# Fundamental inversion problem for the magnetic-force microscopy of superconductors

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A fundamental inversion problem for the magnetic-force microscopy (MFM) of a semi-infinite superconductor in the Meissner state is formulated. Under certain assumptions on the MFM tip and superconductor geometry, a unique layer-dependent penetration depth $\lambda(z)$ can be recovered from one-dimensional force measurements. This development opens new possibilities for the nondestructive evaluation of superconducting crystals and films. Some of the implications of the detailed knowledge of the temperature dependence of the penetration depth at low temperature on the superconducting pairing state are noted. [S0163-1829(98)04818-8]

## INTRODUCTION

In magnetic-force microscopy[1–3] a magnetized tip senses the stray field of a sample. A flexible cantilever, often with an optical sensor of displacement,[1–3] is used to measure the force. In particular, magnetic-force microscopes (MFMs) operating at cryogenic temperatures provide a tool for measuring the electromagnetic fields of superconductors.[2,3] This paper is concerned with the inversion of MFM force data for superconductors, where the measured stray field is induced by the tip.

In the direct MFM modeling problem for superconductors, material properties such as the penetration depth and magnetic permeability are assumed given. On the other hand, in the inverse problem the material characteristics are to be extracted from the measurements. In the direct problem, calculated fields give the expected force, while in the inverse problem the force is to be used to recover the material properties.

It may be useful to recall some broad characteristics of inverse problems, as these lie at the core of the observational problem in physics. These problems are among the most challenging in all of mathematical physics. General procedures for solution are limited to very special classes of problems and in practice each procedure must be fashioned to the specifics of the application. The traditional theoretical (direct) approach is to make an informed guess about the properties of a system and then see what consequences follow. However, the more fundamental, and generally more difficult approach, as taken in this paper, is to recover the system properties from the observables.

In general, the MFM inversion problem is ill posed, with difficulties of nonuniqueness and instability, a feature of this class of problems.[4] This paper focuses on a semi-infinite superconductor, probed by a point tip of specific magnetization, where force data at all heights above the surface are assumed known. The result of inversion is a unique layer-dependent penetration depth function $\lambda(z)$.

The purpose of this paper is to demonstrate in detail that in principle $\lambda=\lambda(z)$ can be recovered. However, the stability of the algorithm needs to be further explored and improved. In particular, the question of numerical inverse Laplace transformation needs to be addressed.[5] In addition, curve-fitting procedures need to be considered in expanding the wave number-dependent kernel function. Alternatives to this approach exist, which are secondary to the current theoretical aim.

The results of this research have many potential technological applications. This new approach to the analysis of MFM data opens the way to a new method of characterizing superconducting crystals and films. The function $\lambda(z)$ reflects the quality of the sample, with $\lambda$ diverging for normal material. The inversion procedure provides a contactless method to access a property throughout the thickness of the sample.

In particular, multilayer superconducting structures are now routinely made by processes including laser ablation/deposition and sputtering techniques.[6] The data from such structures with alternating superconductor and substrate could provide input to the algorithm. In any case, in first testing the algorithm, synthetic data can and should be generated for analytically solvable models. The discussion section of this paper touches on this point. Experimental data with the concomitant noise bring another set of issues.

The inverse MFM problem also has many implications for the basic physics of superconductivity. The detailed temperature dependence of the penetration depth, especially at temperatures close to absolute zero, can shed light on the basic mechanism. In particular, the deviation of $\lambda(T)$ from a constant at low temperature, of exponential vs power-law form, is of interest. For $s$-wave superconductivity,[7] $\Delta\lambda/\lambda(0) \equiv[\lambda(T)-\lambda(0)]/\lambda(0)\approx 3.33(T_c/T)^{1/2}\exp(-1.76T_c/T)$ for low temperature, where $T_c$ is the transition temperature. However, other pairing states can lead to an algebraic temperature dependence at low temperature; powers $T$, $T^2$, $T^3$, and $T^4$ are possible depending upon the type of nodes in the energy gap.[8] Furthermore, the specific power, if this case holds, can give insight into the role of disorder in the material. A study of $YBa_2Cu_3O_{7-\delta}$ data interpreted a crossover in $\Delta\lambda$ from $T^2$ to $T$ at approximately $0.27T_c$ as a result of disorder-induced modifications in the superconducting state.[9] A calculation for a disordered, two-dimensional $d$-wave superconductor may support this view.[10]

