Growth of the normal-flow instability of a vortex array in two-component HeII

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It has been known for some time that there is a hydrodynamic instability in rotating vortex-permeated superfluid HeII. This instability sets in at a critical relative velocity of the two components (normal and superfluid). The first theoretical treatment assumed a uniform constant velocity of the normal component, thus limiting the dynamics to the superfluid. Growth rates were found. A little later, it was shown that the critical velocity for the *onset* of the instability so obtained survived a more rigorous two-component calculation. However, *growth rates* obtained from the one fluid model are a different story and remain questionable (the original calculation is inconsistent with an expansion in the ratio of the two densities). Here the problem is solved by considering the two-component model. Generally speaking, growth rates are somewhat different from those hitherto used. This is important in view of recent experiments in which the superfluid matches the vorticity of the normal fluid. [S0163-1829(98)01621-X]

I. INTRODUCTION

There is a well known streaming instability in a vortex-permeated, rotating superfluid helium.[1,2] More generally, two component media tend to support streaming instabilities, usually when the relative velocity of the two fluids is large enough to feed energy to the system.

Assume two components with constant densities ρ_s and ρ_n, $\rho=\rho_s+\rho_n$ (superfluid and normal). The coordinate system is rotating with angular velocity Ω around the z axis. We consider a relative velocity u between the two components, also along z. The instability in question owes its existence to two factors: the relative velocity and the existence of vortex lines along z, permeating the system and coupling the two fluids. Perturbations considered here will propagate along the vortex lines $\mathbf{k}=k\mathbf{e}_z$. The vortices will then undergo infinitesimal helical deformations which initially grow exponentially, until nonlinear effects saturate them. The calculation of Ref. 1, in which a one-component model is used, has been utilized to explain experimental observations.[3] Here we will try to improve on Ref. 1 and obtain growth rates consistent with the physical behavior of the frictional coefficients. It is a common mistake to just take these coefficients as given, whereas they can and often do depend on parameters crucial to a given calculation.

We take as our basis the equations of motion of the two components as given by Hall.[4] These generalize the classic ones of Landau to a rotational fluid. They are, in a coordinate system rotating with angular frequency $\mathbf{\Omega}$:

$$\frac{\partial \mathbf{v}_s}{\partial t}=\nabla\phi+2\mathbf{v}_s\times\mathbf{\Omega}+\alpha\hat{\boldsymbol{\lambda}}\times[\boldsymbol{\lambda}\times(\mathbf{v}_s-\mathbf{v}_n)]+\beta\boldsymbol{\lambda}\times(\mathbf{v}_s-\mathbf{v}_n)-\alpha\nu\boldsymbol{\lambda}\times(\boldsymbol{\lambda}\cdot\nabla)\hat{\boldsymbol{\lambda}}+\nu(1-\beta)(\boldsymbol{\lambda}\cdot\nabla)\boldsymbol{\lambda}, \tag{1}$$

$$\frac{\partial \mathbf{v}_n}{\partial t}+(\mathbf{v}_n\cdot\nabla)\mathbf{v}_n=\nabla\psi+2\mathbf{v}_n\times\mathbf{\Omega}-\alpha(\rho_s/\rho_n)\hat{\boldsymbol{\lambda}}\times[\boldsymbol{\lambda}\times(\mathbf{v}_s-\mathbf{v}_n)]-\beta(\rho_s/\rho_n)\boldsymbol{\lambda}\times(\mathbf{v}_s-\mathbf{v}_n)+\nu\alpha(\rho_s/\rho_n)\boldsymbol{\lambda}\times(\boldsymbol{\lambda}\cdot\nabla)\hat{\boldsymbol{\lambda}}+\nu\beta(\rho_s/\rho_n)(\boldsymbol{\lambda}\cdot\nabla)\hat{\boldsymbol{\lambda}}. \tag{2}$$

Here we linearize in $\mathbf{v}_s$ and this explains the absence of the streaming term on the left hand side of Eq. (1).

We take $\boldsymbol{\lambda}=\nabla\times\mathbf{v}_s+2\Omega$, $\hat{\boldsymbol{\lambda}}=\boldsymbol{\lambda}/\lambda$, α and β are proportional to the mutual friction coefficients, and ϕ and ψ are scalar terms involving pressure, temperature, etc.

We assume ρ_s and ρ_n to be constant, thus adding

$$\nabla\cdot\mathbf{v}_n=0,\quad \nabla\cdot\mathbf{v}_s=0 \tag{3}$$

to the above equations.

