Path-integral approach to the scattering theory of quantum transport

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The scattering theory of quantum transport relates transport properties of disordered mesoscopic conductors to their transfer matrix $\boldsymbol{T}$. We introduce an approach to the statistics of transport quantities which expresses the probability distribution of $\boldsymbol{T}$ as a path integral. The path integal is derived for a model of conductors with broken time-reversal invariance in arbitrary dimensions. It is applied to the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation which describes quasi-one-dimensional wires. We use the equivalent channel model whose probability distribution for the eigenvalues of $\boldsymbol{T}\boldsymbol{T}^{\dagger}$ is equivalent to the DMPK equation independent of the values of the forward scattering mean free paths. We find that infinitely strong forward scattering corresponds to diffusion on the coset space of the transfer-matrix group. It is shown that the saddle-point of the path integral corresponds to ballistic conductors with large conductances. We solve the saddle-point equation and recover random-matrix theory from the saddle-point approximation to the path integral. [S0163-1829(98)11319-X]

I. INTRODUCTION

Advances in microfabrication technology led to the realization of mesoscopic electronic devices. In such devices the mean free path for inelastic electron scattering exceeds the dimension of the device. As a consequence, phase coherence is maintained, which leads to quantum interference effects like universal conductance fluctuations, persistent currents, and Aharanov-Bohm oscillations in rings, or weak localization.[1] The phase coherence also has serious theoretical implications. It causes large conductance fluctuations which are related to the problem of high gradient operators in the field-theoretic description of the metal insulator transition.[2–5] These fluctuations manifest themselves in the metallic regime as logarithmic normal tails of the conductance probability distribution. As the critical regime is approached, the conductance probability distribution becomes increasingly broader, until it reaches a logarithmic normal form in the insulating regime.[6]

A common approach to transport quantities of mesoscopic conductors is the scattering theory of quantum transport.[7,8] It models the conductor by a disordered region which is connected to a number of ideal leads which support propagating wave modes. The number of leads corresponds to the number of measurement terminals. Here only two terminal geometries will be considered. The scattering matrix relates the amplitudes I_k, I_k' of the incoming with the amplitudes O_k, O_k' $(k=1,\dots,N)$ of the scattered propagating wave modes at the Fermi energy,

$$\begin{pmatrix} O \\ O' \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} I \\ I' \end{pmatrix}, \tag{1}$$

where

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{r} & \boldsymbol{t}' \\ \boldsymbol{t} & \boldsymbol{r}' \end{pmatrix}. \tag{2}$$

$\boldsymbol{t}$ and $\boldsymbol{r}$ are the transmission and reflection matrices for incident waves from the left, and $\boldsymbol{t}'$ and $\boldsymbol{r}'$ are the transmission and reflection matrices for incident waves from the right. The dimensionless two-probe conductance $g=G/(e^2/h)$ in terms of the transmission eigenvalues T_k of $\boldsymbol{t}\boldsymbol{t}^{\dagger}$ is

$$g = \sum_{k=1}^{N} T_k. \tag{3}$$

There are three universality classes which correspond to different physical situations. Conductors with time-reversal invariance lie in the orthogonal universality class. The unitary universality class corresponds to conductors in which the time-reversal symmetry is broken, e.g., by a magnetic field. Conductors with spin-flip scattering processes but no time-reversal symmetry breaking fall into the symplectic universality class.

Recently, the quasi-one-dimensional wire has attracted considerable attention. The width of a quasi-one-dimensional wire is of the order of the mean free path for elastic electron scattering so that transverse diffusion can be neglected and the cross section of the wire becomes structureless. Interesting nonperturbative results which are valid for all wire lengths have been obtained for this system.[9–12] Furthermore, it has been the ideal playground for ideas in the field of quantum transport.

One of these ideas is the Fokker-Planck (FP) approach to quasi-one-dimensional wires. The FP equation which describes the probability distribution for the transmission eigenvalues is known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation. It has been derived by a number of authors[13–17] who started from various different models. Its form is

$$\frac{\partial p(s;\{\Gamma_k\})}{\partial s} = \frac{2}{\gamma} \sum_{k=1}^{N} \frac{\partial}{\partial \Gamma_k} \left(\frac{\partial p}{\partial \Gamma_k} + \beta p \frac{\partial \Omega(\{\Gamma_k\})}{\partial \Gamma_k} \right), \tag{4}$$

where

$$\Omega(\{\Gamma_k\}) = -\sum_{k<l} \ln|(\cosh\Gamma_k - \cosh\Gamma_l)/2| - 1/\beta \sum_k \ln|\sinh\Gamma_k|, \tag{5}$$

$\gamma = \beta N + 2 - \beta$, and $\cosh\Gamma_k = (2-T_k)/T_k$. The values of β are 1, 2, and 4 for the orthogonal, unitary, and symplectic universality classes, respectively. The DMPK equation has been studied intensively in the past few years.[18–27] Beenakker and Rejaei[28,29] discovered that the variation

$$p(s;\{\Gamma_k\}) = \exp\left\{-\frac{\beta}{2}\Omega(\{\Gamma_k\})\right\}\psi(s;\{\Gamma_k\}) \tag{6}$$

of the Sutherland transformation[30] which is known to solve the Brownian motion model for the circular unitary ensemble,[31] works as well for the DMPK equation. After this transformation, $\psi(s;\{\Gamma_k\})$ obeys a Schrödinger equation for N noninteracting particles. As a consequence the exact form of $p(s;\{\Gamma_k\})$ could be determined. This solution has been the basis for Frahm's exact calculation of the one- and two-point correlation functions of the transmission eigenvalues.[11]

In this paper we present an approach to the scattering theory of quantum transport which expresses the probability distribution of the transfer matrix as a path integral. Our motivation has been the belief that the path-integral technique can be developed into a tool which is more powerful than the FP approach when it comes to the description of higher-dimensional conductors.

