Non-Abelian geometric phases and conductance of spin-$\frac{3}{2}$ holes

Daniel P. Arovas
Department of Physics, University of California at San Diego, La Jolla, California 92093

Yuli Lyanda-Geller
Department of Physics, Loomis Physics Laboratory and Beckman Institute, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

(Received 20 May 1997; revised manuscript received 17 February 1998)

Angular momentum $J=\frac{3}{2}$ holes in semiconductor heterostructures are shown to accumulate non-Abelian geometric phases as a consequence of their motion. We provide a general framework for analyzing such a system and compute conductance oscillations for a simple ring geometry. We also analyze a figure-eight geometry that captures intrinsically non-Abelian interference effects. [S0163-1829(98)13519-1]

I. INTRODUCTION

A nondegenerate quantum state undergoing adiabatic evolution accumulates both a dynamical as well as a geometric (Berry's) phase.[1,2] The geometric phase is responsible for a wide array of interference phenomena, and has been measured in optics,[3] with neutral beams,[4] and by magnetic resonance.[5,6] Most theoretical and experimental work has focussed on the adiabatic evolution of nondegenerate eigenstates, where the geometric phase may be interpreted as arising from a $U(1)$ gauge potential. The canonical example is that of a spin in a constant amplitude magnetic field $\boldsymbol{B}(t)=B\hat{\boldsymbol{n}}$ whose direction $\hat{\boldsymbol{n}}$ varies in a closed path over the unit sphere.[1] The phase is determined by the solid angle subtended by $\hat{\boldsymbol{n}}$ in the course of its evolution.

In certain high-symmetry situations, an entire n-fold degenerate set of levels may adiabatically evolve. Wilczek and Zee[7] showed that in such cases the $U(1)$ geometric phase generalizes to a $U(n)$ matrix,

$$U=\mathcal{P}\exp\left(-i\oint A^i d\lambda_i\right) \tag{1}$$

where $A^i_{\alpha\beta}=-i\langle\alpha|\partial/\partial\lambda_i|\beta\rangle$ is the gauge potential matrix ($|\alpha\rangle$, $|\beta\rangle$ are adiabatic eigenstates), $\{\lambda_i(t)\}$ is a set of slowly evolving parameters, and $\mathcal{P}$ is the path ordering operator. Such a system may exhibit non-Abelian effects in which, e.g., one member of a multiplet evolves into another upon completion of a cycle in parameter space. One example of this phenomenon is in crystalline nuclear quadrupole resonance (NQR), since the quadrupole Hamiltonian $\mathcal{H}=\frac{1}{2}Q_{ij}I_iI_j$ is quadratic in the spin. When the electric-field gradient tensor has cylindrical symmetry, the Hamiltonian can be taken to be $\mathcal{H}=\hbar\omega_Q(\hat{\boldsymbol{n}}\cdot\boldsymbol{I})^2$, where $\hat{\boldsymbol{n}}$ lies in the direction of the principal axis of Q_{ij}. The non-Abelian gauge structure for this problem was discussed in Refs. 8–10 and measured in the $I=\frac{3}{2}$ nucleus ^{35}Cl by Zwanziger, Koenig, and Pines.[11] Paths in which $\hat{\boldsymbol{n}}$ rotates about more than one axis are essential if intrinsically non-Abelian aspects are to be captured.

In this paper we consider an alternative setting for an observation of the non-Abelian geometric phase. We study angular momentum-$\frac{3}{2}$ holes confined to conducting loops embedded in a two-dimensional hole gas of a heterostructure. A hole's momentum $\boldsymbol{p}$ [or coordinate $\phi(t)$ in a ring] acts as an adiabatically changing quantization axis for its angular momentum $\boldsymbol{J}$. For motion around a ring, however, this amounts to a rotation about only one axis. To exhibit the non-Abelian effects lurking here, we propose to effectively place the system in a rotating frame by imposing a static magnetic field $\boldsymbol{H}$ in the plane of the ring.[12] Intrinsically non-Abelian interference effects are measurable in the conductance oscillations of the figure-eight device discussed below (see Fig. 2). Another notable feature is that unlike the case studied in Refs. 8 and 11, both hole doublets manifest a non-Abelian holonomy.

