Magnetic self-field entry into a current-carrying type-II superconductor. III. General criterion of penetration for an external field of arbitrary direction

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The first magnetic-flux penetration into a current-carrying cylindrical type-II superconductor subjected to an external magnetic field is considered in the frame of the London approximation. The lower transverse critical field as well as critical field of the first vortex nucleation at the surface are evaluated on the basis of an exact solution for a vortex of arbitrary flat configuration. Taken together with the previous consideration of the parallel magnetic-field penetration in a current-carrying superconductor [Phys. Rev. B **49**, 6950 (1994) and B **51**, 3686 (1995)] the above results allow us to formulate the general criterium for the first flux entry in the current-carrying samples of arbitrary transverse size subjected to an external field of arbitrary direction. [S0163-1829(98)03902-2]

I. INTRODUCTION

The surface effects play essential role in the electrodynamic behavior of type-II superconductors (SCs) and are especially important in the case of extreme situations with Ginzburg-Landau parameter $\kappa=\lambda/\xi\gg 1$ where λ and ξ are the magnetic-field penetration depth and superconducting correlation length, respectively. Considerable attention was recently paid to the problem of magnetic-flux penetration into the superconducting samples of various geometries.[1–10] The surface Bean-Livingston[11] and geometrical[12] barriers effect on the nucleation of magnetic vortices in bulk SC's turned out to determine their resistive and hysteretic behavior in a wide range of magnetic field and temperature.[1,6,9,10]

One of the consequences of surface effect in type-II SC's is the validity of Silsbee's rule[13] for a samples with a perfect surface[14,15] (the last implies that typical surface defect size $\delta<\xi$). Silsbee's rule, established for the type-I SC's shortly after the discovery of superconductivity itself,[13] stated that the breakdown of the nondissipative state of a macroscopic current-carrying sample occurs when the current self-field at the sample surface first attains the magnitude of the thermodynamic critical field H_c, the only characteristic field for type-I SC's. Contrary to type-I SC's, type-II SC's have two critical fields $H_{c1}\simeq H_c \ln\kappa/\kappa$ and $H_{c2}\simeq\kappa H_c$ and allow magnetic flux to penetrate a sample in the form of magnetic vortices in a wide range of external magnetic field $H_{c1}\ll H\ll H_{c2}$.[16] Though, as was shown by Bean and Livingston,[11] because of the surface potential barrier the vortices may enter a perfect SC sample exposed to an external magnetic field H only if it achieves the above value H_c.

Despite this clear analogy in first flux penetration into type-I and type-II SC's the current self-field was believed for years to first penetrate into a type-II SC once the self-field of the current, H_I, achieves at the surface the value of the lower critical field H_{c1}.[17–19] Exact solutions for the vortex ring[14,20] and vortex helix[15] entry into the current-carrying type-II SC cylinder found recently in the frame of the London approximation have shown that Silsbee's rule holds for the current self-field entry in type-II SC's and is also valid for the case of external magnetic field applied parallel to the current direction. In the last case Silsbee's rule applies to the total magnetic-field value at the surface[15] that is vector sum of the current self-field and external field.

In this paper we continue with the study of the first vortex nucleation in a current-carrying type-II SC cylinder exposed to the magnetic field perpendicular to the transport current direction. For this aim the solution of the London equation inside a SC cylinder of radius $R\gg\xi$ is found for an arbitrarily shaped flat vortex. The general solution allows us to find the first critical field H_{c1} for the SC cylinder in a transverse field and critical conditions for the first flux entry when both field and transport current are applied. Silsbee's rule turns out to be valid for macroscopic samples with $R\gg\lambda$ in a field of any direction but fails for $R<\lambda$.

A general criterium is established for the first flux-line penetration in a perfect current-carrying type-II SC sample of arbitrary transverse size subjected to a magnetic field of ar-

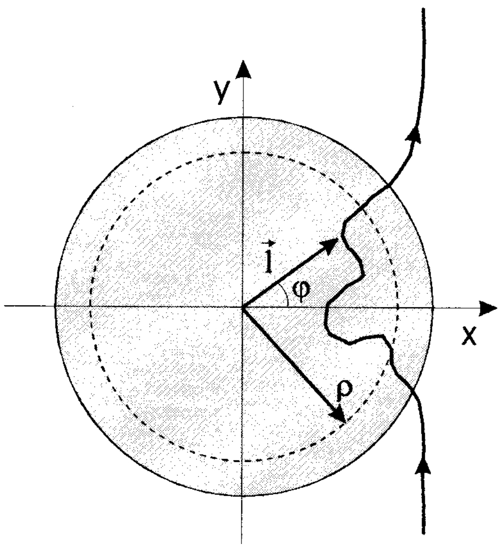

FIG. 1. Magnetic-flux-line (vortex) of an arbitrary form entering a cylindrical superconductor exposed to external transverse magnetic field. There are at least two solutions of the equation $\rho=|\mathbf{l}(\varphi)|$ with respect to angle in the area where vortex is located.

