Impurity bands in photonic insulators

N. Stefanou*
Section of Solid State Physics, University of Athens, Panepistimioupolis, GR-157 84 Athens, Greece

A. Modinos
Department of Physics, National Technical University of Athens, Zografou Campus, GR-157 80 Athens, Greece

(Received 12 November 1997)

A chain of impurity cells in a photonic insulator introduces impurity modes of the electromagnetic field over a narrow band of frequencies. We introduce a model of this band in the manner of a tight-binding description of impurity bands in semiconductors, and use it to describe waveguiding along the chain, and, in particular, across a corner of 90°. We also point out the possibility of using impurity bands in photonic insulators to study wave propagation along an effectively one-dimensional disordered chain. [S0163-1829(98)11919-7]

I. INTRODUCTION

Photonic crystals are composite materials with a dielectric function that is a periodic function of the position. The period of this variation or, equivalently, the lattice constant of the photonic crystal, will determine the frequency regions where (absolute) gaps in the frequency spectrum of the electromagnetic (EM) field will (or might) occur. The theoretical study of photonic crystals has advanced considerably over the last few years[1–4] and it is certain that if the appropriate structures can be constructed, these will exhibit absolute gaps in the desired region of the EM spectrum.

On the experimental side, it is now possible to fabricate photonic crystals with frequency gaps in the region up to 4THz (Ref. 5) and further progress to higher frequencies is expected. For a recent review of the subject the reader is referred to Ref. 6.

One possible application of photonic crystals, suggested by Yablonovitch[7] is the possibility of a resonant cavity, which accepts an almost monochromatic mode of the EM field within it.[8,9] Such arises when the material within a unit cell of a photonic crystal exhibiting a gap is modified so as to produce a state of the EM field (a solution of Maxwell equations) with a frequency $\widetilde{\omega}$ within the above gap, which is localized within the modified cell (we shall refer to it as the impurity cell) decaying exponentially outside that cell. The idea is that an emitter, e.g., an excited atom, capable of emitting photons in the frequency region of the gap, placed in the above cavity will only emit the ''right'' frequency photons, permitted by the cavity. In practical applications one would of course like to transfer the emitted photons from their point of creation to an appropriate receiver placed, one assumes, at another point in the photonic crystal at some distance from the initial point. One way of doing this would be to have a chain of impurity cells, each interacting with its neighbors along the chain, transferring the EM energy along the way. This creates automatically a band of frequencies (we shall refer to it as an impurity band), which implies a widening of the single frequency associated with the single impurity. A complete description of a single impurity in a photonic crystal is complicated but it is possible.[8,10] Planar defects in an otherwise perfect crystal have also been considered.[11] It is evident from the above calculations that an exact treatment of wave propagation along a chain of impurity cells is a formidable theoretical problem. It is the purpose of the present paper to introduce a simple model of impurity photonic bands, which demonstrates some of the essential physics of the given problem. The model applies to photonic insulators which can be described by a real negative dielectric function $\epsilon(\omega)$. The obvious example are metals in the frequency region from the near infrared up to the plasmon frequency (in the visible or ultraviolet part of the EM spectrum). In these systems there is a small dissipation of energy but in the noted region of frequency this is usually very small and can be neglected. So that, to a very good approximation, $\epsilon(\omega)$ is real and negative in the above frequency region. Artificial metals that appear to be well described by a real negative dielectric function in the GHz band have recently been proposed by Pendry *et al.*[12] and our model may be applicable to these as well.

Our model, apart from its usefulness in the study of waveguiding as suggested above, can also serve as a model for the study of disorder in such systems: the impurity cells along the chain can vary randomly in some of their properties. It will be seen that the mathematical formalism which describes the propagation of EM waves along a chain of impurity cells is practically identical with that which describes the transport of an electron along a chain of atoms and, therefore, known theoretical results for the electronic problem can be transferred to the photonic one. The advantage of the photonic situation, which has been pointed out already by other authors,[13] is that correlation effects, which complicate matters in the electronic situation, are absent in the photonic one. In the present paper we shall introduce the model and discuss some of its properties, especially in relation to waveguiding, but we shall reserve a more detailed analysis of the model in relation to disorder for a subsequent paper. Waveguiding in photonic crystals has also been discussed from a different point of view by Mekis *et al.*[14]

II. MODEL

We consider an extended (infinite) photonic insulator described by a dielectric function $\epsilon(\omega)$, which is a real nega-

