbf{k}}(\mathbf{q})D^*_{\mathbf{k}}(-\mathbf{q}). \tag{C14}$$

Solving these equations and using the fact that the dielectric function is just the ratio of the external divided by the effective potential we get

$$\epsilon(\mathbf{q},\omega) = \frac{U_{\rm ext}(\mathbf{q},t)}{U_{\rm eff}(\mathbf{q},t)} = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\,\frac{n_F(\mathbf{k}+\mathbf{q}/2) - n_F(\mathbf{k}-\mathbf{q}/2)}{\omega - \mathbf{k}\cdot\mathbf{q}/m}, \tag{C15}$$

which is nothing but the RPA dielectric function of Bohm and Pines.

APPENDIX D

In this appendix we use the equation of motion approach to solve for the momentum distribution and compare it with the solution obtained via exact diagonalization as described in the main text. The equations of motion for the Bose propagators read as

$$\left(i\frac{\partial}{\partial t} - \omega_{\mathbf{k}}(\mathbf{q})\right)\frac{-i\langle Ta^t_{\mathbf{k}}(\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}$$
$$= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}\delta(t) + \left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\sum_{\mathbf{k}''}\left[\Lambda_{\mathbf{k}''}(-\mathbf{q})\frac{-i\langle Ta^t_{\mathbf{k}''}(\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle} + \Lambda_{\mathbf{k}''}(\mathbf{q})\frac{-i\langle Ta^{\dagger t}_{\mathbf{k}''}(-\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}\right], \tag{D1}$$

$$\left(i\frac{\partial}{\partial t} + \omega_{\mathbf{k}}(-\mathbf{q})\right)\frac{-i\langle Ta^{\dagger t}_{\mathbf{k}}(-\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}$$
$$= -\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(\mathbf{q})\sum_{\mathbf{k}''}\left[\Lambda_{\mathbf{k}''}(\mathbf{q})\frac{-i\langle Ta^{\dagger t}_{\mathbf{k}''}(-\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle} + \Lambda_{\mathbf{k}''}(-\mathbf{q})\frac{-i\langle Ta^t_{\mathbf{k}''}(\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}\right]. \tag{D2}$$

The boundary conditions on these propagators may be written down as (for interacting systems $\mu_B=0$)

$$\frac{-i\langle Ta^{\dagger t}_{\mathbf{k}}(-\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle} = \frac{-i\langle Ta^{\dagger(t-i\beta)}_{\mathbf{k}}(-\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}, \tag{D3}$$

$$\frac{-i\langle Ta^t_{\mathbf{k}}(\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle} = \frac{-i\langle Ta^{(t-i\beta)}_{\mathbf{k}}(\mathbf{q})a^\dagger_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}, \tag{D4}$$

$$\delta(t)=\left(\frac{1}{-i\beta}\right)\sum_{n}\ \exp(\omega_n t), \tag{D5}$$

$$\theta(t)=\left(\frac{1}{-i\beta}\right)\sum_{n}\ \frac{\exp(\omega_n t)}{\omega_n}. \tag{D6}$$

The boundary conditions imply that we may write

$$\frac{-i\langle Ta^{t}_{\mathbf{k}}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}=\sum_{n}\ \exp(\omega_n t)\frac{-i\langle Ta^{n}_{\mathbf{k}}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}, \tag{D7}$$

$$\frac{-i\langle Ta^{\dagger t}_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}=\sum_{n}\ \exp(\omega_n t)\frac{-i\langle Ta^{\dagger n}_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}, \tag{D8}$$

and, $\omega_n=(2\pi n)/\beta$. Thus,

$$[i\omega_n-\omega_{\mathbf{k}}(\mathbf{q})]\frac{-i\langle Ta^{n}_{\mathbf{k}}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}$$
$$=\frac{\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}}{-i\beta}+\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\sum_{\mathbf{k}''}\left[\Lambda_{\mathbf{k}''}(-\mathbf{q})\frac{-i\langle Ta^{n}_{\mathbf{k}''}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}+\Lambda_{\mathbf{k}''}(\mathbf{q})\frac{-i\langle Ta^{\dagger n}_{\mathbf{k}''}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}\right], \tag{D9}$$

$$[i\omega_n+\omega_{\mathbf{k}}(-\mathbf{q})]\frac{-i\langle Ta^{\dagger n}_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}$$
$$=-\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(\mathbf{q})\sum_{\mathbf{k}''}\left[\Lambda_{\mathbf{k}''}(\mathbf{q})\frac{-i\langle Ta^{\dagger n}_{\mathbf{k}''}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}+\Lambda_{\mathbf{k}''}(-\mathbf{q})\frac{-i\langle Ta^{n}_{\mathbf{k}''}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}\right], \tag{D10}$$

Define

$$\sum_{\mathbf{k}}\ \Lambda_{\mathbf{k}}(-\mathbf{q})\frac{-i\langle Ta^{n}_{\mathbf{k}}(\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}=G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n), \tag{D11}$$

$$\sum_{\mathbf{k}}\ \Lambda_{\mathbf{k}}(\mathbf{q})\frac{-i\langle Ta^{\dagger n}_{\mathbf{k}}(-\mathbf{q})a^{\dagger}_{\mathbf{k}'}(\mathbf{q}')\rangle}{\langle T1\rangle}=G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n). \tag{D12}$$