II. SCATTERING MODEL

We use the transfer matrix $\boldsymbol{T}$ instead of the $\boldsymbol{S}$ matrix to model the scattering properties of the disordered conductor. The transfer matrix relates the scattering amplitudes in the left lead with the scattering amplitudes in the right lead:

$$\begin{pmatrix} O' \\ I' \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} I \\ O \end{pmatrix}. \tag{7}$$

It has the advantage that it obeys the multiplication law

$$\boldsymbol{T}(L+\delta L,0) = \boldsymbol{T}(L+\delta L,L)\boldsymbol{T}(L,0), \tag{8}$$

which leads to the simple Langevin equation

$$\dot{\boldsymbol{T}}(x) \equiv \frac{d\boldsymbol{T}(x,0)}{dx} = \boldsymbol{\varepsilon}(x)\boldsymbol{T}(x,0) \equiv \begin{pmatrix} \boldsymbol{\varepsilon}^{11}(x) & \boldsymbol{\varepsilon}^{12}(x) \\ \boldsymbol{\varepsilon}^{21}(x) & \boldsymbol{\varepsilon}^{22}(x) \end{pmatrix}\boldsymbol{T}(x,0) \tag{9}$$

for the stochastic evolution of the transfer matrix. The disorder is generated by the multiplicative noise $\boldsymbol{\varepsilon}$.

In this paper we consider only conductors in the unitary universality class. Then $\boldsymbol{T}$ obeys the symmetry constraint

$$\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T} = \mathbf{1}, \tag{10}$$

which ensures flux conservation, where

$$\boldsymbol{\Sigma}_z = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{11}$$

A convenient parametrization of the transfer matrix is the polar decomposition[15,32]

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{u}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{u}_3 \end{pmatrix}\begin{pmatrix} \cosh(\boldsymbol{\Gamma}/\mathbf{2}) & \sinh(\boldsymbol{\Gamma}/\mathbf{2}) \\ \sinh(\boldsymbol{\Gamma}/\mathbf{2}) & \cosh(\boldsymbol{\Gamma}/\mathbf{2}) \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{u}_4 \end{pmatrix}, \tag{12}$$

where $\boldsymbol{\Gamma}$ is a real, diagonal $N\times N$ matrix, and $\boldsymbol{u}_i$ ($i=1$, 2, 3, and 4) are unitary $N\times N$ matrices.

Relation (10) implies that $\boldsymbol{\Sigma}_z\boldsymbol{\varepsilon}^\dagger\boldsymbol{\Sigma}_z + \boldsymbol{\varepsilon} = 0$, leading to the symmetries

$$\begin{aligned} \boldsymbol{\varepsilon}^{11\dagger} &= -\boldsymbol{\varepsilon}^{11}, \\ \boldsymbol{\varepsilon}^{22\dagger} &= -\boldsymbol{\varepsilon}^{22}, \\ \boldsymbol{\varepsilon}^{12\dagger} &= \boldsymbol{\varepsilon}^{21} \end{aligned} \tag{13}$$

for the noise. The stochastics properties of $\boldsymbol{\varepsilon}$ could be derived from a microscopic Hamiltonian.[33,34] Here we adopt a simple model[17,35] which assumes Gaussian white noise, such that

$$\begin{aligned} \langle \varepsilon_{kl}(x)\rangle &= 0, \\ \langle \varepsilon^{11}_{kl}(x)\varepsilon^{11*}_{k'l'}(x')\rangle &= \frac{1}{l^f_{kl}}\delta_{kk'}\delta_{ll'}\delta(x-x'), \\ \langle \varepsilon^{22}_{kl}(x)\varepsilon^{22*}_{k'l'}(x')\rangle &= \frac{1}{l'^f_{kl}}\delta_{kk'}\delta_{ll'}\delta(x-x'), \\ \langle \varepsilon^{12}_{kl}(x)\varepsilon^{12*}_{k'l'}(x')\rangle &= \frac{1}{l^b_{kl}}\delta_{kk'}\delta_{ll'}\delta(x-x'), \end{aligned} \tag{14}$$

and all other independent second moments are zero. The mean free paths l^f_{kl}, l'^f_{kl}, and l^b_{ij}, l'^b_{kl} for forward and backward scattering, respectively, are defined by the limits of the disorder averages

$$\begin{aligned} \frac{1}{l^f_{kl}} &\equiv \lim_{\delta L\to 0}\frac{\langle |t_{kl}-\delta_{kl}|^2\rangle_{\delta L}}{\delta L}, \\ \frac{1}{l'^f_{kl}} &\equiv \lim_{\delta L\to 0}\frac{\langle |t'_{kl}-\delta_{kl}|^2\rangle_{\delta L}}{\delta L}, \\ \frac{1}{l^b_{kl}} &\equiv \lim_{\delta L\to 0}\frac{\langle |r_{kl}|^2\rangle_{\delta L}}{\delta L}, \\ \frac{1}{l'^b_{kl}} &\equiv \lim_{\delta L\to 0}\frac{\langle |r'_{kl}|^2\rangle_{\delta L}}{\delta L} \end{aligned} \tag{15}$$

for a short piece of conductor with length δL. Note that the symmetries (13) imply the relation $l^b_{kl} = l'^b_{lk}$.

We want a path-integral representation of the stochastic process (9) in terms of the transfer matrix $\boldsymbol{T}$. The derivation technique which is most suited for that purpose derives the path-integral directly from the Langevin equation (see Chap.

4 in Ref. 36). The symmetry constraints (10) on $\boldsymbol{T}$ will be taken into account by δ functions, which leads naturally to the invariant measure of the transfer-matrix group as the path integration measure. We illustrate the essential ideas of the derivation technique with the simple example of diffusion on a circle before we deal with the transfer matrix.

III. DIFFUSION ON THE CIRCLE AS A SIMPLE EXAMPLE

Let the angle φ determine the position on a circle. The analog of the Langevin equation (9) is

$$\dot{u} \equiv \frac{du(t)}{dt} = \varepsilon(t)u(t) \tag{16}$$

where $u=\exp(i\varphi)$. The symmetry $\varepsilon^*=-\varepsilon$ implies $d(uu^*)/dt=0$, which ensures that u remains a phase. Choosing Gaussian white noise for the imaginary part of ε, such that

$$\langle \varepsilon(t)\rangle = 0,$$

$$\langle \varepsilon(t)\varepsilon(t')^*\rangle = 2D\,\delta(t-t') \tag{17}$$

leads to the FP equation

$$\frac{\partial p(t;\varphi)}{\partial t} = D\frac{\partial^2 p(t;\varphi)}{(\partial\varphi)^2}, \tag{18}$$

which describes diffusion on the circle.