Therefore information on $\Delta\lambda(T)$ can be used to discriminate the angular momentum of the pairing state of superconducting holes or electrons. The reproducibility of MFM measurements and the ability of the MFM to scan through a range of temperatures makes it a very appropriate tool for such a test. Furthermore, the local, or microscopic, nature of magnetic force measurement makes it possible to avoid the effects of grain boundaries and related sample imperfections on $\lambda(T)$ in high-$T_c$ materials. Other methods, such as microwave techniques, effectively give an averaged penetration depth. It is recalled that the short coherence length of high-$T_c$ superconductors makes them unusually sensitive to structural imperfections.

For the inverse problem it is especially important to review the necessary changes to the governing partial differential equations when a position-dependent penetration depth is present. After a condensed consideration of this topic, the inversion procedure is developed. In considering the magnetic boundary value problem for stratified superconductors, the solution technique builds on earlier work of the author.[11,13]

In this paper the solution of a MFM inverse problem whose associated forward problem is also nonlinear is avoided. This especially applies to a consideration of using the coupled Ginzburg-Landau (GL) equations. It is recalled that these equations are nonlinear in both the complex order parameter and vector potential (or magnetic field). It seems that not even a partial mathematical solution of the inverse problem for the GL equations is available. Given the significant mathematical difficulties of the inverse GL problem, these equations still lack physically: they strictly hold only near the transition temperature $T_c$. Additionally, in the high-$T_c$ materials, with their short coherence lengths, fluctuations play a major role near $T_c$. Therefore a GL approach appears to be fraught with many difficulties, whose mathematical intricacies may not be easily physically justifiable. In this paper the starting point for the forward problem is linear London theory.

## BASIC SUPERCONDUCTOR EQUATIONS

The following treatment will assume axisymmetry for the superconducting half space problem. The important simplifications that arise for this geometry are emphasized here. In particular, the modification of the London equation for a layer-dependent penetration depth $\lambda(z)$ is considered. From the London relation $\mathbf{j}_s(\mathbf{x}) = -\mathbf{A}(\mathbf{x})/[\mu_0\lambda^2(\mathbf{x})]$, where $\mathbf{j}_s$ is the supercurrent density and $\mathbf{A}$ the vector potential, the relation for the magnetic induction $\mathbf{B}=\nabla\times\mathbf{A}$, and Ampere's law, it follows that

$$\nabla\times(\nabla\times\mathbf{B}) = -\frac{1}{\lambda^2}\,\mathbf{B}+\mathbf{A}\times\nabla\left(\frac{1}{\lambda^2}\right). \tag{1}$$

In obtaining this equation the total current density has been taken to be the supercurrent density and $\mathbf{B}$ has been taken equal to $\mu_0\mathbf{H}$ for simplicity. Of note here in the vector equation for the magnetic induction is the last term of Eq. (1). This term complicates the solution for this field, especially in non-Cartesian coordinate systems.

Since $\mathbf{B}$ is the principal field for MFM force calculation and we desire to solve boundary value problems for it, the axisymmetry assumption is introduced. Here we assume, in terms of cylindrical coordinates, that the penetration depth depends on $z$ alone, that $\mathbf{B}$ has only radial and vertical components, and that $\mathbf{A}$ can be taken to be azimuthal. The important point for us here is that the vertical component of Eq. (1) reduces to the usual London equation, with variable coefficient,

$$\nabla^2 B_z(\rho,z) = \frac{1}{\lambda^2(z)}\,B_z(\rho,z). \tag{2}$$

Since $\mathbf{B}$ is divergenceless, $B_z$ can be employed as a scalar potential.[13]