0163-1829/98/57(19)/12127(7)/$15.00

tive quantity. Moreover, we shall assume that over a limited region of frequency of interest to us, we can replace $\epsilon(\omega)$ by a constant $\overline{\epsilon}<0$. Impurity cells in the above insulator are nonabsorbing dielectric spheres of an appropriate dielectric constant ϵ_s. The wave field at frequency ω in and around a single sphere of radius S is described by an electric-field component $\boldsymbol{E}(\boldsymbol{r})\exp(-i\omega t)$, where $\boldsymbol{E}(\boldsymbol{r})$ is given in the usual manner as follows (see, e.g., Ref. 15). Inside the sphere ($r<S$) we have

$$\boldsymbol{E}(\boldsymbol{r})=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{i}{\kappa_s}a_{\ell m}^{E(I)}\nabla\times j_\ell(\kappa_s r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}}) + a_{\ell m}^{H(I)} j_\ell(\kappa_s r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}})\right), \tag{1}$$

where $\kappa_s=\omega\sqrt{\epsilon_s\epsilon_0\mu_0}$, j_ℓ is a spherical Bessel function, and $\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}})$ is a vector spherical harmonic. We shall not write down the explicit form for the corresponding magnetic-field component, which can be obtained in the manner described in Ref. 15. Outside the sphere the electric field is given by

$$\boldsymbol{E}(\boldsymbol{r})=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{i}{\kappa}a_{\ell m}^{E(0)}\nabla\times j_\ell(\kappa r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}}) + a_{\ell m}^{H(0)} j_\ell(\kappa r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}}) + \frac{i}{\kappa}a_{\ell m}^{E(+)}\nabla\times h_\ell^{+}(\kappa r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}}) + a_{\ell m}^{H(+)} h_\ell^{+}(\kappa r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}})\right), \tag{2}$$

where h_ℓ^{+} is a spherical Hankel function. The first two terms in the above equation describe an incident wave and the last two terms a scattered wave. The wave number κ in the present case of negative dielectric constant is a purely imaginary number $\kappa=iq=i\omega\sqrt{-\overline{\epsilon}\epsilon_0\mu_0}$. Because of the spherical symmetry of the scatterer we obtain

$$a_{\ell m}^{E(+)}=T_\ell^{E}a_{\ell m}^{E(0)},$$
$$a_{\ell m}^{H(+)}=T_\ell^{H}a_{\ell m}^{H(0)}, \tag{3}$$

where

$$T_\ell^{E}(\omega)=\left[\frac{j_\ell(\kappa_s r)\frac{\partial}{\partial r}[r j_\ell(\kappa r)]\epsilon_s - j_\ell(\kappa r)\frac{\partial}{\partial r}[r j_\ell(\kappa_s r)]\overline{\epsilon}}{h_\ell^{+}(\kappa r)\frac{\partial}{\partial r}[r j_\ell(\kappa_s r)]\overline{\epsilon} - j_\ell(\kappa_s r)\frac{\partial}{\partial r}[r h_\ell^{+}(\kappa r)]\epsilon_s}\right]_{r=S} \tag{4}$$

with a corresponding expression for $T_\ell^{H}(\omega)$.[15] In the case of a single sphere in a homogeneous medium of negative dielectric function there can be no incident wave: $a_{\ell m}^{E(0)}=a_{\ell m}^{H(0)}=0$ and, therefore, nontrivial states of the EM field, of given ℓm, will exist at a frequency $\tilde{\omega}$ if

$$T_\ell^{E}(\tilde{\omega})=\infty \quad \text{or} \quad T_\ell^{H}(\tilde{\omega})=\infty. \tag{5}$$

In what follows we shall assume that over a region of frequency of interest to us, only the first of Eqs. (5) is satisfied and then only for $\ell=1$. For ω in the neighborhood of $\tilde{\omega}$, we can then write

$$T_1^{E}(\omega)\simeq\frac{A}{\omega-\tilde{\omega}}\equiv\frac{1}{\Omega}, \tag{6}$$

where A is a real quantity, and

$$T_{\ell\neq 1}^{E}(\omega)=T_\ell^{H}(\omega)=0. \tag{7}$$

The value of the constant A in Eq. (6) is determined for a given sphere as follows: near $\omega=\tilde{\omega}$ the denominator of T_1^{E} given by Eq. (4) has the form *constant*$(\omega-\tilde{\omega})$ and the numerator can be approximated by its value at $\tilde{\omega}$. We refer to the above approximation as the electric-dipole approximation and we shall omit hereafter the superscript E in the relevant quantities. We have then, at $\omega=\tilde{\omega}$, a state of the EM field localized on the given sphere. The electric-field component of this state is given by

$$\boldsymbol{E}(\boldsymbol{r})=\frac{1}{\tilde{q}}\sum_{m=-1}^{1}a_{1m}^{(+)}\nabla\times h_1^{+}(i\tilde{q}r)\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}) \tag{8}$$

for $r>S$ and

$$\boldsymbol{E}(\boldsymbol{r})=\frac{i}{\tilde{\kappa}_s}\sum_{m=-1}^{1}a_{1m}^{(I)}\nabla\times j_1(\tilde{\kappa}_s r)\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}) \tag{9}$$

for $r<S$, where $\tilde{q}=q(\tilde{\omega})$ and $\tilde{\kappa}_s=\kappa_s(\tilde{\omega})$. We note that, asymptotically, $h_1^{+}(i\tilde{q}r)\simeq i\exp(-\tilde{q}r)/\tilde{q}r$, as expected for a localized state. The state described by Eqs. (8) and (9) refers to a sphere centered at the origin of the coordinates.