The probability distribution of u can be formally expressed as

$$p(t;u) = \langle \delta[u-\overline{u}(t)]\rangle \tag{19}$$

where $u\equiv u^{(1)}+iu^{(2)}$, $\delta(u)\equiv\delta(u^{(1)})\delta(u^{(2)})$, and $\overline{u}(t)$ is the value of u which is acquired at time t for a certain realization of the noise and the initial value $\overline{u}(0)=u_0$. The brackets $\langle\cdots\rangle$ denote the average over all possible noise configurations. The path-integral representation is derived by inserting a product of δ functions

$$p(t;u) = \left\langle \int \prod_{t'=0}^{t} du(t')\,\delta[u(t')-\overline{u}(t')]\,\delta[u(t)-u]\right\rangle, \tag{20}$$

where $du\equiv du^{(1)}du^{(2)}$. The δ function $\delta[u(t')-\overline{u}(t')]$ restricts the value of $u(t')$ to $\overline{u}(t')$. Since $\overline{u}(t')$ is not explicitly known, we enforce this constraint implicitly by the relation $\dot{u}(t)u^{-1}(t)-\varepsilon(t)=0$, which follows from the Langevin equation (16). That leads to

$$p(t;u) = \left\langle \int \prod_{t'=0}^{t} du(t')|\det\hat{\mathcal{A}}|\,\delta[\dot{u}(t')u^{-1}(t')-\varepsilon(t')] \times \delta[u(t)-u]\right\rangle, \tag{21}$$

where the operator $\hat{\mathcal{A}}$ is defined by the functional derivative

$$\mathcal{A}_{jj'}(t,t') = \frac{\delta[\dot{u}(t)u^{-1}(t)-\varepsilon(t)]^{(j)}}{\delta u^{(j')}(t')}. \tag{22}$$

The average over the Gaussian probability measure

$$P[\varepsilon]\prod_{x=0}^{L} d\varepsilon(x) = \frac{1}{\mathcal{N}}\exp\left\{-\int_0^L dx\,\frac{\varepsilon(x)\varepsilon^*(x)}{4D}\right\}\prod_{x=0}^{L} d\varepsilon(x), \tag{23}$$

where

$$d\varepsilon = d\varepsilon^{(1)}d\varepsilon^{(2)}\delta(\varepsilon+\varepsilon^*), \tag{24}$$

yields

$$p(t;u) = \frac{1}{\mathcal{N}}\int \prod_{t'=0}^{t} du(t')\,\delta[\dot{u}(t')u^{-1}(t') + \dot{u}^*(t')u^{-1*}(t')]|\det\hat{\mathcal{A}}|\exp\{-S\}, \tag{25}$$

where

$$S = \frac{1}{4D}\int_0^t dt'\,\dot{u}(t')u^{-1}(t')[\dot{u}(t')u^{-1}(t')]^*, \tag{26}$$

and the path summation includes all paths which start at u_0 and end at u.

The property that $\dot{w}(t)w^{-1}(t)+\dot{w}^*(t)w^{-1*}(t) =\dot{u}(t)u^{-1}(t)+\dot{u}^*(t)u^{-1*}(t)$ if $w(t)=u(t)v(t)$, where $v(t)$ is a phase, suggests that $\Pi_{t'=0}^{t}du(t')\,\delta[\dot{u}(t')u^{-1}(t') +\dot{u}^*(t')u^{-1*}(t')]$ is proportional to $\Pi_{t'=0}^{t}d\mu[u(t')]$, where $d\mu(u)$ is the invariant measure on $U(1)$. This becomes explicit if the δ function is introduced via an auxiliary field $\kappa(t')$

$$p(t;u) = \int \prod_{t'=0}^{t} du(t')d\kappa(t')|\det\hat{\mathcal{A}}|\exp\{-\tilde{S}\}, \tag{27}$$

where

$$\tilde{S} = S + i\int_0^t dt'\,\kappa(t')[\dot{u}(t')u^{-1}(t')+\dot{u}^*(t')u^{-1*}(t')]$$

$$= S + i\int_0^t dt'\,\kappa(t')\frac{d}{dt'}\ln[u(t')u^*(t')]. \tag{28}$$

Partial integration yields

$$\tilde{S} = S + i\int_0^t dt'\,\lambda(t')\ln[u(t')u^*(t')], \tag{29}$$

where $\lambda(t)=-\dot{\kappa}(t)$. The Jacobian of the transformation $\lambda(t)=-\dot{\kappa}(t)$ is an irrelevant constant which can be incorporated into the normalization factor. Hence

$$p(t;u) = \mathcal{N}^{-1}\int \prod_{t'=0}^{t} d\mu[u(t')]|\det\hat{\mathcal{A}}|\exp\{-S\}, \tag{30}$$

since $du\;\delta[\ln(uu^*)]=du\;\delta(uu^*-1)$, which is proportional to the invariant measure $d\mu(u)$.[34] The restriction to $uu^*=1$ in the invariant measure simplifies action (26),

$$S=\frac{1}{4D}\int_0^t dt'\dot{u}(t')\dot{u}^*(t'). \tag{31}$$

To calculate det$\mathcal{A}$, we evaluate Eq. (22), which gives

$$\begin{aligned}\mathcal{A}_{11}(t,t')&=[a(t,t')+a^*(t,t')]/2,\\ \mathcal{A}_{12}(t,t')&=i[a(t,t')-a^*(t,t')]/2,\\ \mathcal{A}_{21}(t,t')&=-i[a(t,t')-a^*(t,t')]/2,\\ \mathcal{A}_{22}(t,t')&=[a(t,t')+a^*(t,t')]/2,\end{aligned} \tag{32}$$

where

$$a(t,t')=u^{-1}(t)\left(\frac{d}{dt}\delta(t-t')-\delta(t-t')\dot{u}(t)u^{-1}(t)\right). \tag{33}$$