The coupling of $\boldsymbol{p}$ to $\boldsymbol{J}$, which arises naturally within a $\boldsymbol{k}\cdot\boldsymbol{p}$ treatment of conduction electron and valence hole states, is analogous to the spin-orbit interaction. It is qualitatively different, however, from the spin-orbit splitting of electron states.[13,14] For electrons in zinc-blende crystals, the spin states are split because of the inversion asymmetry, which in a quantum well or heterostructure leads to a linear coupling between spin and momentum;[15] another source of linear coupling is the asymmetry of the quantum well or heterojunction itself.[16] The electron's momentum then acts as an in-plane component of the magnetic field, and as the electron moves around a ring its spin quantization axis traverses the unit sphere at a colatitude $\theta=\tan^{-1}(H_z/\lambda p_\phi)$, where H_z is the physical magnetic field (oriented perpendicular to the plane of the ring), p_ϕ is the azimuthal component of the electron's momentum, and λ is a coupling constant.[17] (The Abelian geometric phase due to an effective momentum-dependent field predicted in Ref. 17 has recently been experimentally observed in Ref. 18.) Although the spin-orbit coupling is by nature a relativistic effect, it is effectively enhanced in a crystalline environment, and the fictitious in-plane component of the field can be of considerable magnitude,[19] although the splitting of ↑ and ↓ states is still much less than the kinetic energy of the electrons.

This situation is quite different for holes in group IV or III-V semiconductors, which are characterized by a 4×4

matrix Luttinger Hamiltonian, acting on states in the Γ_8 representation of double groups of T_d or O_h.[13,14,20] The effective Hamiltonian for bulk holes contains a term $(\boldsymbol{p}\cdot\boldsymbol{J})^2$, which distinguishes between light- ($J^z=\pm\frac{3}{2}$) and heavy- ($J^z=\pm\frac{1}{2}$) hole branches. This term is large and is present in bulk centrosymmetric materials. In contrast to the electron case, characterized by the effective magnetic field, the hole Hamiltonian is thus characterized by the effective quadrupole tensor field. This leads to non-Abelian effects, which can be probed in the conductance of the double loop device, discussed below.

II. HAMILTONIAN AND ITS GAUGE STRUCTURE

The effective Hamiltonian for the valence band is written in terms of a spin-$\frac{3}{2}$ operator $\boldsymbol{J}$ and the crystal momentum $\boldsymbol{p}$:[14,21]

$$\mathcal{H}=-(A+\tfrac{5}{4}B)\boldsymbol{p}^2+B(\boldsymbol{p}\cdot\boldsymbol{J})^2, \tag{2}$$

where A (B) is given by $\frac{1}{4}\hbar^2(m_{\mathrm{lh}}^{-1}\pm m_{\mathrm{hh}}^{-1})$, in terms of the light hole and heavy hole masses. For simplicity, in this work we take the Hamiltonian for bulk holes in the spherical approximation; we also neglect all terms in the Hamiltonian that arise due to the absence of inversion symmetry. Consider now a geometry in which holes are confined to a ring lying within the plane perpendicular to the $\hat{\boldsymbol{z}}\|(001)$ axis. The radial coordinate r is constrained to lie between R and $R+a$, with $a\ll R$ and the coordinate z is also confined. Then the spin-dependent part of the effective Hamiltonian reads

$$\mathcal{H}_{\mathrm{eff}}=\langle p_z^2\rangle BJ_z^2+\tfrac{1}{4}\langle p_r^2\rangle B(J^+e^{-i\phi}+J^-e^{i\phi})^2, \tag{3}$$

valid to order a/R, with $\langle p_r^2\rangle\approx\pi^2\hbar^2/a^2$, and $\langle p_z^2\rangle$ depending on the confining potential. We now treat the ϕ motion semiclassically and let $\phi(t)$ be a prescribed function of time, with $d\phi/dt=\omega$ for motion around a ring.[22] The spin Hamiltonian becomes $\mathcal{H}=K(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^2+\Xi J_z^2$, where $K=B\langle p_r^2\rangle$, $\Xi=B\langle p_z^2\rangle$, and $\hat{\boldsymbol{n}}=\hat{\boldsymbol{x}}\cos\phi(t)+\hat{\boldsymbol{y}}\sin\phi(t)$ is the time-varying principle quadrupole axis for our problem. Now this quadrupole field rotates about only one axis, and in order to extract non-Abelian effects from this setting, we must effectively introduce another axis of rotation by applying an *in-plane* magnetic field $\boldsymbol{H}=H_x\hat{\boldsymbol{x}}$, which adds a term $\mathcal{H}'=-g\mu_BH_xJ_x/\hbar$ to the Hamiltonian. Note that there is no perpendicular component, hence no orbital effects of the magnetic field. We then eliminate H_x by shifting to a rotating basis via the gauge transformation $|\psi\rangle=\exp(-i\Omega tJ_x/\hbar)|\widetilde{\psi}\rangle$. In this basis, the Hamiltonian becomes