Multiplying the above equations with $\Lambda_{\mathbf{k}}(-\mathbf{q})$ and summing over $\mathbf{k}$ one arrives at simple formulas for G_1 and G_2:

$$G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=\Lambda_{\mathbf{k}'}(-\mathbf{q})\frac{\delta_{\mathbf{q},\mathbf{q}'}}{-i\beta[i\omega_n-\omega_{\mathbf{k}'}(\mathbf{q})]}+f_n(\mathbf{q})[G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)+G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)] \tag{D13}$$

and

$$G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=f^{*}_n(-\mathbf{q})[G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)+G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)], \tag{D14}$$

$$G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=\frac{f^{*}_n(-\mathbf{q})}{[1-f^{*}_n(-\mathbf{q})]}G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n),$$

$$G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)+G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)/[1-f^{*}_n(-\mathbf{q})],$$

$$G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=\left(\frac{1}{-i\beta}\right)\frac{[1-f^{*}_n(-\mathbf{q})]\Lambda_{\mathbf{k}'}(-\mathbf{q})\delta_{\mathbf{q},\mathbf{q}'}}{[1-f^{*}_n(-\mathbf{q})-f_n(\mathbf{q})][i\omega_n-\omega_{\mathbf{k}'}(\mathbf{q})]}, \tag{D15}$$

$$G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=\left(\frac{1}{-i\beta}\right)\frac{f^{*}_n(-\mathbf{q})\Lambda_{\mathbf{k}'}(-\mathbf{q})\delta_{\mathbf{q},\mathbf{q}'}}{[1-f^{*}_n(-\mathbf{q})-f_n(\mathbf{q})][i\omega_n-\omega_{\mathbf{k}'}(\mathbf{q})]}, \tag{D16}$$

$$G_1(\mathbf{q},\mathbf{k}',\mathbf{q}';n)+G_2(\mathbf{q},\mathbf{k}',\mathbf{q}';n)=\left(\frac{1}{-i\beta}\right)\frac{\Lambda_{\mathbf{k}'}(-\mathbf{q})\,\delta_{\mathbf{q},\mathbf{q}'}}{[1-f_n^*(-\mathbf{q})-f_n(\mathbf{q})][i\omega_n-\omega_{\mathbf{k}'}(\mathbf{q})]},$$

$$\frac{-i\langle Ta_{\mathbf{k}}^n(\mathbf{q})a_{\mathbf{k}'}^\dagger(\mathbf{q}')\rangle}{\langle T1\rangle}=\frac{\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'}}{-i\beta(i\omega_n-\omega_{\mathbf{k}}(\mathbf{q}))}+\left(\frac{1}{-i\beta}\right)\left(\frac{v_{\mathbf{q}}}{V}\right)\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{[i\omega_n-\omega_{\mathbf{k}}(\mathbf{q})]}\,\frac{\Lambda_{\mathbf{k}'}(-\mathbf{q})\,\delta_{\mathbf{q},\mathbf{q}'}}{[1-f_n^*(-\mathbf{q})-f_n(\mathbf{q})][i\omega_n-\omega_{\mathbf{k}'}(\mathbf{q})]}. \tag{D17}$$

The zero-temperature correlation function of significance here is

$$-i\langle a_{\mathbf{k}'}^\dagger(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q})\rangle. \tag{D18}$$

This may be obtained from the above formulas as

$$-i\langle a_{\mathbf{k}'}^\dagger(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q})\rangle=-\left(\frac{v_{\mathbf{q}}}{V}\right)\Lambda_{\mathbf{k}}(-\mathbf{q})\Lambda_{\mathbf{k}'}(-\mathbf{q})\,\delta_{\mathbf{q},\mathbf{q}'}\int_C\frac{d\omega}{2\pi i}\,\frac{1}{[i\omega-\omega_{\mathbf{k}}(\mathbf{q})][i\omega-\omega_{\mathbf{k}'}(\mathbf{q})][1-f_n^*(-\mathbf{q})-f_n(\mathbf{q})]}, \tag{D19}$$

where C is the positively oriented contour that encloses the upper half-plane [upper half-plane, because we need $\langle a_{\mathbf{k}'}^\dagger(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q})\rangle$ and not $\langle a_{\mathbf{k}}(\mathbf{q})a_{\mathbf{k}'}^\dagger(\mathbf{q}')\rangle$]. Thus the problem now reduces to computing all the zeros of $[1-f_n^*(-\mathbf{q})-f_n(\mathbf{q})]$ that have positive imaginary parts. It may be shown quite easily that

$$\epsilon_{\mathrm{RPA}}(\mathbf{q},i\omega_n)=1-f_n^*(-\mathbf{q})-f_n(\mathbf{q}). \tag{D20}$$

In 1D, the dielectric function is evaluated as follows:

$$1-f_n^*(-q)-f_n(q)=1+v_q\left(\frac{1}{2\pi}\right)\left(\frac{m}{q}\right)\ln\left[\frac{(k_f+q/2)^2+(m\omega/q)^2}{(k_f-q/2)^2+(m\omega/q)^2}\right]=0. \tag{D21}$$