The decomposition $\hat{\mathcal{A}}=\hat{\mathcal{B}}\hat{\mathcal{C}}\hat{\mathcal{D}}$ into a product of three operators

$$[\hat{\mathcal{B}}](t,t')=\frac{1}{\sqrt{2}}\begin{pmatrix}\delta(t-t') & -i\,\delta(t-t')\\ -i\,\delta(t-t') & \delta(t-t')\end{pmatrix},$$

$$[\hat{\mathcal{C}}](t,t')=\begin{pmatrix}a(t,t') & 0\\ 0 & a^*(t,t')\end{pmatrix}, \tag{34}$$

$$[\hat{\mathcal{D}}](t,t')=\frac{1}{\sqrt{2}}\begin{pmatrix}\delta(t-t') & i\,\delta(t-t')\\ i\,\delta(t-t') & \delta(t-t')\end{pmatrix},$$

implies that det$\hat{\mathcal{A}}=$det$\hat{\mathcal{C}}=$det$\hat{a}$det$\hat{a}^*$, since det$\hat{\mathcal{B}}=$det$\hat{\mathcal{D}}=1$. The operator $\hat{a}$ can be as well factorized into $\hat{a}=\hat{a}_1\hat{a}_2\hat{a}_3$, where

$$\begin{aligned}a_1(t,t')&=u^{-1}(t)\,\delta(t-t'),\\ a_2(t,t')&=\frac{d}{dt}\delta(t-t'),\\ a_3(t,t')&=\delta(t-t')-\theta(t-t')\dot{u}(t')u^{-1}(t').\end{aligned} \tag{35}$$

The determinant of $\hat{a}_1\hat{a}_1^*$ is 1 since the δ function in the path integration measure enforces that $u(t)u^*(t)=1$. The determinant of $\hat{a}_2$ is an irrelevant constant which contributes only to the normalization. Using det$=$exp tr ln and $\ln(1+x)=\Sigma_{k=1}^{\infty}(-1)^{k+1}x^k/k$ to evaluate det$\hat{a}_3$ yields

$$\det\hat{a}_3=\exp\left\{-\int_0^t dt'\,\theta(0)\dot{u}(t')u^{-1}(t')+\cdots\right\}. \tag{36}$$

The higher-order terms which are indicated by the dots vanish due to products of θ functions. The quantity $\theta(0)$ is not yet defined, which can be traced back to the multiplicative noise in the Langevin equation (16). The correct choice is $\theta(0)=\frac{1}{2}$ (see the discussion in Chap. 4 of Ref. 36). Here this choice does not matter, since $\dot{u}(t')u^{-1}(t')+\dot{u}^*(t')u^{-1*}(t')=0$, which implies that det$\hat{a}_3det\hat{a}_3^*=1$, leading to the final form

$$p(t;u)=\mathcal{N}^{-1}\int\prod_{t'=0}^{t}d\mu[u(t')]\exp\{-S\} \tag{37}$$

of the path-integral representation of the stochastic process (16).

IV. PATH INTEGRAL FOR THE TRANSFER MATRIX

The analog of Eq. (20) for the transfer matrix is

$$p(L;\boldsymbol{T})=\int\left\langle\int\prod_{x=0}^{L}d\boldsymbol{T}(x)\,\delta[\boldsymbol{T}(x)-\overline{\boldsymbol{T}}(x)]\,\delta[\boldsymbol{T}(L)-\boldsymbol{T}]\right\rangle, \tag{38}$$

where

$$d\boldsymbol{T}\equiv\prod_{k,l}dT_{kl}^{(1)}dT_{kl}^{(2)},$$

$$\delta(\boldsymbol{T}-\overline{\boldsymbol{T}})\equiv\prod_{k,l}\delta(T_{kl}^{(1)}-\overline{T}_{kl}^{(1)})\,\delta(T_{kl}^{(2)}-\overline{T}_{kl}^{(2)}). \tag{39}$$

Enforcing $\overline{\boldsymbol{T}}(x)$ by $\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)-\boldsymbol{\varepsilon}(x)=0$, which follows from the Langevin equation (9), yields

$$p(L;\boldsymbol{T})=\int\left\langle\int\prod_{x=0}^{L}dT(x)|\det\mathcal{A}|\,\delta[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)-\boldsymbol{\varepsilon}(x)]\times\delta[\boldsymbol{T}(L)-\boldsymbol{T}]\right\rangle, \tag{40}$$

where the operator $\hat{\mathcal{A}}$ is defined by the functional derivative

$$\mathcal{A}_{kl,k'l'}^{jj'}(x,x')=\frac{\delta[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)-\boldsymbol{\varepsilon}(x)]_{kl}^{(j)}}{\delta T_{k'l'}^{(j')}(x')}. \tag{41}$$

Performing the average over the Gaussian probability measure

$$\begin{aligned}P[\boldsymbol{\varepsilon}]\prod_{x=0}^{L}d\boldsymbol{\varepsilon}(x)=\frac{1}{\mathcal{N}}\exp\Big\{-\tfrac{1}{2}\int_0^L dx[&l_{kl}^{f}\varepsilon_{kl}^{11}(x)\varepsilon_{kl}^{11*}(x)\\ &+l'^{f}_{kl}\varepsilon_{kl}^{22}(x)\varepsilon_{kl}^{22*}(x)+l_{kl}^{b}\varepsilon_{kl}^{12}(x)\varepsilon_{kl}^{12*}(x)\\ &+l'^{b}_{kl}\varepsilon_{kl}^{21}(x)\varepsilon_{kl}^{21*}(x)]\Big\}\prod_{x=0}^{L}d\varepsilon(x),\end{aligned} \tag{42}$$

where

$$d\boldsymbol{\varepsilon}\equiv\prod_{i,j,k,l} d\varepsilon_{kl}^{ij(1)}\varepsilon_{kl}^{ij(2)}\delta_{\mathrm{S}}(\boldsymbol{\varepsilon}),$$