$$\widetilde{\mathcal{H}}=V^\dagger\widetilde{\mathcal{H}}_0V(t),$$

$$V(t)=\exp[i\phi(t)J_z/\hbar]\,\exp(i\Omega tJ_x/\hbar),$$

$$\widetilde{\mathcal{H}}_0=KJ_x^2+\Xi J_z^2, \tag{4}$$

with $\Omega=g\mu_BH_x/\hbar$. This Hamiltonian is similar to that explored in Refs. 6, 8 and 11 in the context of $J=\frac{3}{2}$ NQR, although $\widetilde{\mathcal{H}}_0$ in our case is anisotropic.

We next compute the non-Abelian gauge potential matrix $A_{\alpha\beta}(t)$:

$$A_{\alpha\beta}(\phi)=-i\langle\widetilde{\alpha}(t)|\frac{d}{d\phi}|\widetilde{\beta}(t)\rangle, \tag{5}$$

where $|\widetilde{\alpha}(t)\rangle=V^\dagger|\alpha\rangle$ is an adiabatic eigenstate of $\widetilde{\mathcal{H}}$. The eigenstates of $\widetilde{\mathcal{H}}_0$, expressed in eigenstates of J_z, form two degenerate blocks ($\sigma=\pm$):

$$|1,\sigma\rangle=u_\sigma|-\tfrac{3}{2}\rangle+v_\sigma|+\tfrac{1}{2}\rangle,$$

$$|2,\sigma\rangle=u_\sigma|+\tfrac{3}{2}\rangle+v_\sigma|-\tfrac{1}{2}\rangle,$$

$$E_\sigma=\tfrac{5}{4}(K+\Xi)+\sigma\sqrt{K^2+\Xi^2-K\Xi}, \tag{6}$$

with $u_+=v_-=\cos\frac{1}{2}\vartheta$, $v_+=-u_-=\sin\frac{1}{2}\vartheta$, and where $\tan\vartheta=\sqrt{3}K/(2\Xi-K)$. The 4×4 gauge potential matrix $A_{\alpha\beta}$ is block diagonal in this basis with 2×2 subblocks

$$A_\omega^\pm(\phi)=\begin{pmatrix} a & b\\ b^* & -a\end{pmatrix},$$

$$a=(\tfrac{1}{2}\pm\cos\vartheta), \tag{7}$$

$$b=\mp\sqrt{\frac{3}{2}}\frac{\Omega}{\omega}\sin\,\vartheta e^{-i\phi}-\tfrac{1}{2}\frac{\Omega}{\omega}(1\mp\cos\vartheta)e^{+i\phi}.$$

This gauge potential determines the adiabatic evolution of wave functions of holes. Finally, the $U(2)$ phase accrued by a state evolving according to the Hamiltonian $\mathcal{H}+\mathcal{H}'$ is

$$U(t_1,t_0)=e^{ik_FL}e^{-i\phi_1J_z/\hbar}\Lambda_\vartheta^\dagger W_\omega(\phi_1,\phi_0)\Lambda_\vartheta e^{+i\phi_0J_z/\hbar},$$

$$W_\omega(\phi_1,\phi_0)=\mathcal{P}\exp\left(-i\int_{\phi_0}^{\phi_1}d\phi A_\omega(\phi)\right), \tag{8}$$

where Λ_ϑ transforms from the J^z eigenbasis to the basis of Eq. (6), k_F is the Fermi momentum of the holes, L is the distance traveled, and where the path ordering operator places earlier *times* to the right. Adiabaticity is satisfied provided $\omega,\Omega\ll\sqrt{K^2+\Xi^2-K\Xi}$.