Until now, the growth rates of the instability were known only in the one-fluid model, in which $\mathbf{v}_n$ is constant and is just $u\mathbf{e}_z$. This limit corresponds to $\rho_s/\rho_n\rightarrow 0$. However, the coefficients α and β tend to infinity in this limit.[4] Therefore, keeping α and β fixed in such a calculation does not seem to be justified. Here we will find the growth rates from both Eqs. (1) and (2), and then discuss various limits. Incidentally, the value of u corresponding to the *onset* of instability happens to be given correctly by the one-fluid calculation.[5]

II. THE STABILITY CALCULATION

We take ϕ, ψ, $\mathbf{v}_s$, and $\mathbf{v}_n-\mathbf{u}$ to be proportional to $\exp[i(kz+\omega t)]$, and $\mathbf{\Omega}$ and $\mathbf{u}$ along z. Both $\mathbf{v}_s$ and $\mathbf{v}_n-\mathbf{u}$ are assumed small.

It is known[5] that, for given k, the critical velocity for the onset of instability is u_0:

$$u_0=(2\Omega+\nu k^2)/k. \tag{4}$$

This value is independent of ρ_s/ρ_n. Not so the growth rates, however. Here we will consider $u>\sqrt{8\Omega\nu}$. Equation (4) leads us to expect instability for k in the interval

$$k_1<k<k_2,$$

$$k_{1,2}=\frac{u\mp\sqrt{u^2-8\Omega\nu}}{2\nu},$$

$$u=u_0(k_1)=u_0(k_2). \tag{5}$$

Equations for v_{sx}, v_{sy}, v_{nx}, and v_{ny} form a closed set. They simplify if we work with $\mathbf{v}_s$ and $\mathbf{v}=(\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n)/\rho$.

The consistency condition of the system is found in the form of a dispersion relation

$$\tilde{\omega}^4+a_3\,\tilde{\omega}^3+a_2\,\tilde{\omega}^2+a_1\,\tilde{\omega}+a_0=0, \tag{6}$$

where

$$\tilde{\omega}=\omega+uk,$$

$$a_3=-2ia,$$

$$a_2=-(a^2+b^2+4\Omega^2[1+(\rho_s/\rho_n)d]),$$

$$a_1=4i[2a\Omega^2+\Omega(\rho_s/\rho_n)(2\Omega c+ad-bc)],$$

$$a_0=4\Omega^2[a^2+b^2+(\rho_s/\rho_n)^2(\alpha^2+\beta^2)(\nu^2k^4-u^2k^2)+2(\rho_s/\rho_n)(ac+bd)]$$

and

$$a=\alpha u_0k+iuk(\beta-1)+2\Omega(\rho_s/\rho_n)\alpha,$$

$$b=-i\alpha uk+u_0k(\beta-1)+2\Omega(\rho_s/\rho_n)\beta,$$

$$c=\alpha\nu k^2+i\beta uk,$$

$$d=\beta\nu k^2-i\alpha uk.$$

This dispersion relation yields exactly one unstable root in the k interval $[k_1,k_2]$. Growth rates $\gamma=-\mathrm{Im}\,\omega$ are drawn for this root in Fig. 1, using MATHEMATICA. We will discuss the implications in Sec. IV.

III. RECOVERY OF KNOWN RESULTS

If we take $\rho_s/\rho_n\to 0$ formally, artificially keeping α and β fixed, we can solve Eq. (6) to obtain four distinct roots:

$$\tilde{\omega}_n=ia+b;\ \ ia-b;\ \ 2\Omega;\ \ -2\Omega,\qquad n=1,\ldots,4. \tag{7}$$

The first two, ω_1 and ω_2, coincide with those given in Ref. 1. However, this recovery is not a physical vindication of that calculation, as the fact that α and β grow large for small ρ_s was not taken into account.

For $u=u_0$, we find that $a_0=0$ and all roots are stable; the $\tilde{\omega}=0$ root of course marginally so. We see that this mode is marginally stable when the fluid velocity is equal to the phase velocity of that mode. Thus for the important root, $\tilde{\omega}(k_1)=\tilde{\omega}(k_2)=0$, and k_1,k_2 are independent of ρ_s/ρ_n, see Fig. 1.

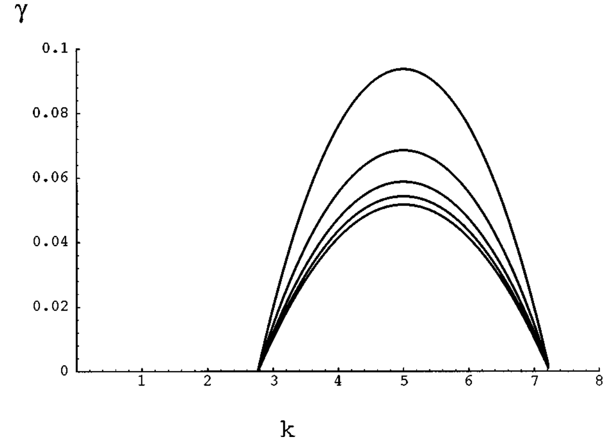

FIG. 1. Growth rates of the instability for $\alpha=\beta=0.1$; $\nu=0.1\ \mathrm{cm^2/s}$; $u=1$ cm/s; $\Omega=1$ rad/s, and ρ_s/ρ_n (moving up the figure): 0.2, 0.5, 1, 2, 5. All critical values of k coincide, as predicted by Ostermeier and Glaberson (Ref. 5).