$$\delta_{\mathrm{S}}(\boldsymbol{\varepsilon})\equiv\prod_{k<l}\{\delta[(\varepsilon_{kl}^{11}+\varepsilon_{lk}^{11*})^{(1)}]\delta[(\varepsilon_{kl}^{11}+\varepsilon_{lk}^{11*})^{(2)}]$$
$$\times\delta[(\varepsilon_{kl}^{22}+\varepsilon_{lk}^{22*})^{(1)}]\delta[(\varepsilon_{kl}^{22}+\varepsilon_{lk}^{22*})^{(2)}]\}$$
$$\times\prod_{k}\delta[(\varepsilon_{kk}^{11})^{(1)}]\delta[(\varepsilon_{kk}^{22})^{(1)}]$$
$$\times\prod_{k,l}\delta[(\varepsilon_{kl}^{12}-\varepsilon_{lk}^{21*})^{(1)}]\delta[(\varepsilon_{lk}^{21*}-\varepsilon_{kl}^{12})^{(2)}] \tag{43}$$

yields

$$p(L;\boldsymbol{T})=\mathcal{N}^{-1}\int\prod_{x=0}^{L} d\boldsymbol{T}(x)\,\delta_S[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]$$
$$\times|\det\hat{\mathcal{A}}|\exp\{-S\}, \tag{44}$$

where

$$S=\frac{1}{2}\int_0^L dx\{l_{kl}^{f}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{11}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{11*}$$
$$+l'^{f}_{kl}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{22}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{22*}+l_{kl}^{b}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{12}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{12*}$$
$$+l'^{b}_{kl}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{21}[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]_{kl}^{21*}\}. \tag{45}$$

In analogy with Sec. III, we expect that $\Pi_{x=0}^{L}d\boldsymbol{T}(x)\,\delta_S[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]$ is proportional to $\Pi_{x=0}^{L}d\mu[\boldsymbol{T}(x)]$, where $d\mu(\boldsymbol{T})$ is the invariant measure of the transfer-matrix group. This will be proven in the Appendix. The form of the invariant measure in terms of the polar coordinates (12) is

$$d\mu(\boldsymbol{T})=\prod_{k<l}(\cosh\Gamma_k-\cosh\Gamma_l)^2\prod_{k}\sinh\Gamma_k d\Gamma_k\prod_{k=1}^{4}d\mu(\boldsymbol{u}_k), \tag{46}$$

where $d\mu(\boldsymbol{u}_k)$ is the the invariant measure on the unitary group.[15]

We proceed with the calculation of $\det\mathcal{A}$. Using $\partial/\partial T_{kl}^{(1)}=\partial/\partial T_{kl}+\partial/\partial T_{kl}^{*}$, $\partial/\partial T_{kl}^{(2)}=i(\partial/\partial T_{kl}-\partial/\partial T_{kl}^{*})$, and $\partial T_{kl}^{-1}/\partial T_{k'l'}=-T_{kk'}^{-1}T_{l'l}^{-1}$ to evaluate Eq. (41) yields

$$[\hat{\mathcal{A}}]^{11}=(\hat{\boldsymbol{A}}+\hat{\boldsymbol{A}}^{*})/2,$$
$$[\hat{\mathcal{A}}]^{12}=i(\hat{\boldsymbol{A}}-\hat{\boldsymbol{A}}^{*})/2,$$
$$[\hat{\mathcal{A}}]^{21}=-i(\hat{\boldsymbol{A}}-\hat{\boldsymbol{A}}^{*})/2, \tag{47}$$
$$[\hat{\mathcal{A}}]^{22}=(\hat{\boldsymbol{A}}+\hat{\boldsymbol{A}}^{*})/2,$$

where

$$A_{kl,k'l'}(x,x')=\delta_{km}T_{nl}^{-1}(x)\bigg(\frac{d}{dx}\delta(x-x')\,\delta_{mk'}\delta_{nl'}$$
$$-\delta(x-x')[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]_{mk'}\delta_{nl'}\bigg). \tag{48}$$

The decomposition $\hat{\mathcal{A}}=\hat{\mathcal{B}}\hat{\mathcal{C}}\hat{\mathcal{D}}$ into a product of three operators,

$$\hat{\mathcal{B}}=\frac{1}{\sqrt{2}}\begin{pmatrix}\hat{\mathbf{1}} & -i\hat{\mathbf{1}}\\ -i\hat{\mathbf{1}} & \hat{\mathbf{1}}\end{pmatrix},$$
$$\hat{\mathcal{C}}=\begin{pmatrix}\hat{\boldsymbol{A}} & 0\\ 0 & \hat{\boldsymbol{A}}^{*}\end{pmatrix}, \tag{49}$$
$$\hat{\mathcal{D}}=\frac{1}{\sqrt{2}}\begin{pmatrix}\hat{\mathbf{1}} & i\hat{\mathbf{1}}\\ i\hat{\mathbf{1}} & \hat{\mathbf{1}}\end{pmatrix},$$

where $[\hat{\mathbf{1}}]_{kl,k'l'}(x,x')=\delta(x-x')\,\delta_{kk'}\delta_{ll'}$, implies that $\det\hat{\mathcal{A}}=\det\hat{\mathcal{C}}=\det\hat{\boldsymbol{A}}\det\hat{\boldsymbol{A}}^{*}$, since $\det\hat{\mathcal{B}}=\det\hat{\mathcal{D}}=1$. The operator $\hat{\boldsymbol{A}}$ can be as well factorized into $\hat{\boldsymbol{A}}=\hat{\boldsymbol{A}}_1\hat{\boldsymbol{A}}_2\hat{\boldsymbol{A}}_3$, where

$$A_{1;kl,k'l'}(x,x')=[\mathbf{1}\otimes(\boldsymbol{T}^{-1})^{T}(x)]_{kk',ll'}\delta(x-x'),$$
$$A_{2;kl,k'l'}(x,x')=\frac{d}{dx}\delta(x-x')\,\delta_{kk'}\delta_{ll'}, \tag{50}$$
$$A_{3;kl,k'l'}(x,x')=[\delta(x-x')\mathbf{1}\otimes\mathbf{1}-\theta(x-x')$$
$$\times\dot{\boldsymbol{T}}(x')\boldsymbol{T}^{-1}(x')\otimes\mathbf{1}]_{kl,k'l'}.$$

The product $\det\hat{\boldsymbol{A}}_1\det\hat{\boldsymbol{A}}_1^{*}$ is 1 since the determinant of the transfer matrix is a phase. The determinant of $\hat{\boldsymbol{A}}_2$ is an irrelevant constant which contributes only to the normalization. Using det = exp tr ln and $\ln(1+x)=\Sigma_{k=1}^{\infty}(-1)^{k+1}x^k/k$ to evaluate $\det\hat{\boldsymbol{A}}_3$ yields

$$\det\hat{\boldsymbol{A}}_3=\exp\bigg\{-N\theta(0)\int_0^L dx\ \mathrm{tr}[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]\bigg\}. \tag{51}$$

The symmetries of the transfer matrix imply that $\mathrm{tr}(\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)+[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]^{*})=0$, which gives $\det\hat{\boldsymbol{A}}_3\det\hat{\boldsymbol{A}}_3^{*}=1$.