Let us now consider a chain of spheres, and let us assume that the nth sphere along the chain interacts with its nearest neighbors along the chain. We then expect the resonance $\tilde{\omega}$ of the single sphere to widen into a band of frequencies, and energy supplied at one part of the chain to spread throughout the chain. The wave outgoing from the nth sphere, centered at $\boldsymbol{R}_n$, will in this case be given by

$$\boldsymbol{E}_n(\boldsymbol{r})=\frac{1}{q}\sum_{m=-1}^{1}a_{1m}(n)\nabla\times h_1^{+}(iqr_n)\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}_n), \tag{10}$$

where $\boldsymbol{r}_n \equiv \boldsymbol{r} - \boldsymbol{R}_n$ and we have dropped the superscript $(+)$ from the coefficient $a_{1m}^{(+)}$ of Eq. (8). The wave (10) is determined through the first of Eqs. (3), by the wave incident on the nth sphere, which is constituted by the waves outgoing from the $(n-1)$th and the $(n+1)$th spheres, given by Eq. (10) with n replaced by $(n-1)$ and $(n+1)$, respectively. Expanding these waves about the center of the nth sphere (the relevant formulas can be found in Ref. 15) and using Eq. (6) to describe the scattering by the nth sphere (A and $\tilde{\omega}$ may depend on n in the general case) we obtain

$$\Omega(n)\mathbf{a}^T(n) = \mathbf{U}(\boldsymbol{R}_n - \boldsymbol{R}_{n+1})\mathbf{a}^T(n+1) + \mathbf{U}(\boldsymbol{R}_n - \boldsymbol{R}_{n-1})\mathbf{a}^T(n-1), \tag{11}$$

where $\mathbf{a}^T(n)$ is the transpose of a row vector $\mathbf{a}(n)$ of three components $a_{1m}(n)$, $m=-1,0,1$ and $U_{mm'}$ is given by

$$\mathbf{U}(\boldsymbol{R}) = h_0^+(i\tilde{q}R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{h_2^+(i\tilde{q}R)}{2}\begin{pmatrix} \sqrt{\frac{4\pi}{5}}Y_{20}(\hat{\boldsymbol{R}}) & \sqrt{\frac{12\pi}{5}}Y_{21}(\hat{\boldsymbol{R}}) & \sqrt{\frac{24\pi}{5}}Y_{22}(\hat{\boldsymbol{R}}) \\ \sqrt{\frac{12\pi}{5}}Y_{21}^*(\hat{\boldsymbol{R}}) & -2\sqrt{\frac{4\pi}{5}}Y_{20}(\hat{\boldsymbol{R}}) & -\sqrt{\frac{12\pi}{5}}Y_{21}(\hat{\boldsymbol{R}}) \\ \sqrt{\frac{24\pi}{5}}Y_{22}^*(\hat{\boldsymbol{R}}) & -\sqrt{\frac{12\pi}{5}}Y_{21}^*(\hat{\boldsymbol{R}}) & \sqrt{\frac{4\pi}{5}}Y_{20}(\hat{\boldsymbol{R}}) \end{pmatrix}, \tag{12}$$

which is obviously a Hermitian matrix. We note that in Eq. (12) we have replaced q by $\tilde{q}$, which is valid for $\omega \simeq \tilde{\omega}$ in the spirit of our approximation. We note also that

$$\mathbf{U}(\boldsymbol{R}) = \mathbf{U}(-\boldsymbol{R}) \tag{13}$$

because of the property of the spherical harmonics: $Y_{2m}(\hat{\boldsymbol{R}}) = Y_{2m}(-\hat{\boldsymbol{R}})$. If we assume all spheres to be in the xz plane with a constant distance R between the centers of successive spheres, $\mathbf{U}$ simplifies as follows:

$$\mathbf{U}(\boldsymbol{R}) = h_0^+(i\tilde{q}R)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{h_2^+(i\tilde{q}R)}{2}\begin{pmatrix} \frac{1}{2}(3\cos^2\vartheta - 1) & -3\frac{\sqrt{2}}{2}\sin\vartheta\cos\vartheta & \frac{3}{2}\sin^2\vartheta \\ -3\frac{\sqrt{2}}{2}\sin\vartheta\cos\vartheta & 1-3\cos^2\vartheta & 3\frac{\sqrt{2}}{2}\sin\vartheta\cos\vartheta \\ \frac{3}{2}\sin^2\vartheta & 3\frac{\sqrt{2}}{2}\sin\vartheta\cos\vartheta & \frac{1}{2}(3\cos^2\vartheta - 1) \end{pmatrix} \tag{14}$$

where ϑ is the polar angle of $\boldsymbol{R}$.