IV. A USEFUL MODEL

The quartic of Eq. (6) is not very useful when contemplating a simple discussion of various approaches such as the one-fluid model, or else limits such as $\rho_s\to 0$ or $\rho_n\to 0$. We therefore found a simple approximation to γ $(k_1\leqslant k\leqslant k_2)$:

$$\gamma_{\mathrm{app}}=\frac{\alpha(u-u_0)k}{(1-\beta\rho_s/\rho_n)^2+(\alpha\rho_s/\rho_n)^2}. \tag{8}$$

This formula gives correct slopes at k_1 and k_2, and is almost indistinguishable from γ in between these limits for all but the most extreme parameters, see Fig. 2.

From now on we will base our discussion on Eq. (8), in our opinion the more important result here.

V. CONCLUSIONS OF THEORY

For $\rho_s/\rho_n\to 0$ we have[2,4] $\alpha\to\infty$, $\beta\to-\infty$ as negative powers of $T_\lambda-T$. We can find from the second reference of[4] how $\alpha\rho_s/\rho_n$ and $\beta\rho_s/\rho_n$ will behave very near $T_\lambda=2.172$ K. When these limits are substituted into Eq. (8) we find that, in that limit,

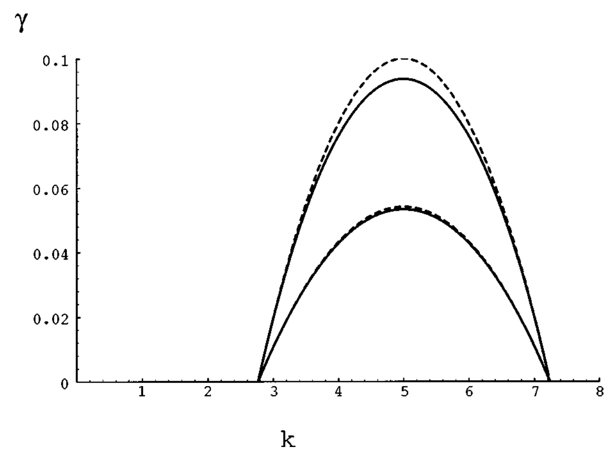

FIG. 2. Comparison of growth rates as given by the γ_{app} model (broken lines) with values following from Eq. (6). Here ρ_s/ρ_n is 0.4 and 5. Other parameters as in the previous figure.

$$\gamma \simeq \alpha (u - u_0) k. \tag{9}$$

This is what we would expect, as in this limit the dynamics must be dominated by the normal fluid.

We can also see from Ref. 2, p. 93, that $\beta > 0$ for temperatures below 2.06 K and $\beta < 0$ between this temperature and T_λ. Thus, for temperatures only slightly below T_λ, growth rates are definitely smaller than those following from the one-fluid model Eq. (9). Below 2.06 K, however, they can be larger.

For large ρ_s/ρ_n the need to use Eq. (8) instead of Eq. (9) is self-evident. When this ratio tends to infinity, Eq. (8) reduces to

$$\gamma \simeq \frac{\alpha \rho_n^2 (u - u_0) k}{(\alpha^2 + \beta^2) \rho_s^2} \to 0.$$

The fact that the growth rate becomes very small is important because of the implications for vorticity matching discussed below. All in all, Eqs. (6) and (8) should prove useful when interpreting recent and future experiments.

The next step would be to generalize our calculation to arbitrary angles of propagation of the perturbation. Formulas would cease to be simple, as they are not so even in the limit of Ref. 1. Very similar problems arise in two-component plasmas.[6]

VI. IMPLICATIONS FOR RECENT EXPERIMENTS

There is current interest in the study of intense superfluid turbulence and in the connection between superfluid and classical turbulence. We know that classical turbulence is not just random disorder. The vorticity appears to be concentrated in ''vorticity tubes'' which appear spontaneously in the flow and have finite lifetimes. We therefore expect that vorticity tubes are present in the turbulence of the normal fluid. A numerical simulation of vortex lines in a model of normal-fluid turbulence with these ''vorticity tubes'' has been performed. The instability considered here (extending Ostermeier and Glaberson) is seen to play a key role in creating ''superfluid vortex bundles.'' These bundles allow the superfluid to match the vorticity of the normal fluid. Vorticity matching is an effect observed in recent experiments. It is at this point that the issue of growth rate of the instability discussed here becomes important. If the growth rate is not large enough, the normal-fluid vortex tubes will die out before the superfluid vortex tubes can be formed. Vorticity matching cannot then take place.[7]

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