## INVERSE BOUNDARY VALUE PROBLEM

We now wish to develop the inverse boundary value problem for a semi-infinite superconductor, with surface the plane $z=0$, in the Meissner state, in the presence of a vertical point magnetic dipole of moment $\mathbf{m}$ at height $a$. The case of a point magnetic charge tip (monopole tip) is considered in Ref. 14. Most practical tips are found to lie somewhere between the dipole and monopole cases. The latter situation tends to hold for longer vertical tips.[14]

The governing partial differential equations in conjoined half spaces become

$$\nabla^2 B_z(\rho,z) = V_z(\rho,z) = \mu_0 m\,\delta(z-a)\nabla^2_{2D}\delta_{2D}(\rho), \quad z\geqslant 0, \tag{3a}$$

$$[\nabla^2 - \lambda^{-2}(z)]B_z = 0, \quad z\leqslant 0. \tag{3b}$$

Two-dimensional Fourier transformation[15] of Eqs. (3) gives

$$(\partial_z^2 - k^2)B_z(k,z) = V_z(k,z), \quad z\geqslant 0, \tag{4a}$$

$$[\partial_z^2 - \gamma^2(z)]B_z(k,z) = 0, \quad z\leqslant 0, \tag{4b}$$

where the (unknown) coefficient function $\gamma^2(z)\equiv k^2 + \lambda^{-2}(z)$. The transform of the source is

$$V_z(k,z) = \int_0^\infty \rho J_0(k\rho)V_z(\rho,z)\,d\rho = -\frac{\mu_0 m}{2\pi}\,\delta(z-a)k^2. \tag{4c}$$

Here and throughout $J_n$ denotes the Bessel function of order $n$ of the first kind.[17]

A particular solution of Eq. (3a) is

$$B_{1z}(\rho,z) = \frac{\mu_0 m}{4\pi}\,\frac{[2(z-a)^2-\rho^2]}{[\rho^2+(z-a)^2]^{5/2}}. \tag{5}$$

In the upper half space $\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2$ where the vertical component of the induced field is taken to have the form

$$B_{2z}(\rho,z\geqslant 0) = \int_0^\infty F_1(k)e^{-kz}J_0(k\rho)k\,dk. \tag{6}$$

The function $B_{2z}(\rho,z)$ is a homogeneous solution of Eq. (3a); it is a harmonic function in the upper half space. In the lower half space the vertical component is taken to have the form

$$B_z(\rho,z\leqslant 0) = \int_0^\infty F_2(k)Z_2(z,k)J_0(k\rho)k\,dk. \tag{7}$$

The function $Z_2$ solves Eq. (4b) and satisfies the boundary condition $Z_2\to 0$ as $z\to-\infty$. The solution space of Eq. (4b) has as a basis two linearly independent solutions, a growing one and a decaying one. (By Abel's identity, their Wronskian is a constant.) The function $Z_2$ vanishing as $z\to-\infty$ will be the unique monotonically decreasing solution when two conditions are met. First, assuming that $\gamma^2(z)$ is a continuous function, the function $z\gamma^2(z)$ should not be in $L(0,\infty)$. Second, fixing the value of $Z_2$ at a point, say at $z=0$, makes $Z_2$ unique.[16]

The expansion coefficients $F_1$ and $F_2$ are fixed by continuity boundary conditions at $z=0$.[13] Using Eqs. (5), (6), and (7) and the continuity of $B_z$ and $\partial_z B_z$ yields[17,18]

$$\frac{\mu_0 m}{4\pi}\, ke^{-ka}+F_1(k)=F_2(k)Z_2(0,k), \tag{8a}$$

$$\frac{\mu_0 m}{4\pi}\, k^2e^{-ka}-kF_1(k)=F_2(k)\dot{Z}_2(0,k), \tag{8b}$$

where the notation $\dot{Z}(0,k)\equiv[\partial_z Z(z,k)]_{z=0}$ is used. The solution of Eqs. (8) is

$$F_1(k)=\frac{\mu_0 m}{4\pi}\, ke^{-ka}\,\frac{[kZ_2(0,k)-\dot{Z}_2(0,k)]}{[kZ_2(0,k)+\dot{Z}_2(0,k)]}, \tag{9a}$$

$$F_2(k)=\frac{\mu_0 m}{2\pi}\,\frac{k^2e^{-ka}}{[kZ_2(0,k)+\dot{Z}_2(0,k)]}. \tag{9b}$$