III. INFINITE PERIODIC CHAIN OF SPHERES

In the case of an infinite linear chain of identical spheres, $\Omega(n)$ and $\boldsymbol{R}_{n+1} - \boldsymbol{R}_n$ in Eq. (11) are independent of n. In this case, using Bloch's theorem, we can write $\mathbf{a}(n+1) = \exp(ikR)\mathbf{a}(n)$ and, therefore, Eq. (11) reduces to an eigenvalue problem of a 3×3 matrix, as follows:

$$2\cos(kR)\ \mathbf{U}(\boldsymbol{R})\mathbf{a}^T = \Omega\mathbf{a}^T. \tag{15}$$

If the chain lies in the xz plane and makes an angle ϑ with the z axis, the matrix $\mathbf{U}(\boldsymbol{R})$, given by Eq. (14), can be readily diagonalized and we obtain from Eq. (15) two frequency bands: one nondegenerate band (α) and one doubly degenerate band (β) of halfwidths

$$W_\alpha = 2h_0^+(i\tilde{q}R) + 2h_2^+(i\tilde{q}R) = \left(\frac{1}{\tilde{q}R} + \frac{1}{(\tilde{q}R)^2}\right)\frac{6\exp(-\tilde{q}R)}{\tilde{q}R} \tag{16}$$

and

$$W_\beta = 2h_0^+(i\tilde{q}R) - h_2^+(i\tilde{q}R) = -\left(1 + \frac{1}{\tilde{q}R} + \frac{1}{(\tilde{q}R)^2}\right)\frac{3\exp(-\tilde{q}R)}{\tilde{q}R}, \tag{17}$$

respectively. Thus we obtain

$$\Omega_\mu(k) = W_\mu \cos(kR) \tag{18}$$

for $\mu = \alpha, \beta$, with $-\pi < kR < \pi$. The corresponding normalized eigenvectors (Bloch waves) for a given ϑ can be expressed in terms of the unit vectors $\hat{\mathbf{a}}_1 = 1/\sqrt{2}(1,0,-1)$, $\hat{\mathbf{a}}_2 = 1/\sqrt{2}(1,0,1)$, and $\hat{\mathbf{a}}_3 = (0,1,0)$ as follows:

$$\hat{\mathbf{a}}_\alpha^{(\vartheta)} = \hat{\mathbf{a}}_3\cos\vartheta + \hat{\mathbf{a}}_1\sin\vartheta,$$

$$\hat{\mathbf{a}}_\beta^{(\vartheta)} = \begin{cases} \hat{\mathbf{a}}_3\sin\vartheta - \hat{\mathbf{a}}_1\cos\vartheta \\ \hat{\mathbf{a}}_2. \end{cases} \tag{19}$$

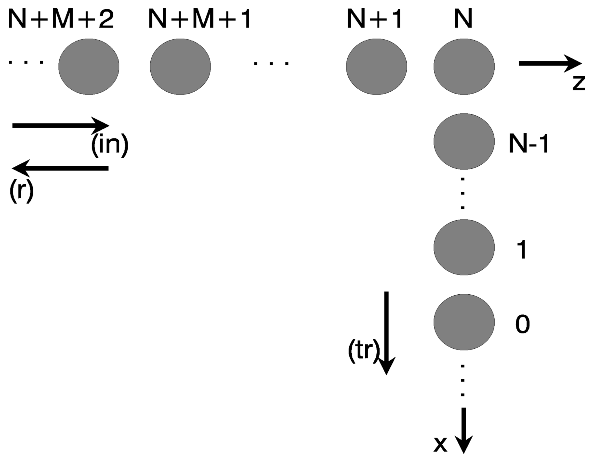

FIG. 1. Waveguiding across a 90°-corner.

Each Bloch wave carries energy along the chain, which is determined by the component of the Poynting vector along the axis of the chain, integrated over a plane normal to this axis. In the Appendix we show that this quantity averaged over a period $2\pi/\omega$ is, for a given mode (a given Bloch wave), given by

$$I_\mu = \frac{1}{4qR\omega\mu_0} \frac{\partial \Omega_\mu(k)}{\partial k} \sum_{m=-1}^{1} |a_{1m;\mu}|^2. \tag{20}$$

Formulas analogous to Eqs. (11) and (18) are well known in the study of electron energy bands in crystals by the tight-binding method. This allows us to transfer already known results from the electronic to the photonic problem. It is evident, for example, that the transfer of energy along a chain implies a certain widening of the resonance frequency associated with a single impurity sphere, for otherwise the group velocity $d\omega_\mu/dk$ associated with this transfer vanishes. We shall return to this analogy in relation to disorder in Sec. V.

IV. TRANSMISSION OF LIGHT ACROSS A CORNER

It follows from Eq. (19) that $\hat{\mathbf{a}}_2$ is an eigenvector of $\mathbf{U}(\boldsymbol{R})$, for any direction of $\boldsymbol{R}$ in the xz plane. Consequently, according to Eq. (11), a mode of this type is totally transmitted through a chain of identical spheres in the xz plane however it bends (the direction of $\boldsymbol{R}_{n+1}-\boldsymbol{R}_n$ may change arbitrarily) from sphere to sphere, provided that the distance between successive spheres remains constant.

We now consider a chain of identical equidistant spheres, stretching from $(0,-\infty)$ to (0,0) along the z axis and then from (0,0) to $(\infty,0)$ along the x axis as shown in Fig. 1.