That leads to the final form

$$p(L;\boldsymbol{T})=\mathcal{N}^{-1}\int\prod_{x=0}^{L}d\mu[\boldsymbol{T}(x)]\exp\{-S\} \tag{52}$$

of the path integral, where S is the action of Eq. (45).

V. DMPK EQUATION

We formulate the DMPK equation in terms of diffusion on the coset space of the transfer-matrix group, as done by Hüffmann.[20] In our context, this can be achieved with the equivalent channel model (ECM). This model was introduced by Mello and Tomsovic for the orthogonal universality class.[37,17] They showed that it is equivalent to the DMPK equation with $\beta=1$, in the sense that the joint probability distributions for $\mathbf{\Gamma}$ of both models are identical. The ECM for the unitary universality class is just model (14), with backscattering mean free paths of the form

$$\frac{1}{l^b_{mn}}=\frac{1}{lN}, \tag{53}$$

and arbitary forward scattering mean free paths. It is equivalent to the DMPK equation with $\beta=2$ in the same sense. The difference between the DMPK equation and the ECM is that the unitary matrices need not be isotropically distributed, and that there can be correlations between them and $\mathbf{\Gamma}$.

We choose forward scattering to be infinitely strong so that the mean free paths l^f_{mn} and l'^f_{mn} are zero. Then action (45) simplifies to

$$S=\frac{Nl}{2}\int_0^L dx\ \mathrm{tr}\{[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]^{12}([\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]^{12})^\dagger + [\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]^{21}([\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}]^{21})^\dagger\}. \tag{54}$$

Using that $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}=-\boldsymbol{T}\dot{\boldsymbol{T}^{-1}}$, and the symmetries of $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$, one can simplify further:

$$\begin{aligned}S&=\frac{Nl}{8}\int_0^L dx\ \mathrm{tr}\{[\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}+(\dot{\boldsymbol{T}}\boldsymbol{T}^{-1})^\dagger]^2\},\\ &=\frac{Nl}{8}\int_0^L dx\ \mathrm{tr}\{2\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}(\dot{\boldsymbol{T}}\boldsymbol{T}^{-1})^\dagger-\dot{\boldsymbol{T}}\dot{\boldsymbol{T}^{-1}}-\dot{\boldsymbol{T}}^\dagger\dot{\boldsymbol{T}^{-1\dagger}}\}\\ &=-\frac{Nl}{8}\int_0^L dx\ \mathrm{tr}(\dot{\boldsymbol{M}}\dot{\boldsymbol{M}^{-1}}),\end{aligned} \tag{55}$$

where $\boldsymbol{M}=\boldsymbol{T}^\dagger\boldsymbol{T}$ which does not depend on $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$ anymore. The infinite strong forward scattering immediately randomizes the probability distribution of $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$, so that they become isotropically distributed. Note that the space which is formed by the matrices $\boldsymbol{M}$ is isomorphic to the coset space of the transfer-matrix group. The path integral describes diffusion on the coset space, since the action is the classical action for free motion on this space.[38,39]

Introducing the dimensionless length $s=x/(Nl)$ yields

$$S=-\tfrac{1}{8}\int_0^{1/g_{\mathrm{cl}}} ds\ \mathrm{tr}(\dot{\boldsymbol{M}}\dot{\boldsymbol{M}^{-1}}), \tag{56}$$

where the dot now stands for the derivative with respect to s, and $g_{\mathrm{cl}}\equiv Nl/L$ is the classical (bare) conductance[24,22] in units of e^2/h. Hence large conductances correspond to the ''short-time'' regime of the path integral, which justifies a saddle-point approach for good conductors. The variation $\boldsymbol{M}(s)+\delta\boldsymbol{M}(s)=\delta\boldsymbol{T}^\dagger(s)\boldsymbol{M}(s)\,\delta\boldsymbol{T}(s)$, where $\delta\boldsymbol{T}=\mathbf{1}+\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}$ obeys the symmetries (13) leads to the saddle-point equation

$$0=\delta S\propto\int_0^{1/g_{cl}} ds\ \mathrm{tr}[(\boldsymbol{\varepsilon}^\dagger\boldsymbol{M}+\boldsymbol{M}\varepsilon)\ddot{\boldsymbol{M}^{-1}} - \ddot{\boldsymbol{M}}(\boldsymbol{\varepsilon}\boldsymbol{M}^{-1}+\boldsymbol{M}^{-1}\boldsymbol{\varepsilon}^\dagger)]. \tag{57}$$

One can verify easily that $\boldsymbol{M}_{sp}(s)=\exp\{s\boldsymbol{X}\}$ is the solution for a path which starts at $\boldsymbol{M}(0)=\mathbf{1}$ and ends at $\boldsymbol{M}=\exp\{\boldsymbol{X}/g_{cl}\}$. Evaluation of the saddle-point action yields the transfer-matrix probability measure in the saddle-point approximation,

$$\begin{aligned}p(L;\boldsymbol{T})d\mu(\boldsymbol{T})&\approx\prod_k \exp\left\{-\frac{Nl}{4L}\Gamma_k^2\right\}d\mu(\boldsymbol{T})\\ &=\prod_{k<l}(\cosh\Gamma_k-\cosh\Gamma_l)^2\prod_k \exp\left\{-\frac{Nl}{4L}\Gamma_k^2\right\}\\ &\quad\times\prod_k \sinh\Gamma_k d\Gamma_k\prod_{k=1}^4 d\mu(\boldsymbol{u}_k).\end{aligned} \tag{58}$$

This is just the random-matrix theory probability distribution measure which has been proposed for the transfer matrix.[40,41,28] Since it is known that random transfer-matrix theory describes the stochastic properties of ballistic conductors,[42] we conclude that the saddle point of the path integral correctly describes the ballistic regime of the conductor.