Then follows the components of the magnetic induction[13]

$$B_z(\rho,z\geqslant 0)=B_{1z}(\rho,z)+\frac{\mu_0 m}{4\pi}\int_0^\infty k^2dk\,J_0(k\rho)e^{-k(z+a)}\times\frac{[1-K(k)]}{[1+K(k)]}, \tag{10a}$$

$$B_z(\rho,z\leqslant 0)=\frac{\mu_0 m}{2\pi}\int_0^\infty k^2dk\,\frac{J_0(k\rho)Z_2(z,k)e^{-ka}}{[Z_2(0,k)+\dot{Z}_2(0,k)/k]}, \tag{10b}$$

$$B_\rho(\rho,z\geqslant 0)=B_{1\rho}(\rho,z)+\frac{\mu_0 m}{4\pi}\int_0^\infty k^2dk\,J_1(k\rho)e^{-k(z+a)}\times\frac{[1-K(k)]}{1+K(k)}, \tag{10c}$$

$$B_\rho(\rho,z<0)=-\frac{\mu_0 m}{2\pi}\int_0^\infty k^2dk\,\frac{J_1(k\rho)\dot{Z}_2(z,k)e^{-ka}}{[kZ_2(0,k)+\dot{Z}_2(0,k)]}, \tag{10d}$$

where the kernel function

$$K(k)\equiv\frac{\dot{Z}_2(0,k)}{kZ_2(0,k)} \tag{11}$$

has been introduced. The azimuthal supercurrent density can be computed from Ampere's law and Eqs. (10b) and (10d):

$$j_{s\theta}(\rho,z\leqslant 0)=-\frac{m}{2\pi}\int_0^\infty k^2dk\,J_1(k\rho)e^{-ka}\times\frac{[\ddot{Z}_2(z,k)-k^2Z_2(z,k)]}{[kZ_2(0,k)+\dot{Z}_2(0,k)]}. \tag{12}$$

It will be seen that the wave-number-dependent function $K(k)$ is central to the inversion algorithm. From Eqs. (10b) and (10d) and Hankel inversion the equivalence of the kernel function and interface quantities is known:

$$K(k)=-\int_0^\infty B_\rho(\rho,0)J_1(k\rho)\rho\,d\rho\Big/\int_0^\infty B_z(\rho,0)J_0(k\rho)\rho\,d\rho. \tag{13}$$

Even more important here is the connection between $K$ and the measured MFM force $\mathbf{F}$. The magnetostatic self-interaction energy for the vertical dipole is $U(a)=-(\frac{1}{2})mB_{2z}(\rho=0,z=a)$. Using Eq. (10a) this becomes

$$U(a)=-\frac{\mu_0 m^2}{8\pi}\int_0^\infty k^2dk\,e^{-2ka}\,\frac{[1-K(k)]}{[1+K(k)]}. \tag{14}$$

Then the lifting force $F_z=-\partial U/\partial a$ is

$$F_z(a)=-\frac{\mu_0 m^2}{4\pi}\int_0^\infty k^3dk\,e^{-2ka}\,\frac{[1-K(k)]}{[1+K(k)]}. \tag{15}$$

The above results can be checked in the special case of the direct problem with $\lambda=$const. Then $Z_2(z,k)=\exp(\gamma z)$, $kZ_2(0,k)+\dot{Z}_2(0,k)=k+\gamma$, and $K(k)=\gamma/k$.