III. CONDUCTANCE OSCILLATIONS IN A LOOP

We next consider the ballistic transport of holes in the upper ($\sigma=+$) doublet through a ring confined to the two-dimensional hole gas. High-mobility hole gases have been investigated experimentally in Ref. 23. The mobility of $\approx10^6$ cm^2/V s in these samples makes ballistic transport in such constrictions feasible. We assume that the ring is connected to leads through two antipodally placed T junctions, each described by the S matrix,[24]

$$S=\begin{pmatrix} r & \frac{1}{\sqrt{2}}\sqrt{1-r^2} & \frac{1}{\sqrt{2}}\sqrt{1-r^2}\\ \frac{1}{\sqrt{2}}\sqrt{1-r^2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{\sqrt{2}}\sqrt{1-r^2} & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}, \tag{9}$$

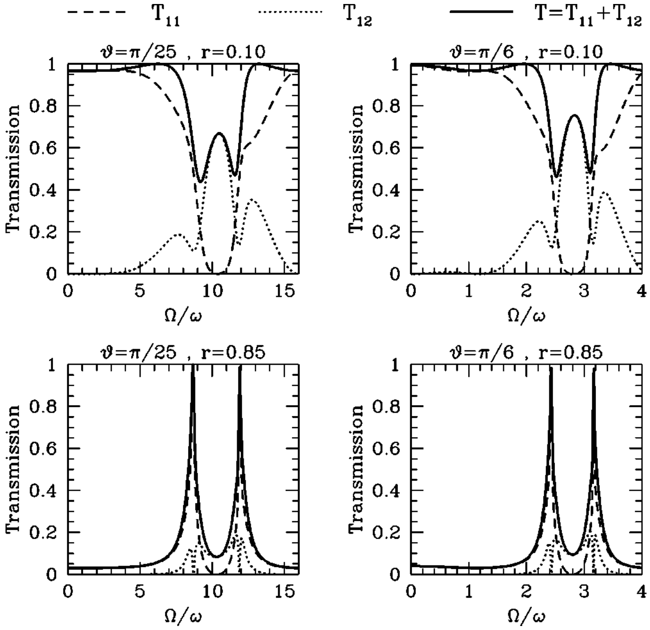

FIG. 1. Transmission coefficients T_{11} and T_{12} as a function of Ω/ω for a simple ring geometry.

where r is the reflection amplitude for a wave incident from the incoming lead, and where, for simplicity, we assume that S is real and is diagonal in the basis of Eq. (6). In Fig. 1 we plot the transmission probabilities $T_{\sigma\sigma'}=T_{\sigma'\sigma}$—due to the non-Abelian geometric phase an incoming hole in state $|1+\rangle$ may be transformed with probability T_{12} to the state $|2+\rangle$. The conductance of the device is given by $G=(e^2/h)\,\Sigma_{\sigma\sigma'}T_{\sigma\sigma'}=(2e^2/h)\,T$ (since both degenerate levels are occupied in the incoming lead). In our computations we assumed that holes are confined to a plane thickness 100 Å, the Fermi energy (for holes) is 2 meV, the ring radius is $1\,\mu$m, and the width of the ring and leads is 400 Å. This corresponds to a rotation frequency $\omega=p_F/m_{\rm h}R\sim4.84\times10^{10}$ Hz.

The $T_{\sigma\sigma'}$ are plotted for $k_FR=0.370$ (mod π)—the qualitative results are roughly insensitive to this parameter—as a function of Ω/ω for two values of ϑ at both weak ($r=0.10$) and strong ($r=0.85$) coupling of leads to ring. With $g=2.6$ as in GaAs, we have $\Omega/H_x=2.28\times10^{12}$ Hz/T, so a ratio of $\Omega/\omega=4$ corresponds to a field of 0.85 T. The resonances arise due to interference effects both Abelian and non-Abelian in origin—the intrinsically non-Abelian effects manifested in T_{12} are not possible to isolate in this geometry.

IV. FIGURE-EIGHT DEVICE

The device depicted in Fig. 2 probes non-Abelian interference effects. Holes incident from lead A may scatter into branch B of the figure eight or else continue on to lead F. Neglecting the effect of the magnetic field on the contacts themselves, we assume a time-reversal invariant (i.e., symmetric) S matrix for the *ABEF* vertex of the form[25]

$$\begin{pmatrix} A_{\rm out}\\ B_{\rm out}\\ E_{\rm out}\\ F_{\rm out}\end{pmatrix}=\begin{pmatrix} u & v & 0 & t\\ v & -u & t & 0\\ 0 & t & u & -v\\ t & 0 & -v & -u\end{pmatrix}\begin{pmatrix} A_{\rm in}\\ B_{\rm in}\\ E_{\rm in}\\ F_{\rm in}\end{pmatrix}, \qquad (10)$$