This leads to the root

$$\omega=i\left(\frac{|q|}{m}\right)\sqrt{\frac{(k_f+q/2)^2-(k_f-q/2)^2\exp(-(2\pi q/m)(1/v_q))}{1-\exp(-(2\pi q/m)(1/v_q))}}. \tag{D22}$$

Therefore the final result may be written as

$$\langle a_{\mathbf{k}'}^\dagger(\mathbf{q}')a_{\mathbf{k}}(\mathbf{q})\rangle=\left(\frac{1}{V}\right)\frac{\Lambda_k(-q)\Lambda_{k'}(-q)\,\delta_{q,q'}}{[\omega_R(q)+\omega_k(q)][\omega_R(q)+\omega_{k'}(q)](m/q^2)(1/2\pi k_f)2(m/q)^2\omega_R(q)[\cosh(\lambda(q))-1]}, \tag{D23}$$

where

$$\lambda(q)=\left(\frac{2\pi q}{m}\right)\left(\frac{1}{v_q}\right), \tag{D24}$$

$$\omega_R(q)=\left(\frac{|q|}{m}\right)\sqrt{\frac{(k_f+q/2)^2-(k_f-q/2)^2\exp(-\lambda(q))}{1-\exp(-\lambda(q))}}. \tag{D25}$$

In other words,

$$\langle c_k^\dagger c_k\rangle=n_F(k)+(2\pi k_f)\int_{-\infty}^{+\infty}\frac{dq_1}{2\pi}\,\frac{\Lambda_{k-q_1/2}(-q_1)}{2\omega_R(q_1)[\omega_R(q_1)+\omega_{k-q_1/2}(q_1)]^2(m^3/q_1^4)[\cosh(\lambda(q_1))-1]}$$
$$-(2\pi k_f)\int_{-\infty}^{+\infty}\frac{dq_1}{2\pi}\,\frac{\Lambda_{k+q_1/2}(-q_1)}{2\omega_R(q_1)[\omega_R(q_1)+\omega_{k+q_1/2}(q_1)]^2(m^3/q_1^4)[\cosh(\lambda(q_1))-1]}. \tag{D26}$$

Note that the above formula possesses a nonanalytic dependence in the coupling strength $[\cosh(2\pi q/m)(1/v_q)-1]$, an unmistakable signature of a nondigrammatic result. Next, we would like to provide formulas for the momentum distribution when we use the correct interacting expectation values in the Fermi-bilinear sea-boson correspondence. The results obtained from these formulas are likely to be very different from the weakly nonideal case, which in any case is not very interesting. The answers are given below:

$$\overline{n}_{\mathbf{k}}=\frac{n^\beta(\mathbf{k})}{S_1(\mathbf{k})}+\frac{S_2(\mathbf{k})}{S_1(\mathbf{k})}, \tag{D27}$$

where

$$S_1(\mathbf{k})=1+\sum_{\mathbf{q},i}\frac{\overline{n}_{\mathbf{k}-\mathbf{q}}}{[\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m-\epsilon_{\mathbf{q}}]^2}g_i^2(-\mathbf{q})+\sum_{\mathbf{q},i}\frac{1-\overline{n}_{\mathbf{k}+\mathbf{q}}}{[\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m+\epsilon_{\mathbf{q}}]^2}g_i^2(-\mathbf{q}), \tag{D28}$$

$$S_2(\mathbf{k})=\sum_{\mathbf{q},i}\frac{\overline{n}_{\mathbf{k}-\mathbf{q}}}{[\tilde{\omega}_i(-\mathbf{q})+\mathbf{k}\cdot\mathbf{q}/m-\epsilon_{\mathbf{q}}]^2}g_i^2(-\mathbf{q}), \tag{D29}$$

also the form of the ''RPA'' dielectric function and its zeros $\tilde{\omega}_i(\mathbf{q})$ are now different. The RPA dielectric function is given by

$$\epsilon_{\mathrm{RPA}}(\mathbf{q},\omega)=1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\overline{n}_{\mathbf{k}+\mathbf{q}/2}-\overline{n}_{\mathbf{k}-\mathbf{q}/2}}{\omega-\mathbf{k}\cdot\mathbf{q}/m}, \tag{D30}$$

$$g_i(\mathbf{q})=\left[\sum_{\mathbf{k}}\frac{\overline{n}_{\mathbf{k}-\mathbf{q}/2}-\overline{n}_{\mathbf{k}+\mathbf{q}/2}}{(\tilde{\omega}_i(\mathbf{q})-\mathbf{k}\cdot\mathbf{q}/m)^2}\right]^{-1/2}. \tag{D31}$$

The commutators are given as before, except for three changes. In the new approach

$$\Lambda_{\mathbf{k}}(\mathbf{q})=\sqrt{\overline{n}_{\mathbf{k}+\mathbf{q}/2}(1-\overline{n}_{\mathbf{k}-\mathbf{q}/2})}. \tag{D32}$$