Recognizing Eq. (15) as a Laplace transform, the kernel function can be obtained in terms of the inverse Laplace transform of the force

$$K\left(\frac{k}{2}\right)=\frac{1+(c_m/k^3)\,\mathcal{L}^{-1}[F_z(a)]}{1-(c_m/k^3)\,\mathcal{L}^{-1}[F_z(a)]}, \tag{16}$$

where $c_m\equiv 64\pi/\mu_0 m^2$. Alternatively, the kernel function could be found from the potential energy function, Eq. (14). It remains to show how $\lambda^{-2}(z)$ can be recovered from the kernel function.

## RECOVERY OF THE PENETRATION DEPTH PROFILE

The penetration depth function can be recovered from the kernel function by using the form of the Schrödinger-like equation (4b), $\ddot{Z}_2/Z_2-k^2=\lambda^{-2}(z)$, and an infinite series solution. For large wave numbers, $Z_2$ approaches $\exp(kz)$, in which case $K(k)\to 1$. Therefore the kernel function has the expansion

$$K(k)=\sum_{n=0}^{\infty}\frac{a_n}{k^n}, \tag{17}$$

where $a_0=1$. If we introduce the logarithmic derivative function $v(z,k)=\partial_z \ln Z_2(z,k)$, then it is seen that $K(k)$

$=v(0,k)/k$. In a sense the function $v$ serves to extend the kernel away from $z=0$. Since for large $k$, $v\rightarrow k$, this function has an expansion

$$v(z,k)=k\sum_{n=0}^{\infty}\frac{\alpha_n(z)}{k^n}, \tag{18}$$

where $\alpha_0=1$ and each coefficient has the boundary condition $\alpha_j(z)\rightarrow a_j$ as $z\rightarrow 0$. Based upon the (direct problem) special case that $\lambda=$const, where

$$K(k)\rightarrow\left(1+\frac{1}{\lambda^2k^2}\right)^{1/2}=1+\frac{1}{2\lambda^2k^2}-\frac{1}{8}\frac{1}{\lambda^4k^4}+\frac{1}{16}\frac{1}{\lambda^6k^6}-\frac{5}{128}\frac{1}{\lambda^8k^8}+\cdots, \tag{19}$$

we anticipate that $\lambda^{-2}(z)$ can be found as $2\alpha_2(z)$. If we integrate $v(z,k)$ from $z$ to zero, we have

$$\ln[Z_2(0,k)/Z_2(z,k)]=k\sum_{n=0}^{\infty}\frac{1}{k^n}\int_z^0\alpha_n(z')dz'. \tag{20}$$

When $v$ is substituted into Eq. (4b), a Ricatti equation results:

$$v^2+\dot{v}=k^2+\lambda^{-2}(z). \tag{21}$$

Equation (21) manifests the well-known connection between the second-order, linear Schrödinger equation and the first-order, nonlinear Ricatti equation.[4] When the expansion (18) is substituted into Eq. (21), we find that $\alpha_1=0$, that indeed

$$2\alpha_2(z)=\lambda^{-2}(z), \tag{22}$$

and the recursion relation

$$\sum_{n=0}^{l}\alpha_n\alpha_{l-n}=-\dot{\alpha}_{l-1}(z),\quad l>2. \tag{23}$$

Equation (23) expresses the derivatives $\dot{\alpha}_l(z)$ in terms of the functions $\alpha_j(z)$ themselves. From this recursion relation one extracts the function

$$\alpha_2(z)=\sum_{n=0}^{\infty}\frac{\alpha_2^{(n)}(0)}{n!}z^n. \tag{24}$$

The higher derivatives of $\alpha_2$ at $z=0$ are also computable from Eq. (23). In fact, we have

$$\alpha_2(0)=a_2,\quad -\dot{\alpha}_2(0)=2a_3,$$
$$\ddot{\alpha}_2(0)/2!=2a_4+a_2^2,\quad -\alpha_2^{(3)}(0)/3!=4a_5/3+8a_2a_3/3,$$
$$\alpha_2^{(4)}(0)/4!=2a_6/3+2a_2a_4+5a_3^2/3+2a_2^3/3, \tag{25}$$

so that the beginning of the series for $\lambda^{-2}(z)$ is

$$\lambda^{-2}(z)\sim 2a_2-4a_3z+2(2a_4+a_2^2)z^2-(\tfrac{8}{3})(a_5+2a_2a_3)z^3+\cdots. \tag{26}$$

The correctness of the relations (25) can be checked in the special case $\lambda=$const, from Eq. (19), where $a_{2n+1}=0$.