VI. CONCLUSION

In summary, we have presented a path-integral approach to the stochastic properties of mesoscopic disordered conductors. Its application to quasi-one-dimensional wires in the ballistic regime led to the random transfer-matrix theory probability distribution. We believe that known results for the quasi-one-dimensional wire could be recovered by a systematic perturbation expansion in powers of $1/g_{\mathrm{cl}}$. At the moment it is not clear to us whether the short-time regime of the path integral in higher dimensions corresponds as well to conductors with large conductances. That still has to be clarified. The further development of the path-integral technique also remains to be done.

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APPENDIX: INVARIANT MEASURE OF THE TRANSFER-MATRIX GROUP

The invariant measure on the transfer-matrix group does not change under multiplication with a fixed transfer matrix $\boldsymbol{T}_0$ from the left or the right,

$$d\mu(\boldsymbol{T})=d\mu(\boldsymbol{T}_0\boldsymbol{T})=d\mu(\boldsymbol{T}\boldsymbol{T}_0). \tag{A1}$$

In this appendix we prove the claim of Sec. IV that $\Pi_{x=0}^L d\boldsymbol{T}(x)\,\delta_S[\dot{\boldsymbol{T}}(x)\boldsymbol{T}^{-1}(x)]$ is proportional to $\Pi_{x=0}^L d\mu[\boldsymbol{T}(x)]$.

Since the inverse of $\boldsymbol{T}$ in δ_S cannot be handled as easily as u^{-1} in the example of diffusion on the circle, we show first that $\delta_S(\boldsymbol{\varepsilon}) \propto \delta_S(\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{\varepsilon} \boldsymbol{T})$ up to a Jacobian. This will allow us to replace $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$ in the argument of δ_S by $\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \dot{\boldsymbol{T}}$.

Writing the δ function in terms of its Fourier representation yields

$$\delta_S(\boldsymbol{\varepsilon}) = \frac{1}{(2\Pi)^{4N^2}} \int d\boldsymbol{\kappa} \exp\left\{ \frac{i}{2} \mathrm{tr}[\boldsymbol{\kappa}(\boldsymbol{\varepsilon} + \boldsymbol{\Sigma}_z \boldsymbol{\varepsilon}^\dagger \boldsymbol{\Sigma}_z)] \right\}, \tag{A2}$$

where

$$\boldsymbol{\kappa} = \begin{pmatrix} \boldsymbol{\kappa}^{11} & \boldsymbol{\kappa}^{12} \\ \boldsymbol{\kappa}^{21} & \boldsymbol{\kappa}^{22} \end{pmatrix}, \tag{A3}$$

$$\boldsymbol{\kappa}^{11\dagger} = \boldsymbol{\kappa}^{11}, \quad \boldsymbol{\kappa}^{22\dagger} = \boldsymbol{\kappa}^{22}, \quad \boldsymbol{\kappa}^{12\dagger} = -\boldsymbol{\kappa}^{21}, \tag{A4}$$

and

$$d\boldsymbol{\kappa} = \prod_{k<l} d\kappa_{kl}^{11(1)} d\kappa_{kl}^{11(2)} d\kappa_{kl}^{22(1)} d\kappa_{kl}^{22(2)} \times \prod_k d\kappa_{kk}^{11(1)} d\kappa_{kk}^{22(1)} \prod_{k,l} d\kappa_{kl}^{12(1)} d\kappa_{kl}^{12(2)}. \tag{A5}$$

Then the linear transformation

$$\boldsymbol{\varepsilon}' = \boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{\varepsilon} \boldsymbol{T} \tag{A6}$$

of $\boldsymbol{\varepsilon}$ can be absorbed into $\boldsymbol{\kappa}$,

$$\delta_S(\boldsymbol{\varepsilon}') = \frac{1}{(2\Pi)^{4N^2}} \int d\boldsymbol{\kappa} \exp\left\{ \frac{i}{2} \mathrm{tr}[\boldsymbol{\kappa}'(\boldsymbol{\varepsilon} + \boldsymbol{\Sigma}_z \boldsymbol{\varepsilon}^\dagger \boldsymbol{\Sigma}_z)] \right\}, \tag{A7}$$

where

$$\boldsymbol{\kappa}' = \boldsymbol{T} \boldsymbol{\kappa} \boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z . \tag{A8}$$

Since $\boldsymbol{\kappa}'$ has the same symmetries as κ, it follows that

$$\delta_S(\boldsymbol{\varepsilon}') = \delta_S(\boldsymbol{\varepsilon}) / |\mathcal{J}(\boldsymbol{T})|, \tag{A9}$$

where $\mathcal{J}(\boldsymbol{T})$ is the Jacobian of the linear transformation (A8). Hence replacement of the argument $\dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$ in δ_S by $\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \dot{\boldsymbol{T}}$ via the linear transformation (A6) yields

$$\prod_{x=0}^{L} \delta_S[\dot{\boldsymbol{T}}(x) \boldsymbol{T}^{-1}(x)] \propto \int \prod_{x=0}^{L} d\boldsymbol{\kappa}(x) |\mathcal{J}[\boldsymbol{T}(x)]| \times \exp\left\{ i \int_{x=0}^{L} dx \, \mathrm{tr}\left[\boldsymbol{\kappa} \frac{d}{dx} (\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T}) \right] \right\}. \tag{A10}$$

Partial integration and using that $\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T} = \boldsymbol{1}$ at the end points gives

$$\prod_{x=0}^{L} \delta_S[\dot{\boldsymbol{T}}(x) \boldsymbol{T}^{-1}(x)] \propto \int \prod_{x=0}^{L} d\boldsymbol{\kappa}(x) |\mathcal{J}[\boldsymbol{T}(x)]| \times \exp\left\{ -i \int_{x=0}^{L} dx \, \mathrm{tr}[\dot{\boldsymbol{\kappa}}(\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T} - \boldsymbol{1})] \right\}. \tag{A11}$$