Next, the zeros are slightly different. The collective mode has to be computed self-consistently, whereas the particle-hole mode may be written down as described earlier,

$$\tilde{\omega}_i(\mathbf{q})=\omega_{\mathbf{k}_i}(\mathbf{q})+\left(\frac{v_{\mathbf{q}}}{V}\right)\frac{\Lambda_{\mathbf{k}_i}^2(-\mathbf{q})}{1+(v_{\mathbf{q}}/V)\sum_{\mathbf{k}}\frac{\Lambda_{\mathbf{k}}^2(\mathbf{q})}{\omega_{\mathbf{k}_i}(\mathbf{q})+\omega_{\mathbf{k}}(-\mathbf{q})}-\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}\neq\mathbf{k}_i}\frac{\Lambda_{\mathbf{k}}^2(-\mathbf{q})}{\omega_{\mathbf{k}_i}(\mathbf{q})-\omega_{\mathbf{k}}(\mathbf{q})}}. \tag{D33}$$

The last change is in the form of $U_0(\mathbf{q})$, here we have to make sure we use the finite-temperature noninteracting values. The other issue that is also of interest is whether the momentum distribution evaluated using the Fermi-bilinear/sea-boson correspondence is the same as that suggested by the full propagator. We have found that the answer to this is difficult and probably in the negative. This does not mean that the whole program is wrong. Some comfort and confidence in these manipulations may still be retained by demonstrating that the expression for the number operator is consistent with the RPA form of the Fermi creation operator. Again here, we have to be content with a weak form of this requirement. We take the point of view that it is sufficient to show that the commutator between the total momentum of the electrons and the field operator comes out the same in both the original Fermi language and in the sea-boson language. The total momentum of the electrons has the form:

$$\mathbf{P}=\sum_{\mathbf{k}}\mathbf{k}c_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}. \tag{D34}$$

In the sea-boson language, it takes the form

$$\mathbf{P}=\sum_{\mathbf{k},\mathbf{q}}\mathbf{q}a_{\mathbf{k}}^{\dagger}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q}). \tag{D35}$$

Therefore in the original Fermi language we have

$$[\mathbf{P},\psi^{\dagger}(\mathbf{x})]=i\nabla\psi^{\dagger}(\mathbf{x}). \tag{D36}$$

In the sea-boson language we have

$$\begin{aligned}[\mathbf{P},\psi^{\dagger}(\mathbf{x})]&=\sum_{\mathbf{k},\mathbf{q}}\mathbf{q}[a_{\mathbf{k}}^{\dagger}(\mathbf{q})a_{\mathbf{k}}(\mathbf{q}),\psi^{\dagger}(\mathbf{x})]\\&=\sum_{\mathbf{k},\mathbf{q}}\mathbf{q}a_{\mathbf{k}}^{\dagger}(\mathbf{q})[-g_{\mathbf{k},\mathbf{q}}(\mathbf{x})]\psi^{\dagger}(\mathbf{x})\\&\quad+\sum_{\mathbf{k},\mathbf{q}}\mathbf{q}[-g_{\mathbf{k},\mathbf{q}}^{*}(\mathbf{x})]\psi^{\dagger}(\mathbf{x})a_{\mathbf{k}}(\mathbf{q})\\&=i\nabla\psi^{\dagger}(\mathbf{x}),\end{aligned}$$

as it should be. All this points to the fact that the answers for the momentum distribution and propagators should not be taken too literally, rather one must be content with the qualitative predictions that are most likely accurate, which also seem to contradict conventional wisdom.

APPENDIX E

In the late 1970s and early 1980s, attempts were made to write down field theories that describe scalar mesons in terms of observables like currents and densities rather than the creation and annhilation operators. The motivation for doing this stems from the fact that a theory cast directly in terms of observables was more physically intuitive than the more traditional approach based on raising and lowering operators on the Fock space. This attempt however, raised a number of technical questions, among them was how to make sense of the various identities connecting say the kinetic energy density to the currents and particle densities and so on. Elaborate mathematical machinery was erected by the authors who started this line of research[9] to address these issues. However, it seems gaps still remain especially with regard to the crucial question of how one goes about writing down a formula for the annhilation operator (Fermi or Bose) alone in terms of bilinears like currents and densities. The bilinears in question namely currents and densities satisfy a closed algebra known as the current algebra.[9] This algebra is insensitive to the nature of the statistics of the underlying fields. On the other hand, if one desires information about single-particle properties, it is necessary to relate the annhilation operator (whose commutation rules determine the statistics) to bilinears like currents and densities. That such a correspondence is possible was demonstrated by Goldin, Menikoff, and Sharp.[9] However, they have not explicitly written down such a formula nor have they clarified some important issues such as whether this formula changes when one consider interacting fields rather than free fields. The general belief is that these formulas are different for interacting fields. It is shown here that this is in fact not the case, interactions in the system merely cause a change in the Hamiltonian but do not affect how the annhilation operator is related to local currents and densities. The attempts made here are partly based on the work of Goldin *et al.*,[9] Ligouri and Mintchev on generalized statistics,[23] and the series by Reed and Simon.[24] As has been demonstrated earlier, for the Bose case we had to choose $\Phi=0$. We argued that this choice was unique. In the Fermi case the choice was different but was also unique due to the necessity of recovering the free theory. In this section, we write down a mathematically rigorous statement of this uniqueness criterion. This exercise also settles the issue regarding the delicate question of multiplying two operator-valued distributions at the same point and other related issues, like the meaning of the square root of the density distribution. For this we prove this lemma.