Let us consider the transmission of a mode $\hat{\mathbf{a}}_3$. A Bloch wave of this type propagating along the z leg of the chain has a frequency within the nondegenerate band. At the corner it is partly reflected and partly transmitted into the x leg of the chain, propagating in the positive x direction as a Bloch wave of the doubly degenerate band. To find the corresponding transmission coefficient we proceed as follows: we assign a value $\mathbf{a}(0)=(0,1,0)$ to a sphere (the zeroth sphere) of the x leg and, because we are considering a Bloch wave of the doubly degenerate band, $\mathbf{a}(1)=\exp(-i\phi_\beta)\mathbf{a}(0)$, where $\phi_\mu = kR$ is to be determined for the given frequency from Eq. (18) for $\mu=\alpha,\beta$. Accordingly, the field at the Nth sphere (the sphere at the corner) is given by $\mathbf{a}(N) =\exp(-iN\phi_\beta)\mathbf{a}(0)$. We obtain the field at the $(N+1)$th sphere using Eq. (11). The result is: $\mathbf{a}(N+1) =(W_\beta/W_\alpha)\exp(-i(N+1)\phi_\beta)\mathbf{a}(0)$. We then iterate the recurrence relation (11) along the z leg of the chain starting from $\mathbf{a}(N)$, $\mathbf{a}(N+1)$. This generates a sequence, the general term of which can be expressed in terms of the type-II Chebyshev polynomials,[16] and in this way we obtain the field at the $(N+M+1)$th sphere,

$$\mathbf{a}(N+M+1) = \left(\frac{W_\beta}{W_\alpha} \exp[-i(N+1)\phi_\beta] \frac{\sin M\phi_\alpha}{\sin \phi_\alpha} - \exp(-iN\phi_\beta) \frac{\sin (M-1)\phi_\alpha}{\sin \phi_\alpha} \right) \mathbf{a}(0). \tag{21}$$

This field matches continuously a wave traveling to the right, of amplitude $\mathbf{a}^{(\mathrm{in})}$ (incident wave) and a wave traveling to the left, of amplitude $\mathbf{a}^{(\mathrm{r})}$ (reflected wave). At two neighboring spheres we have $\mathbf{a}(N+M+1)=\mathbf{a}^{(\mathrm{in})}(N+M+1)+\mathbf{a}^{(\mathrm{r})}(N+M+1)$ and $\mathbf{a}(N+M+2)=\mathbf{a}^{(\mathrm{in})}(N+M+2)+\mathbf{a}^{(\mathrm{r})}(N+M+2)$. On the other hand, Bloch's theorem implies: $\mathbf{a}^{(\mathrm{in})}(N+M+1)=\mathbf{a}^{(\mathrm{in})}(N+M+2)\exp(i\phi_\alpha)$ and $\mathbf{a}^{(\mathrm{r})}(N+M+1) =\mathbf{a}^{(\mathrm{r})}(N+M+2)\exp(-i\phi_\alpha)$. Therefore, the amplitudes of the reflected and the incident waves can be readily obtained from $\mathbf{a}(N+M+1)$ and $\mathbf{a}(N+M+2)$, given by Eq. (21) and, as it turns out, their magnitudes are independent of N and M, as expected. After calculating $\mathbf{a}^{(\mathrm{in})}$ and $\mathbf{a}^{(\mathrm{r})}$ in the above manner we obtain the energy fluxes associated with the incident and reflected waves using Eq. (20). The transmitted flux is calculated in the same way from the given transmitted wave. Finally we obtain for the ratio $t(\Omega)$ of the transmitted to the incident flux, and for the ratio $r(\Omega)$ of the reflected to the incident flux the following formulas:

$$t(\Omega) = \frac{4\sqrt{W_\alpha^2-\Omega^2}\sqrt{W_\beta^2-\Omega^2}}{(\sqrt{W_\alpha^2-\Omega^2}+\sqrt{W_\beta^2-\Omega^2})^2},$$

$$r(\Omega) = \frac{(\sqrt{W_\alpha^2-\Omega^2}-\sqrt{W_\beta^2-\Omega^2})^2}{(\sqrt{W_\alpha^2-\Omega^2}+\sqrt{W_\beta^2-\Omega^2})^2}. \tag{22}$$

It can be readily verified that $t(\Omega)+r(\Omega)=1$, as it must. The same results are obtained for a mode of type $\hat{\mathbf{a}}_1$.

In Fig. 2 we show the two bands $\Omega_\mu(k)$, $\mu=\alpha,\beta$ for a specific example corresponding to $\tilde{q}R=3$ calculated from Eqs. (16), (17), and (18), and the corresponding transmission coefficient for an incident wave of mode $\hat{\mathbf{a}}_1$ or $\hat{\mathbf{a}}_3$ calculated from the first of Eqs. (22).