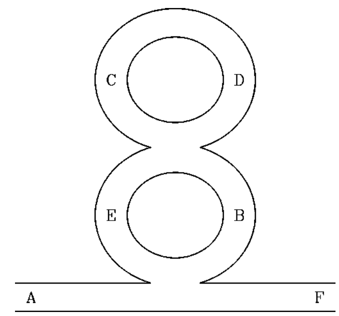

FIG. 2. Figure-eight device.

where $A_{\rm in}$ is a two-component vector describing the incident flux of holes in the upper doublet. The real parameters t,u,v satisfy $t^2+u^2+v^2=1$ and are assumed to be the same for both states of the doublet. $t\approx1$ corresponds to weak coupling between the leads and the figure eight, and $S_{1,3}=S_{2,4}=0$ means that there is negligible backscattering between E and A and between B and F. The vertex at the center of the figure eight is assumed to pass B to C and D to E with negligible scattering into other channels, i.e., the $EDCB$ S matrix corresponds to that of Eq. (10) with $t=1$. This ensures that holes that enter the figure eight through branch B will execute a $BDCE$ circuit before entering the lead F or being rescattered into B. Such a scattering matrix for the $BECD$ contact is realized when this contact is collimating, i.e., it conserves the momentum of holes. The conservation of momentum in the course of transmission through the contact holds if dimensions of the contact are larger than the wavelength of holes, so that diffraction effects are suppressed. Such contacts are technologically feasible and were studied in electron transport (for a review, see Ref. 27). Under these conditions, we may write the relation between B and E as

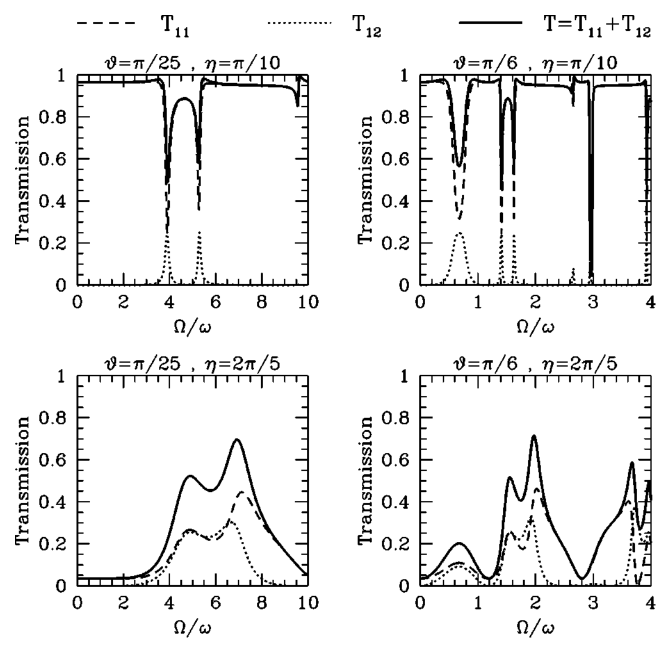

FIG. 3. Transport through the figure eight. The parameters of Eq. (10) are related to η by $t=\cos\eta$, $u=v=1/\sqrt{2}\sin\eta$. $k_FL=0.19\pi\,{\rm mod}\,\pi$ in all cases, where $L=L_B+L_C+L_D+L_E$ is the distance traveled in the figure eight.

$$E_{\rm in}=e^{ik_FL}W_{\omega}(0,-\pi)W_{-\omega}(-\pi,\pi)W_{\omega}(\pi,0)B_{\rm out},$$
$$E_{\rm out}=e^{-ik_FL}W_{-\omega}(0,\pi)W_{\omega}(\pi,-\pi)W_{-\omega}(-\pi,0)B_{\rm in}. \quad (11)$$

This result, in conjunction with Eq. (10), determines the ballistic conductance of the figure-eight device. It is easy to see that when $\Omega=0$ (no in-plane field), the gauge potential A_ω is diagonal and there are no non-Abelian effects—the quadrupole field rotates only about the $\hat{\boldsymbol{z}}$ axis. This reduces the products of W matrices in Eq. (11) to the unit matrix, so that the only interference between the paths AF and $ABCDEF$ is due to the difference in their lengths. In Fig. 3, resonances in the transmission probability for the figure-eight device are shown (we use parameters identical to those for the ring). We find pronounced oscillations with the variation of magnetic field. These oscillations are a non-Abelian effect, because the path executed by the holes corresponds to a zero net solid angle subtended by the effective quadrupole field,[26] meaning that Abelian effects are canceled.