Lemma. Let $\mathcal{F}$ be a smooth function from a bounded subset of the real line on to the set of reals. Also let f and g be smooth functions from some bounded subset of $\mathcal{R}^d$ to reals. Let us further assume that the range of these functions are such that it is always possible to find compositions such as $\mathcal{F}o f$ and they will also be smooth functions with sufficiently big domains. They possess Fourier transforms since they are well behaved. If

$$\mathcal{F}(f(\mathbf{x}))=g(\mathbf{x}) \tag{E1}$$

and

$$f(\mathbf{x})=\sum_{\mathbf{q}}\tilde{f}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}, \tag{E2}$$

$$g(\mathbf{x})=\sum_{\mathbf{k}}\tilde{g}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{E3}$$

then the following also holds:

$$\left[\mathcal{F}\left(\sum_{\mathbf{q}}\tilde{f}_{\mathbf{q}}T_{-\mathbf{q}}(\mathbf{k})\right)\right]\delta_{\mathbf{k},\mathbf{0}}=\tilde{g}_{\mathbf{k}}, \tag{E4}$$

where $T_{\mathbf{q}}(\mathbf{k})=\exp(\mathbf{q}\cdot\nabla_{\mathbf{k}})$. Here the operator $T_{\mathbf{q}}(\mathbf{k})$ acts on the $\mathbf{k}$ in the Kronecker delta on the extreme right, and every time it translates the $\mathbf{k}$ by an amount $\mathbf{q}$.

Proof. Proof is by brute force expansion. We know

$$\mathcal{F}(y)=\sum_{n=0}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}y^n, \tag{E5}$$

therefore

$$\mathcal{F}(f(\mathbf{x}))=\mathcal{F}(0)+\sum_{n=1}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\sum_{\{\mathbf{q}_i\}}\tilde{f}_{\mathbf{q}_1}\tilde{f}_{\mathbf{q}_2}\dots\tilde{f}_{\mathbf{q}_n}\times\exp\left(i\left(\sum_{i=1}^{n}\mathbf{q}_i\right)\cdot\mathbf{x}\right)=\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{g}_{\mathbf{k}}. \tag{E6}$$

This means (take the inverse Fourier transform)

$$\mathcal{F}(0)\,\delta_{\mathbf{k},0}+\sum_{n=1}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\sum_{\{\mathbf{q}_i\}}\tilde{f}_{\mathbf{q}_1}\tilde{f}_{\mathbf{q}_2}\dots\tilde{f}_{\mathbf{q}_n}\delta_{\left(\mathbf{k}-\sum_{i=1}^{n}\mathbf{q}_i\right),\mathbf{0}}=\tilde{g}_{\mathbf{k}}. \tag{E7}$$

This may also be cleverly rewritten as

$$\mathcal{F}(0)\,\delta_{\mathbf{k},0}+\sum_{n=1}^{\infty}\frac{\mathcal{F}^{(n)}(0)}{n!}\sum_{\{\mathbf{q}_i\}}\tilde{f}_{\mathbf{q}_1}\tilde{f}_{\mathbf{q}_2}\dots\tilde{f}_{\mathbf{q}_n}T_{-\mathbf{q}_1}(\mathbf{k})\times T_{-\mathbf{q}_2}(\mathbf{k})\dots T_{-\mathbf{q}_n}(\mathbf{k})\,\delta_{\mathbf{k},\mathbf{0}}=\tilde{g}_{\mathbf{k}} \tag{E8}$$

and therefore,

$$\tilde{g}_{\mathbf{k}}=\left[\mathcal{F}\left(\sum_{\mathbf{q}}\tilde{f}_{\mathbf{q}}T_{-\mathbf{q}}(\mathbf{k})\right)\right]\delta_{\mathbf{k},\mathbf{0}} \tag{E9}$$

and the *Proof* is now complete.

Now we would like to capture the notion of the Fermi density operator. Physicists define it to be $\rho(x)=\psi^{*}(x)\psi(x)$. Multiplication of two Fermi fields at the same point is a delicate issue and we would like to make more sense out of it. For this we have to set our single-particle Hilbert Space:

$$\mathcal{H}=L_p(\mathcal{R}^3)\otimes\mathcal{W}.$$

Here, $L_p(\mathcal{R}^3)$ is the space of all periodic functions with period L in each space direction. $\mathcal{W}$ is the spin space spanned by two vectors. An orthonormal basis for $\mathcal{W}$ is

$$\{\xi_{\uparrow},\xi_{\downarrow}\}.$$

A typical element of $\mathcal{H}$ is given by $f(\mathbf{x})\otimes\xi_{\downarrow}$. A basis for $\mathcal{H}$ is given by

$$\mathcal{B}=\left\{\sqrt{\frac{1}{L^3}}\exp(i\mathbf{q_n}\cdot x)\otimes\xi_s\right\};\mathbf{n}=(n_1,n_2,n_3)\in\mathcal{Z}^3,s\in\{\uparrow,\downarrow\}.$$