A rather interesting interpretation of the above result can be obtained by a simple model similar to that proposed by Mekis *et al.*[14] In the case, say, of an $\hat{\mathbf{a}}_3$ incident mode of frequency ω, the incident and reflected waves in the z leg of the chain are propagating Bloch waves with a group velocity, $v_\alpha = d\omega_\alpha/dk$, determined from the slope of the nondegenerate band α at ω. Similarly, the transmitted Bloch wave in the x leg of the chain propagates with a group velocity, $v_\beta =d\omega_\beta/dk$, determined from the slope of the doubly degen-

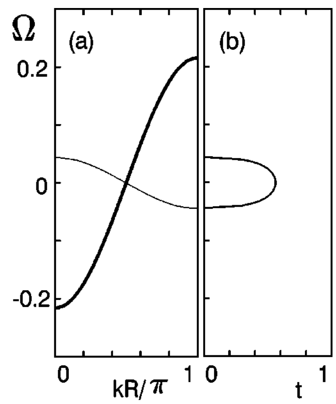

FIG. 2. (a) Nondegenerate (thin line) and doubly degenerate (thick line) bands of an infinite chain of spheres with $\tilde{q}R=3$. (b) Transmittance of an $\hat{\mathbf{a}}_1$ or $\hat{\mathbf{a}}_3$ mode across the corner of Fig. 1.

erate band β at ω. We can then view the transmission through the corner as follows: the $\hat{\mathbf{a}}_3$ mode, propagating with a wave vector $k_\alpha(\omega)=n_\alpha(\omega)/c$ in a homogeneous effective medium of refractive index $n_\alpha(\omega)=c/v_\alpha(\omega)$, is partly transmitted into a different homogeneous effective medium of refractive index $n_\beta(\omega)=c/v_\beta(\omega)$, with a wave vector $k_\beta(\omega)=n_\beta(\omega)/c$, where c is the velocity of light in vacuum. The transmission and reflection coefficients for this simple scattering problem are well known; we have $t=4k_\alpha k_\beta/(k_\alpha+k_\beta)^2$, $r=(k_\alpha-k_\beta)^2/(k_\alpha+k_\beta)^2$. Substituting into these equations the effective wave vectors k_α, k_β as described above, we recover Eqs. (22).

In the case of a corner of arbitrary angle or, for that matter, of a number of such corners, we proceed in similar manner. However, in the general case, starting with a single, $\hat{\mathbf{a}}_1$ or $\hat{\mathbf{a}}_3$, transmitted mode in the x leg of the chain, the recurrence relation (11) finally leads to a field that mixes these two modes. At a sphere N' in the z leg of the chain, the field has in general the form of a linear combination of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$,

$$\mathbf{a}(N')=c_\alpha(N')\hat{\mathbf{a}}_3+c_\beta(N')\hat{\mathbf{a}}_1, \tag{23}$$

where the coefficients c_μ satisfy the recurrence relation

$$c_\mu(N'+1)=2\cos\phi_\mu\ c_\mu(N')-c_\mu(N'-1) \tag{24}$$

for $\mu=\alpha,\beta$. This field matches continuously an incident and a reflected wave, as we have previously described; the only difference is that now the incident and reflected waves are linear combinations of two Bloch waves, associated with the modes $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$, respectively. By using an appropriate linear combination of transmitted waves we can obtain the desired single incident mode and the corresponding transmission and reflection coefficients for this mode as in the case of the 90° bend.

V. THE CASE OF DISORDER

We shall consider only the case of a linear chain of spheres (stretching along the z axis). In this case the matrix defined by Eq. (14) is a diagonal matrix and one obtains three decoupled equations in the place of Eq. (11). The corresponding solutions, denoted by $\mu=-1,0,1$, have the form $a_{1m;\mu}(n)=a_\mu(n)\delta_{m\mu}$, and each satisfies an equation as follows:

$$\Omega(n)a_\mu(n)=\chi_\mu(\boldsymbol{R}_n-\boldsymbol{R}_{n+1})a_\mu(n+1)+\chi_\mu(\boldsymbol{R}_n-\boldsymbol{R}_{n-1})a_\mu(n-1), \tag{25}$$

where

$$\chi_{\pm1}(\boldsymbol{R})=h_0^+(i\tilde{q}R)-\frac{h_2^+(i\tilde{q}R)}{2},$$

$$\chi_0(\boldsymbol{R})=h_0^+(i\tilde{q}R)+h_2^+(i\tilde{q}R). \tag{26}$$

Equation (25) can be rewritten, for any given μ, as follows:

$$\begin{pmatrix} a(n) \\ a(n+1) \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -\chi^{-1}(\boldsymbol{R}_n-\boldsymbol{R}_{n+1})\chi(\boldsymbol{R}_n-\boldsymbol{R}_{n-1}) & \chi^{-1}(\boldsymbol{R}_n-\boldsymbol{R}_{n+1})\Omega(n) \end{pmatrix}\begin{pmatrix} a(n-1) \\ a(n) \end{pmatrix}\equiv\mathbf{T}(n)\begin{pmatrix} a(n-1) \\ a(n) \end{pmatrix}, \tag{27}$$

and we have dropped the index μ for the sake of clarity in the presentation. $\mathbf{T}(n)$ is referred to as the transfer matrix. For an ordered chain this is independent of n and one obtains

$$\begin{pmatrix} a(n+n'-1) \\ a(n+n') \end{pmatrix}=\mathbf{T}^n\begin{pmatrix} a(n'-1) \\ a(n') \end{pmatrix}. \tag{28}$$