## DISCUSSION

A brief examination of some very simple but important special cases may help to illuminate the above procedure. When the lower half space is no longer superconducting, $\lambda\rightarrow\infty$, the measured MFM force is zero, its inverse Laplace transform is null, and by Eq. (16) $K(k)=1$, as expected. On the other hand, consider the case of perfect diamagnetism, $\lambda\rightarrow 0$. Then the force as a function of height is $F_z(a)=3\mu_0m^2/32\pi a^4$, which has the inverse Laplace transform $k^3/c_m$. Then by Eq. (16) the kernel function diverges, as expected since $\gamma$ diverges.

A nontrivial example is to use the MFM force data when $\lambda$ is known to be a nonzero constant and to verify consistency. In this case the magnetostatic interaction energy and force are expressible in terms of differences of Struve and Neumann functions, $\mathbf{H}_p-N_p$.[12] Then

$$\mathcal{L}^{-1}[F_z(a/2)]=\frac{16}{c_m}k^3(1+2\lambda^2k^2-2k\lambda^2\sqrt{k^2+\lambda^{-2}}), \tag{27}$$

which, together with some algebra, gives $K(k)=\gamma/k$.

Several extensions of the research reported here can be made, including the consideration of a superconductor with finite thickness, and will be discussed elsewhere.[18] Under the assumptions that the coefficient function $\gamma^2(z)$ in Eq. (4b) is always positive and continuous, a positive monotonically decreasing solution is guaranteed to exist. This justifies the assumption of the existence of a monotonic solution $Z_2(z,k)$ vanishing as $z\rightarrow-\infty$.[16,18] It also makes the definition of the function $v(z,k)$ meaningful.

It is worth describing the nonrelativistic quantum mechanics analogy with the present work. The left-hand side of Eq. (4b), written with terms $(\partial_z^2-\lambda^{-2})B_z$, corresponds to a scaled Hamiltonian operating on a wave function. If we compare this equation in detail with the Schrödinger equation then we can make the correspondences $\lambda^2(z)\rightarrow\hbar^2/2\mu V(z)$ and $k^2\rightarrow-2\mu E/\hbar^2$, where $\mu$ is the particle mass, $V$ is the potential energy, and $E$ the energy eigenvalue. Therefore, as we expect, a weak potential corresponds to a large penetration depth, and vice versa. The negative sign of $E$, which indicates a bound state in quantum mechanics, is connected to the fact that the superconductor problem corresponds to an attenuation, rather than a propagation, problem. In the formulation of this paper, $B_{1z}(k,z)$ acts like an incident field, $B_{2z}(k,z)$ like a reflected field, and $B_z(k,z\leqslant 0)$ like a transmitted field. The function $Z_2(z,k)$, with the defining property

$$\lim_{z\rightarrow-\infty}e^{-kz}Z_2(z,k)=1, \tag{28}$$

might be called a Jost function. By using the Hankel representation of $B_{1z}(\rho,z)$ and Eqs. (10a) and (10b), it is then possible to write

$$R(k)=\frac{K(k)-1}{K(k)+1}, \tag{29}$$

$$T(k)=\frac{2}{1+K(k)}, \tag{30}$$

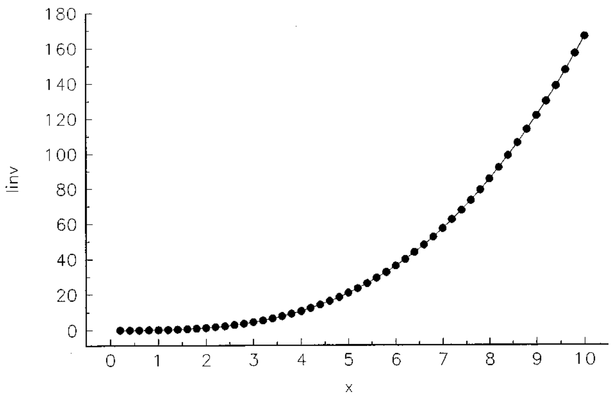


FIG. 1. Numerical inverse Laplace transform of the function $F(p)=1/p^4$, obtained with the Gaver-Stehfest method with $n=18$.