We move on to the definition of the Fermi-density distribution. The Hilbert Space $\mathcal{H}^{\otimes n}$ is the space of all n-particle wave functions with no symmetry restrictions. From this we may construct orthogonal subspaces

$$\mathcal{H}_+^{\otimes n}=P_+\mathcal{H}^{\otimes n},$$

$$\mathcal{H}_-^{\otimes n}=P_-\mathcal{H}^{\otimes n}.$$

Tensors from $\mathcal{H}_+^{\otimes n}$ are orthogonal to tensors from $\mathcal{H}_-^{\otimes n}$. The only exceptions are when $n=0$ or $n=1$. The space $\mathcal{H}_+^{\otimes n}$ is the space of bosonic-wave functions and the space $\mathcal{H}_-^{\otimes n}$ is the space of fermionic wave functions. The definition of the Fermi-density distribution proceeds as follows. Let v be written as

$$v=\sum_{\sigma\in\{\uparrow,\downarrow\}}a(\sigma)\xi_\sigma.$$

The Fermi-density distibution is an operator on the Fock space, given a vector $f\otimes v\in\mathcal{H}$ in the single-particle Hilbert space, and a tensor φ in the n-particle subspace of $\mathcal{F}(\mathcal{H})$, there exists a corresponding operator $\rho(f\otimes v)$ that acts as follows:

$$[\rho(f\otimes v)\varphi]_n(\mathbf{x_1}\sigma_1,\mathbf{x_2}\sigma_2,\ldots,\mathbf{x_n}\sigma_n)=0,$$

if $\varphi\in\mathcal{H}_+^{\otimes n}$ and

$$[\rho(f\otimes v)\varphi]_n(\mathbf{x_1}\sigma_1,\mathbf{x_2}\sigma_2,\ldots,\mathbf{x_n}\sigma_n)=\sum_{i=1}^{n}f(\mathbf{x_i})a(\sigma_i)\varphi_n(\mathbf{x_1}\sigma_1,\mathbf{x_2}\sigma_2,\ldots,\mathbf{x_n}\sigma_n),$$

when $\varphi\in\mathcal{H}_-^{\otimes n}$. The physical meaning of this abstract operator will become clear soon. Let us now define the current density in an analogous fashion, To physicists, it is,

$$\mathbf{J}(\mathbf{x})=\left(\frac{1}{2i}\right)[\psi^\dagger(\nabla\psi)-(\nabla\psi)^\dagger\psi].\tag{E10}$$

To mathematicians it is an operator similar to the density.[24] Given a typical element $f\otimes v$ associated with the underlying single-particle Hilbert space, there is an operator denoted by $J_s(f\otimes v)$, $(s=1,2,3)$ that acts on a typical tensor from the n-particle subspace of the full Fock space as follows:

$$[J_s(f\otimes v)\varphi]_n(\mathbf{x}_1\sigma_1,\mathbf{x}_2\sigma_2,\ldots,\mathbf{x}_n\sigma_n)=0,\tag{E11}$$

if $\varphi\in\mathcal{H}_+^{\otimes n}$, and

$$[J_s(f\otimes v)\varphi]_n(\mathbf{x}_1\sigma_1,\mathbf{x}_2\sigma_2,\ldots,\mathbf{x}_n\sigma_n)=-i\sum_{k=1}^{n}\left\{f(\mathbf{x}_k)a(\sigma_k)\nabla_s^k+\frac{1}{2}[\nabla_s^k f(\mathbf{x}_k)]a(\sigma_k)\right\}\times\varphi_n(\mathbf{x}_1\sigma_1,\mathbf{x}_2\sigma_2,\ldots,\mathbf{x}_n\sigma_n),\tag{E12}$$

if $\varphi\in\mathcal{H}_-^{\otimes n}$. For the bosonic current it is the other way around. Having done all this, we would now like to write the DPVA more rigorously. Now for some notation. As before, let $g=\exp(i\mathbf{k}_m\cdot\mathbf{x})\otimes\xi_r$ (the square root of the volume is not needed as we want all operators in momentum space to be dimensionless). Then, as before

$$\psi(\mathbf{k}_m r)=c(g),\tag{E13}$$

$$\rho(\mathbf{k}_m r)=\rho(g),\tag{E14}$$

$$\delta\rho(\mathbf{k}_m r)=\rho(\mathbf{k}_m r)-N_r^0\delta_{\mathbf{k}_m,\mathbf{0}},\tag{E15}$$

$$j_s(\mathbf{k}_m r)=\mathbf{J}_s(g),\tag{E16}$$

$$\delta j_s(\mathbf{k}_m r)=j_s(\mathbf{k}_m r).\tag{E17}$$

Having done this, we would like to write down another formula for the canonical conjugate:

$$\nabla\Pi(\mathbf{x}\sigma)=(-1/\rho(\mathbf{x}\sigma))\mathbf{J}(\mathbf{x}\sigma)+\nabla\Phi([\rho];\mathbf{x}\sigma)-[-i\Phi,\nabla\Pi].\tag{E18}$$