The Jacobian of the transformation $\tilde{\boldsymbol{\kappa}} = -\dot{\boldsymbol{\kappa}}$ is a constant. Hence

$$\prod_{x=0}^{L} \delta_S[\dot{\boldsymbol{T}}(x) \boldsymbol{T}^{-1}(x)] \propto \prod_{x=0}^{L} |\mathcal{J}[\boldsymbol{T}(x)]| \, \delta_S[\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger(x) \boldsymbol{\Sigma}_z \boldsymbol{T}(x) - \boldsymbol{1}]. \tag{A12}$$

In order to calculate $\mathcal{J}(\boldsymbol{T})$, we introduce the $(4N^2)$ vector notation

$$\vec{\kappa}^T = (\kappa_{11}, \ldots, \kappa_{12N}, \kappa_{21}, \ldots, \kappa_{2N2N}) \tag{A13}$$

of the matrix $\boldsymbol{\kappa}$. Then $\vec{\kappa}' = [\boldsymbol{T} \otimes (\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z)^T] \vec{\kappa}$. There is a complex matrix $\boldsymbol{E}$ such that $\vec{\kappa} = \boldsymbol{E} \vec{\kappa}_{\mathrm{ind}}$, where $\vec{\kappa}_{\mathrm{ind}}$ contains the $4N^2$ real and imaginary parts of the independent matrix elements of $\boldsymbol{\kappa}$. Therefore,

$$\vec{\kappa}'_{\mathrm{ind}} = \boldsymbol{E}^{-1} [\boldsymbol{T} \otimes (\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z)^T] \boldsymbol{E} \vec{\kappa}_{\mathrm{ind}} . \tag{A14}$$

$\mathcal{J}(\boldsymbol{T})$ is the determinant of this linear transformation, which is one since the δ functions in Eq. (A12) enforces $\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z$ to be the inverse of $\boldsymbol{T}$. That leads to

$$\prod_{x=0}^{L} \delta_S[\dot{\boldsymbol{T}}(x) \boldsymbol{T}^{-1}(x)] \propto \prod_{x=0}^{L} \delta_S[\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger(x) \boldsymbol{\Sigma}_z \boldsymbol{T}(x) - \boldsymbol{1}]. \tag{A15}$$

It remains to be shown that

$$d\mu(\boldsymbol{T}) \equiv d\boldsymbol{T} \delta_S(\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T} - \boldsymbol{1}) \tag{A16}$$

has the properties of Eq. (A1), and is therefore the invariant measure.

For multiplication with a transfer matrix $\boldsymbol{T}_0$ from the left, the argument of the δ function does not change, which leads to

$$d\mu(\boldsymbol{T}_0 \boldsymbol{T}) = d\boldsymbol{T} |\mathcal{I}(\boldsymbol{T}_0)| \, \delta_S(\boldsymbol{\Sigma}_z \boldsymbol{T}^\dagger \boldsymbol{\Sigma}_z \boldsymbol{T} - \boldsymbol{1}), \tag{A17}$$

where $\mathcal{I}(\boldsymbol{T}_0)$ is the Jacobian of the linear transformation $\boldsymbol{T}' = \boldsymbol{T}_0 \boldsymbol{T}$. Expressing this transformation in terms of real vectors yields

$$\begin{pmatrix} \vec{T}'^{(1)} \\ \vec{T}'^{(2)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_0^{(1)} \otimes \boldsymbol{1} & -\boldsymbol{T}_0^{(2)} \otimes \boldsymbol{1} \\ \boldsymbol{T}_0^{(2)} \otimes \boldsymbol{1} & \boldsymbol{T}_0^{(1)} \otimes \boldsymbol{1} \end{pmatrix} \begin{pmatrix} \vec{T}^{(1)} \\ \vec{T}^{(2)} \end{pmatrix}. \tag{A18}$$

The Jacobian $\mathcal{I}(\boldsymbol{T}_0)$ is the determinant of the transformation matrix, which can be decomposed into the product

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \boldsymbol{T}_0\otimes\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}_0^*\otimes\mathbf{1} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ i\mathbf{1} & \mathbf{1} \end{pmatrix} \tag{A19}$$

of three matrices. Since $\boldsymbol{\Sigma}_z\boldsymbol{T}_0^\dagger\boldsymbol{\Sigma}_z\boldsymbol{T}_0=\mathbf{1}$ implies that $\det\boldsymbol{T}_0\det\boldsymbol{T}_0^*=1$, one finds that $\mathcal{I}(\boldsymbol{T}_0)=1$, and therefore $d\mu(\boldsymbol{T}_0\boldsymbol{T})=d\mu(\boldsymbol{T})$.

Analogously it can be shown that the Jacobian for the multiplication with $\boldsymbol{T}_0$ from the right is 1 as well, which gives

$$d\mu(\boldsymbol{T}\boldsymbol{T}_0)=d\boldsymbol{T}\delta_S[\boldsymbol{\Sigma}_z\boldsymbol{T}_0^\dagger\boldsymbol{\Sigma}_z(\boldsymbol{\Sigma}_z\boldsymbol{T}^\dagger\boldsymbol{\Sigma}_z\boldsymbol{T}-\mathbf{1})\boldsymbol{T}_0]. \tag{A20}$$

As shown above, $\delta_S(\boldsymbol{\Sigma}_z\boldsymbol{T}_0^\dagger\boldsymbol{\Sigma}_z\boldsymbol{\varepsilon}\boldsymbol{T}_0)=\delta_S(\boldsymbol{\varepsilon})$. Hence

$$\begin{aligned} d\mu(\boldsymbol{T}\boldsymbol{T}_0)&=d\boldsymbol{T}\delta_S(\boldsymbol{\Sigma}_z\boldsymbol{T}^\dagger\boldsymbol{\Sigma}_z\boldsymbol{T}-\mathbf{1})\\ &=d\mu(\boldsymbol{T}), \end{aligned} \tag{A21}$$

which proves our claim.

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