In a disordered chain $\mathbf{T}(n)$ changes randomly from site to site. This may occur either because we have spheres of randomly varying size and/or dielectric function (this affects A and $\tilde{\omega}$) or because the separation between the spheres changes randomly from site to site. Now, the treatment of randomness on the basis of Eq. (27) has attracted a lot of attention; it is well documented (see, e.g., Ref. 17) and in this respect we have nothing to add here. The point we wish to make is the following. So far the treatment of one-dimensional disorder [based on Eq. (27)] has been advanced as a model for electronic motion in effectively one-dimensional systems. A disordered chain of impurity cells in a photonic insulator provides a basis for an alternative verification of existing theoretical results in relation to wave propagation in disordered media, which has the advantage that electron-electron correlation, which complicates matters in the electronic problem, does not arise in the photonic one.

VI. CONCLUSION

In conclusion we may say that the formulas provided can be used to study the propagation of light along a chain of impurity cells (a waveguide) in an extended photonic insulator described by a real negative dielectric function. Our formalism also provides a basis for the study of Anderson localization and related problems along a disordered chain of such cells.

APPENDIX

We consider an infinite, linear and periodic chain of dielectric spheres (along the z axis) in a medium of negative dielectric constant $\overline{\epsilon}$. The electric and magnetic fields in the region between the nth and the $(n+1)$th spheres are given in the electric-dipole approximation by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{q}\sum_{m=-1}^{1}\left[a_{1m}(n)\nabla\times h_1^+(iqr_n)\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}_n) + a_{1m}(n+1)\nabla\times h_1^+(iqr_{n+1})\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}_{n+1})\right],$$

$$\boldsymbol{H}(\boldsymbol{r}) = \sqrt{\frac{\overline{\epsilon}\epsilon_0}{\mu_0}}\sum_{m=-1}^{1}\left[a_{1m}(n)h_1^+(iqr_n)\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}_n) + a_{1m}(n+1)h_1^+(iqr_{n+1})\boldsymbol{X}_{1m}(\hat{\boldsymbol{r}}_{n+1})\right], \qquad \text{(A1)}$$

where $\boldsymbol{r}_n$, $\boldsymbol{r}_{n+1}$ are site-centered coordinates: $\boldsymbol{r}_n = (\boldsymbol{r}_\parallel, z_n)$. Between the nth and the $(n+1)$th spheres we have: $z_n - z_{n+1} = R$. We are interested in the z component of the Poynting vector averaged over a period $2\pi/\omega$ and integrated over a plane normal to the chain at a point between the nth and the $(n+1)$th spheres. We denote this quantity by I. We have

$$I = \frac{1}{2}\mathrm{Re}\int d^2r_\parallel[\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}^*(\boldsymbol{r})]_z. \qquad \text{(A2)}$$

We shall employ the identity

$$h_\ell^+(\kappa r)\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{r}}) = \frac{(-i)^\ell}{2\pi\kappa}\int\frac{d^2p_\parallel}{\sqrt{\kappa^2-p_\parallel^2}}\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{p}}^\pm)\exp(i\boldsymbol{p}^\pm\cdot\boldsymbol{r}) \qquad \text{(A3)}$$

with $\boldsymbol{p}^\pm = \boldsymbol{p}_\parallel \pm \hat{\boldsymbol{z}}\sqrt{(\kappa^2 - p_\parallel^2)}$ where the plus and minus signs are used for $z>0$ and $z<0$, respectively. We note that in the present case $\kappa = iq$ is an imaginary number and that, therefore, $\cos\vartheta_{\boldsymbol{p}^\pm}$ and $\sin\vartheta_{\boldsymbol{p}^\pm}$ in an ordinary spherical harmonic $Y_{\ell m}(\hat{\boldsymbol{p}}^\pm)$ are to be replaced by $\pm\sqrt{(q^2+p_\parallel^2)}/q$ and $-ip_\parallel/q$, respectively. Other than that, $\boldsymbol{X}_{\ell m}(\hat{\boldsymbol{p}}^\pm)$ are given by standard formulas (see, e.g., Ref. 15).

Expanding the fields (A1) in plane waves by the use of Eq. (A3), substituting the resulting expression in Eq. (A2) and using the orthogonality of the plane waves: $\int d^2r_\parallel \exp[i(\boldsymbol{p}_\parallel - \boldsymbol{p}'_\parallel)\cdot\boldsymbol{r}_\parallel] = (2\pi)^2\delta(\boldsymbol{p}_\parallel - \boldsymbol{p}'_\parallel)$, we obtain

$$I = \frac{1}{2q^2\omega\mu_0}\mathrm{Re}\sum_{m,m'}\Bigg(a_{1m}(n)a^*_{1m'}(n+1)\times\int\frac{d^2p_\parallel}{q^2+p_\parallel^2}\{[\boldsymbol{p}^+\times\boldsymbol{X}_{1m}(\hat{\boldsymbol{p}}^+)]\times\boldsymbol{X}^*_{1m'}(\hat{\boldsymbol{p}}^-)\}_z \times\exp(-R\sqrt{q^2+p_\parallel^2}) + a_{1m}(n+1)a^*_{1m'}(n)\times\int\frac{d^2p_\parallel}{q^2+p_\parallel^2}\{[\boldsymbol{p}^-\times\boldsymbol{X}_{1m}(\hat{\boldsymbol{p}}^-)]\times\boldsymbol{X}^*_{1m'}(\hat{\boldsymbol{p}}^+)\}_z \times\exp(-R\sqrt{q^2+p_\parallel^2})\Bigg). \qquad \text{(A4)}$$