as reflection and transmission coefficients, with $R+T=1$. (Here the exponential factor $\exp(-2ka)$ coming from the nonzero height of the MFM tip is omitted.) For the direct problem with $\lambda=$const, these expressions reduce to $R=(\gamma-k)/(\gamma+k)$ and $T=2k/(\gamma+k)$. The inversion procedure of this paper may then be phrased as using knowledge of the reflection coefficient, as known from the MFM force data, to recover the unknown penetration depth as a function of distance.

## NUMERICAL LAPLACE TRANSFORM INVERSION

As mentioned in the Introduction, other approaches to the inverse problem exist. Specifically, an integral equation formulation[18] as opposed to a differential equation formalism as used here can be followed. However, in each case, due to the form of the magnetostatic interaction, Eq. (15), the generally delicate task of numerical Laplace transform inversion must be confronted. This operation is discussed and illustrated in this section. Once the inverse Laplace transform is performed, the kernel function is available for the rest of the inversion method.

The difficulty of inversion of MFM results is further compounded by the fact that the Laplace transform is known only for real values. Therefore the present discussion is limited to considering a continuous Laplace transform function.

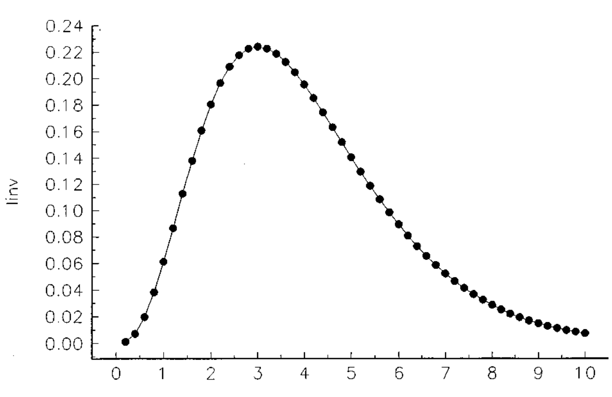


FIG. 2. Numerical inverse Laplace transform of the function $F(p)=1/(p+1)^4$, obtained with the Gaver-Stehfest method with $n=18$.

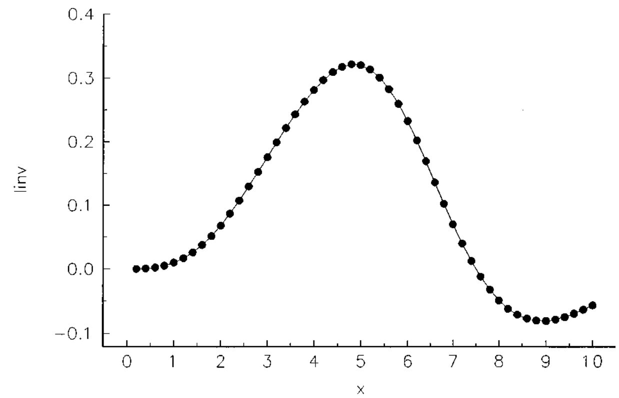


FIG. 3. Numerical inverse Laplace transform of the function $F(p)=1/[p+(p^2+1)^{1/2}]^4$, obtained with the Gaver-Stehfest method with $n=18$.

Two methods known to give fairly accurate and reliable results for real valued functions[19] are those of Piessens based upon the use of Chebyshev polynomials[20] and of Gaver and Stehfest using an extrapolated sample.[21] Both of these methods have been implemented. The second, having fewer numerical parameters, is easier to work with, and is illustrated here.