Then we have [bear in mind here that we have distinguished between the c number N_r^0 and the operator $\rho(\mathbf{0}r)$ whose expectation value is N_r^0]

$$(i\mathbf{q}_m)X_{\mathbf{q}_m r}=-\left(\frac{1}{N_r^0}\right)\frac{1}{1+1/N_r^0\sum_{\mathbf{k}_n}\delta\rho(\mathbf{k}_n r)T_{\mathbf{k}_n}(\mathbf{q}_m)}\times\left[\sum_{\mathbf{p}_n}\delta\mathbf{j}(\mathbf{p}_n r)T_{\mathbf{p}_n}(\mathbf{q}_m)\right]\delta_{\mathbf{q}_m,\mathbf{0}}+\mathbf{F}([\rho];\mathbf{q}_m r),\tag{E19}$$

where

$$\sum_{\mathbf{q}_m}\exp(i\mathbf{q}_m\cdot\mathbf{x})\mathbf{F}([\rho];\mathbf{q}_m r)=\nabla\Phi-[-i\Phi,\nabla\Pi].\tag{E20}$$

As regards the object $X_{\mathbf{0}r}$ that is conjugate to the total number is concerned, we must retain it as it is, since, it will ensure that the total number when commuting with the field operator is the field operator itself rather than the incorrect answer zero. For $\mathbf{q}_m\neq\mathbf{0}$

$$X_{\mathbf{q}_m r}=\left(\frac{1}{q_m^2}\right)\left(\frac{i}{N_r^0}\right)\frac{1}{1+1/N_r^0\sum_{\mathbf{k}_n}\delta\rho(\mathbf{k}_n r)T_{\mathbf{k}_n}(\mathbf{q}_m)}\left[\sum_{\mathbf{p}_n}\mathbf{q}_m\cdot\delta\mathbf{j}(\mathbf{p}_n r)T_{\mathbf{p}_n}(\mathbf{q}_m)\right]\delta_{\mathbf{q}_m,\mathbf{0}}-\frac{i\mathbf{q}_m\cdot\mathbf{F}([\rho];\mathbf{q}_m r)}{q_m^2}. \tag{E21}$$

In order to define $X_{\mathbf{0}r}$ in terms of Fermi fields, we have to make use of the fact that this object does not commute with the total number of fermions. This means it cannot be expressed exclusively in terms of number-conserving Fermi bilinears like currents and densities. This will mean that we merely invert the formula in Eq. (30) and solve for $X_{\mathbf{0}r}$ as

$$X_{\mathbf{0}r}=-\sum_{\mathbf{k}_m\neq\mathbf{0}}X_{\mathbf{k}_m r}+i\sum_{\mathbf{k}_m}\ln\left[\left(\sqrt{N_r^0}+\sum_{\mathbf{q_n}}\delta\psi(\mathbf{q_n}r)T_{-\mathbf{q}_n}(\mathbf{k}_m)\right)\left(N_r^0+\sum_{\mathbf{q}_n}\delta\rho(\mathbf{q}_n r)T_{\mathbf{q}_n}(\mathbf{k}_m)\right)^{-1/2}\right.$$
$$\left.\times\exp\left(-i\sum_{\mathbf{q}_n}\times\phi([\rho];\mathbf{q}_n r)T_{\mathbf{q}_n}(\mathbf{k}_m)\right)\right]\delta_{\mathbf{k}_m,\mathbf{0}}. \tag{E22}$$

Define an operator which is defined to be the formal expansion that the formula itself suggests:

$$\tilde{\psi}(\mathbf{k}_n r)=\exp\left(-i\sum_{\mathbf{q}_m}T_{-\mathbf{q}_m}(\mathbf{k}_n)X_{\mathbf{q}_m r}\right)\times\exp\left(i\sum_{\mathbf{q}_m}T_{\mathbf{q}_m}(\mathbf{k}_n)\phi([\rho];\mathbf{q}_m r)\right)\times\left(N_r^0+\sum_{\mathbf{q}_m}\delta\rho(\mathbf{q}_m r)T_{\mathbf{q}_m}(\mathbf{k}_n)\right)^{1/2}\delta_{\mathbf{k}_n,\mathbf{0}}. \tag{E23}$$

We would now like to write down a statement that would require a proof. This conjecture when proven will vindicate the DPVA.

Conjecture.

There exists a unique functional $\Phi([\rho];\mathbf{x}r)$ and a unique odd (for fermions, even for bosons) integer m such that the following recursion holds:

$$\Phi([\{\rho(\mathbf{y_1}\sigma_1)-\delta(\mathbf{y_1}-\mathbf{x}')\delta_{\sigma_1,\sigma'}\}];\mathbf{x}\sigma)+\Phi([\rho];\mathbf{x}'\sigma')-\Phi([\rho];\mathbf{x}\sigma)-\Phi([\{\rho(\mathbf{y_1}\sigma_1)-\delta(\mathbf{y_1}-\mathbf{x})\delta_{\sigma_1,\sigma}\}];\mathbf{x}'\sigma')=m\pi, \tag{E24}$$

and has the following additional effects. The domain of definition of $\tilde{\psi}(\mathbf{k}_n r)$ (in which the series expansion converges) is the same as that of $\psi(\mathbf{k}_n r)$ and it acts the same way too. In other words