In writing Eq. (A4) we have dropped all terms proportional to $a_{1m}(n)a^*_{1m'}(n)$ and $a_{1m}(n+1)a^*_{1m'}(n+1)$ since the contribution of these terms to I turns out to be of the form Re{*Imaginary quantity*} and therefore vanishes identically. Rewriting the vector products in Eq. (A4) according to the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ and substituting in the resulting formula the explicit expressions for $\boldsymbol{X}_{1m}(\hat{\boldsymbol{p}}^\pm)$ in cylindrical coordinates, we can finally perform the $\boldsymbol{p}_\parallel$ integration analytically. In the case of propagating Bloch waves, when $a_{1m}(n+1) = \exp(ikR)a_{1m}(n)$, we obtain

$$I = \frac{1}{4q\omega\mu_0}\Bigg[(|a_{1-1}(n)|^2 + |a_{11}(n)|^2)\times\left(1+\frac{1}{qR}+\frac{1}{(qR)^2}\right)\frac{3\exp(-qR)}{qR}\sin(kR) - |a_{10}(n)|^2\left(\frac{1}{qR}+\frac{1}{(qR)^2}\right)\frac{6\exp(-qR)}{qR}\sin(kR)\Bigg]. \qquad \text{(A5)}$$

In the spirit of our approximation (Sec. II) we replace q by $\tilde{q}$ within the square brackets of Eq. (A5). By comparing with Eqs. (16), (17), and (18) we can then easily verify that, for given $\mu = \alpha, \beta$, Eq. (A5) gives

$$I_\mu = \frac{1}{4qR\omega\mu_0}\frac{\partial\Omega_\mu(k)}{\partial k}\sum_{m=-1}^{1}|a_{1m;\mu}|^2, \qquad \text{(A6)}$$

where $|a_{1m;\mu}| = |a_{1m;\mu}(n)|$ is independent of n.

*Electronic address: nstefan@atlas.uoa.gr

[1] J. D. Joannopoulos, R. D. Meade, and J. N. Winn, *Photonic Crystals* (Princeton University Press, New York, 1995).

[2] N. Stefanou, V. Karathanos, and A. Modinos, J. Phys.: Condens. Matter **4**, 7389 (1992); A. Modinos, N. Stefanou, and V. Karathanos, in *Photonic Band Gap Materials* (Ref. 6), p. 229.

[3] J. B. Pendry, J. Mod. Opt. **41**, 209 (1993); P. M. Bell, J. B. Pendry, L. Martín-Moreno, and A. Ward, Comput. Phys. Commun. **85**, 306 (1995).

[4] K. Ohtaka and Y. Tanabe, J. Phys. Soc. Jpn. **65**, 2265 (1996); **65**, 2276 (1996); **65**, 2670 (1996).

[5] M. C. Wanke, O. Lehmann, K. Müller, Q. Wen, and M. Stuke, Science **275**, 1284 (1997).

[6] *Photonic Band Gap Materials*, edited by C. M. Soukoulis

(Kluwer Academic, Dordrecht, 1996).
[7] E. Yablonovitch, Phys. Rev. Lett. **58**, 2059 (1987).
[8] R. D. Meade, K. D. Brommer, A. M. Rappe, and J. D. Joannopoulos, Phys. Rev. B **44**, 13 772 (1991).
[9] E. Yablonovitch, T. J. Gmitter, R. D. Meade, K. D. Brommer, A. M. Rappe, and J. D. Joannopoulos, Phys. Rev. Lett. **67**, 3380 (1991).
[10] V. Karathanos, Ph.D. thesis, National Technical University of Athens, 1997.
[11] V. Karathanos, A. Modinos, and N. Stefanou, J. Phys.: Condens. Matter **6**, 6257 (1994).
[12] J. B. Pendry, A. J. Holden, W. J. Stewart, and I. Youngs, Phys. Rev. Lett. **76**, 4773 (1996).
[13] S. John, Phys. Rev. Lett. **58**, 2486 (1987).
[14] A. Mekis, J. C. Chen, I. Kurland, S. Fan, P. R. Villeneuve, and J. D. Joannopoulos, Phys. Rev. Lett. **77**, 3787 (1996).
[15] A. Modinos, Physica A **141**, 575 (1987).
[16] G. Arfken, *Mathematical Methods for Physicists* (Academic Press, New York, 1970).
[17] J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, 1979).