A basic Laplace transform pair for the MFM problem with a point dipole tip is a force function $F(p)=1/p^4$ with inverse $x^3/6$. This pair corresponds to the diamagnetic limit, as discussed above. The numerically obtained inverse function is plotted in Fig. 1. The number of points in the sample $n$, which must be even, has been taken as 18 for all of the example functions presented here. This is a suitable number for double precision arithmetic.[21] If the force should be $F(p)=1/(p+1)^4$, the exact transform is $x^3e^{-x}/6$. The numerically obtained inverse is shown in Fig. 2. The Laplace inverse transform of the function $F(p)=1/[(p^2+1)^{1/2}+p]^4$ is $4J_4(x)/x$, where $J_4$ is the fourth-order Bessel function of the first kind. The result of numerical inversion of $F(p)$ is shown in Fig. 3. The function $F(p)=1/(p^2+1)^2$ has inverse transform $\pi^{1/2}x^{3/2}J_{3/2}(x)/2^{3/2}$ and the numerical result is shown in Fig. 4. It is seen that the numerically computed inverse function in this case significantly degrades for larger values of $x$.

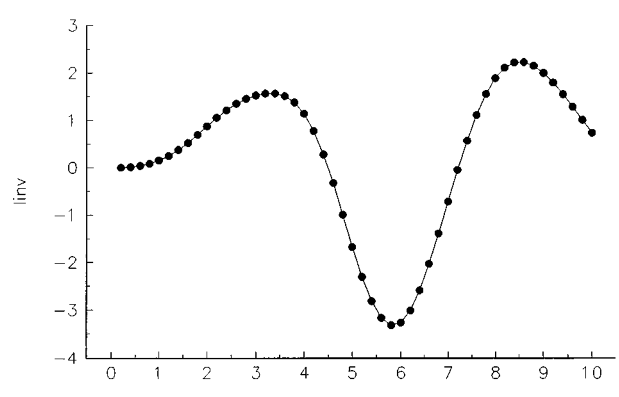


FIG. 4. Numerical inverse Laplace transform of the function $F(p)=1/(p^2+1)^2$, obtained with the Gaver-Stehfest method with $n=18$.

## SUMMARY

A ready application of this work is the contactless assessment of the quality of high-temperature superconducting crystals and films. High-temperature superconducting films are now fabricated from a variety of techniques. These multilayers, to a good first approximation, have the axisymmetry assumed in this paper. In this discussion of the concept of inversion, an infinite width of the sample was also assumed so that Fourier transformation could be applied.

The MFM imaging of high-$T_c$ superconductors is now possible in both the Meissner and mixed states. The low-temperature MFMs are able to reliably image the same area under different temperature and magnetic-field conditions.[3] Force detection at the pico-Newton level has been achieved.[2,3]

The inverse MFM problem also has many implications for the basic physics of superconductivity. The detailed temperature dependence of the penetration depth, especially at temperatures close to absolute zero, can provide information on the symmetry of the pairing wave function. In particular, the deviation of $\lambda(T)$ from a constant at low temperature, of exponential vs power law form, is of interest.

In this work axisymmetry was assumed and two-dimensional Fourier transformation was used to eliminate the $(\rho,\theta)$ coordinates. The solution of the coupled electromagnetic boundary value problem along the $z$ direction was effected by employing a wave number-dependent kernel function $K(k)$. From the continuity boundary conditions at $z=0$, the form of the kernel function was found, Eq. (11). It is assumed that the MFM force data is available for a vertical dipole tip, for all heights $a>0$. Therefore the inversion algorithm can be summarized as follows. (a) Laplace inversion of the MFM force data, as a function of wave number $k$. (b) Calculation of the kernel function $K$ from this information, Eq. (16). (c) Expansion of the kernel function in powers of reciprocal wave number, Eq. (17). The set of numbers $\{a_j\}$ is then known. (d) From this series the auxiliary function $\alpha_2(z)$ appearing in Eqs. (18) and (24) is determined. (e) Finally, from Eq. (22), the layer-dependent penetration depth $\lambda(z)$ is recovered.


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