$$\tilde{\psi}(\mathbf{k}_n r)=\psi(\mathbf{k}_n r). \tag{E25}$$

We know how the ingredients of $\tilde{\psi}(\mathbf{k}_n r)$ namely the current $\mathbf{j}(\mathbf{k}_n r)$ and the density $\delta\rho(\mathbf{q}_n r)$ act on typical elements of the Fock space, and we know how $\psi(\mathbf{k}_n r)$ acts on the Fock space, we just have to show that the complicated $\tilde{\psi}(\mathbf{k}_n r)$ acts the same way as the simple $\psi(\mathbf{k}_n r)$. Moreover, this is true for a unique phase functional Φ. Lastly, we would like to defend the above ''Fourier gymnastics'' by pointing out that the real-space formulation is not well-defined due to the fact that the line integral that appears in the formulas is difficult to define, any attempt is equivalent to the above approach. The other reason for attempting a rigorous formulation is the fact well-known to mathematicians that it is not possible to have a self-adjoint canonical conjugate of a positive definite self-adjoint operator. Since ρ is positive definite, the natural question that arises is whether Π is self-adjoint? We take the naive physicist's approach to this issue, namely we allow for sign changes in ρ and argue that these merely amount to translating the phase functional Φ by constant amounts, thus not altering the general framework. Within this framework, Π is indeed self-adjoint and all is well. It is also worth remarking that the overall conjugate Π has two contributions, one from the position independent part $X_{0\sigma}$, and the other is from terms involving currents and densities. The latter contribution is manifestly self-adjoint. The possible lack of self-adjointness of the overall conjugate stems from the canonical conjugate of the total number which cannot be expressed in terms of Fermi bilinears.

*Electronic address: setlur@mrlxpa.mrl.uiuc.edu

[1] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).

[2] J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).

[3] D. C. Mattis and E. H. Lieb, J. Math. Phys. **6**, 304 (1965).

[4] K. Sawada, Phys. Rev. **106**, 372 (1957).

[5] J. Arponen and E. Pajanne, Ann. Phys. (Paris) **91**, 450 (1975); J. Phys. C **8**, L152 (1975); Ann. Phys. (Paris) **121**, 343 (1979).

[6] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).

[7] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, New York, 1987).

[8] A. Luther, Phys. Rev. B **19**, 320 (1979).

[9] G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. **21**, 650 (1980); R. Menikoff and D. H. Sharp, *ibid.* **18**, 471 (1977); R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).

[10] F. D. M. Haldane, J. Phys. C **14**, 2585 (1981); Helv. Phys. Acta **65**, 152 (1992); *Perspectives in Many-Particle Physics*, Proceedings of the International School of Physics ''Enrico Fermi'' Course CXXI, Varenna, 1992, edited by R. Schreiffer and R. A. Broglia (North-Holland, New York, 1994).

[11] A. H. Castro-Neto and E. Fradkin, Phys. Rev. Lett. **72**, 1393 (1994); Phys. Rev. B **49**, 10 877 (1994); **51**, 4084 (1995).

[12] A. Houghton and J. B. Marston, Phys. Rev. B **48**, 7790 (1993); A. Houghton, H.-J. Kwon, and J. B. Marston, *ibid.* **50**, 1351 (1994);

A. Houghton, H.-J. Kwon, J. B. Marston, and R. Shankar, J. Phys. C **6**, 4909 (1994).

[13] P. Kopietz, J. Hermisson, and K. Schonhammer, Phys. Rev. B **52**, 10 877 (1995).

[14] J. Frohlich, R. Gotschmann, and P. A. Marchetti, J. Phys. A **28**, 1169 (1995); J. Frohlich and R. Gotschmann, Phys. Rev. B **55**, 6788 (1997).

[15] M. Frau, A. Lerda, S. Sciuto, and G. R. Zemba, Int. J. Mod. Phys. A **12**, 4611 (1997); hep-th/9603112 (unpublished).

[16] F. Lesage, V. Pasquier, and D. Serban, Nucl. Phys. B **435**, 585 (1995); P. J. Forrester, *ibid.* **388**, 671 (1992); Z. N. C. Ha, *ibid.* **435**, 604 (1995).

[17] P. K. Ghosh, Phys. Lett. A **229**, 203 (1997).

[18] S. Sunakawa, S. Yamasaki, and T. Kebukawa, Prog. Theor. Phys. **38**, 804 (1967); **41**, 919 (1969); **44**, 565 (1970); **49**, 1802 (1973).

[19] G. S. Setlur and Y.C. Chang (unpublished).

[20] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947); N. N. Bogoliubov and D. N. Zubarev, Sov. Phys. JETP **1**, 83 (1955).

[21] L. Kadanoff and G. Baym, *Quantum Statistical Mechanics* [Addison Wesley (Advanced Book Classics), Redwood City, 1991].

[22] G. D. Mahan, *Many-Particle Physics*, 2nd ed. (Plenum, New York, 1990).

[23] A. Ligouri and M. Mintchev, Commun. Math. Phys. **169**, 635 (1995).

[24] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1980), Vols. I and II.