We note that the interaction of holes moving in constrictions with localized holes or nuclei via spin-spin interactions leads to possibilities of observing non-Abelian phases in nuclear- (spin-) resonance experiments. Yet another option is to study optical transitions of constricted holes.

We are grateful to Alex Pines for interesting discussions of non-Abelian settings and to Richard Webb for very useful remarks about adiabatic contacts. We also thank P. Goldbart, A. Leggett, A. Manohar, and L. J. Sham for discussions. Y.L.G. acknowledges support by the National Science Foundation under Grant No. NSF-DMR 91-57018.

[1] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).

[2] A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989).

[3] A. Tomita and R. Y. Chiao, Phys. Rev. Lett. **57**, 937 (1986).

[4] T. Bitter and D. Dubbers, Phys. Rev. Lett. **59**, 251 (1987).

[5] D. Sutter, G. C. Chingas, R. A. Harris, and A. Pines, Mol. Phys. **61**, 1327 (1987).

[6] R. Tycko, Phys. Rev. Lett. **58**, 2281 (1987).

[7] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).

[8] A. Zee, Phys. Rev. A **38**, 1 (1988).

[9] C. A. Mead, Phys. Rev. Lett. **59**, 161 (1987).

[10] J. E. Avron, L. Sadun, J. Segert, and B. Simon, Phys. Rev. Lett. **61**, 1329 (1988); J. Segert, J. Math. Phys. **28**, 2102 (1987).

[11] J. W. Zwanziger, M. Koenig, and A. Pines, Phys. Rev. A **42**, 3107 (1990).

[12] In the case of a nonplanar conductor, non-Abelian effects arise without the imposition of a magnetic field. However, the requirement of high mobility of holes rules out any such configuration at present.

[13] G. Dresselhaus, Phys. Rev. **100**, 580 (1955).

[14] J. M. Luttinger, Phys. Rev. **102**, 1030 (1956).

[15] B. L. Altshuler, A. G. Aronov, D. E. Khmelnitskii, and A. I. Larkin, Zh. Eksp. Teor. Fiz. **81**, 768 (1981) [Sov. Phys. JETP **54**, 411 (1981)].

[16] Yu. L. Bychkov and E. I. Rashba, J. Phys. C **17**, 6093 (1984).

[17] A. G. Aronov and Y. B. Lyanda-Geller, Phys. Rev. Lett. **70**, 343 (1993).

[18] A. F. Morpurgo, J. P. Heida, T. M. Klapwijk, B. J. van Wees, and G. Borghs, Phys. Rev. Lett. **80**, 1050 (1998).

[19] One can estimate the crystalline enhancement of the spin-orbit interaction by substituting the effective band mass m^* for the bare electron mass m, and the band gap E_g for the vacuum energy $2mc^2$, resulting in an enhancement of several orders of magnitude.

[20] The $J=\frac{1}{2}$ split-off hole states are not considered.

[21] We will refer to the J^z quantum number as spin.

[22] For simplicity, we will analyze the problem assuming that $B<A$. Then actually the same frequency ω characterizes semiclassical motion for both hole doublets of states.

[23] M. B. Santos, Y. W. Suen, M. Shayegan, Y. P. Li, L. W. Engel, and D. C. Tsui, Phys. Rev. Lett. **68**, 1188 (1992).

[24] M. Buettiker, Y. Imry, and M. Ya. Azbel, Phys. Rev. A **30**, 1982 (1984).

[25] The most general $N\times N$ S matrix is specified by N^2 real parameters. Time-reversal invariance implies $S=S^t$; this restriction reduces the dimension of the space to $\frac{1}{2}N(N+1)$. Thus, the S matrices of Eqs. (9) and (10) are nongeneric.

[26] In NQR, a path subtending zero net solid angle (the figure eight), was realized in the investigation of effects on nonadiabaticity for nuclei with spin $\frac{1}{2}$ by S. Appelt, G. Wackerie, and M. Mehring, Phys. Lett. A **204**, 210 (1995). As was mentioned in Ref. 11, such a path under adiabatic conditions would allow the observation of non-Abelian phase factor for spin $\frac{3}{2}$.

[27] S. Washburn and R. Webb, Rep. Prog. Phys. **55**, 1364 (1992).