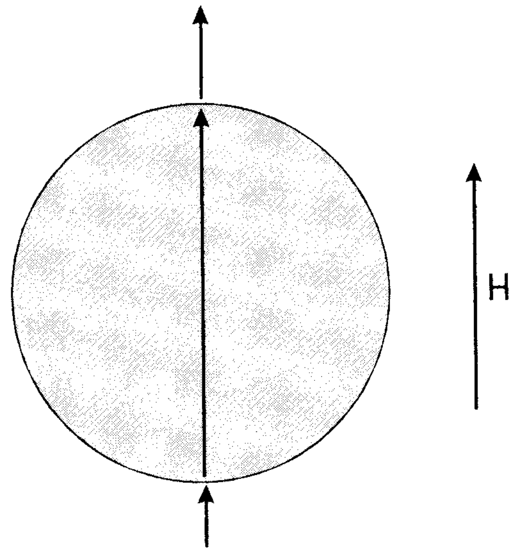

FIG. 2. Magnetic-flux line parallel to the external magnetic field lying along the diameter of the superconducting cylinder.

bitrary direction: the first vortex loop nucleates at the sample surface when (and where) the total current achieves the value of the depairing current.

The structure of the paper is as follows. A detailed description of the theoretical model and results is presented in Secs. II and III. Section II is concerned with the calculation of the structure of an arbitrary vortex inside a SC cylinder. Subsequently the energy, magnetic flux, and moment of definite vortex configurations are evaluated in Sec. III that allows us to find critical parameters of the SC cylinder with respect to the surface effect. The results are summarized and discussed in Sec. IV.

II. STRUCTURE OF A MAGNETIC VORTEX IN A SUPERCONDUCTING CYLINDER

A type-II superconducting cylinder of radius $R \gg \xi$ is considered which extends along the z axis of cylindrical coordinate system (ρ,φ,z). A transverse magnetic field is applied along the positive y direction ($\varphi=\pi/2$) as is shown in Fig. 1. The field is asymptotically uniform (at distances large compared to R) and of magnitude H_0. The situation when a transport current is applied in addition along the z axis will be studied below in Sec. III. Here we address the question of the magnetic structure of the vortex itself. It is supposed to nucleate at some location on the equatorial lines of the cylinder ($\varphi=0$ or $\varphi=\pi$, $\rho=R$) at some critical field H_p and then moves to the center of the cylinder. In the central position (Fig. 2) parallel to $\mathbf{H_0}(\varphi=\pi/2)$ it has the largest magnetic moment that leads to the least Gibbs free energy. This final vortex position is supposed to be stable. During the motion from the cylinder surface to the center the vortex core exhibits an arbitrary form that may be described by position vector $\mathbf{l}(\varphi)$. Here and below we assumed for the sake of simplicity the line $\mathbf{l}$ to be a flat curve (lying in the plane $z=0$) that is favorable from the energy reasons. The problem remains, though, three-dimensional.

We start from the London equation in the form[21]

$$\lambda^2 \mathrm{curl}\,\mathrm{curl}\mathbf{H}+\mathbf{H}=\mathbf{\Phi},\quad \rho\leqslant R \tag{1}$$

and Maxwell equations

$$\mathrm{curl}\mathbf{H}=0,\quad \rho\geqslant R \tag{2}$$

$$\mathrm{div}\,\mathbf{H}=0, \tag{3}$$

where $\mathbf{H}$ is the total magnetic field. The source function on the right-hand side (rhs) of Eq. (1) writes as

$$\mathbf{\Phi}=\Phi_0\int d\mathbf{l}\,\delta(\boldsymbol{\rho}-\mathbf{l}), \tag{4}$$

where $\boldsymbol{\rho}$ is the position vector in the plane $z=0$, Φ_0 is the unit flux quantum, and $d\mathbf{l}$ is the flux-line element. The integration extends along the flux-line (vortex core). The stray field outside the superconductor is described by Maxwell equations (2,3), the last of which is valid in the whole space. The boundary conditions are $\mathbf{H}$ approaches $\mathbf{H_0}$ asymptotically ($\rho\to\infty$), $\mathbf{H}$ is continuous in all components on the circle $\rho=R$.

The solution of Eqs. (1)–(3) may be represented by the superposition of the well-known Meissner response $\mathbf{H_M}$ of a SC cylinder to the transverse field[22] and a field $\mathbf{h}$ of the vortex itself. The field $\mathbf{H_M}$ satisfies the uniform Eqs. (1)–(3) with the zero rhs $\mathbf{\Phi}=0$ and both boundary conditions. Taking into account that the field $\mathbf{h}$ is potential outside the cylinder and may be presented as $\mathbf{h}=\nabla\psi$ we rewrite Eqs. (1),(2) in the form

$$\lambda^2 \mathrm{curl}\,\mathrm{curl}\mathbf{h}+\mathbf{h}=\mathbf{\Phi},\quad \rho\leqslant R, \tag{5}$$

$$\Delta\psi=0,\quad \rho>R \tag{6}$$

with boundary conditions $\mathbf{h}=\nabla\psi$ on the circle $\rho=R$, $\psi\to 0$, $\rho\to\infty$.

The components of the field in cylindrical coordinates $\mathbf{h}=(h_\rho,h_\varphi,h_z)$ may be written with the help of Fourier transformation as

$$h^j(\rho,\varphi,z)=\sum_m \exp(im\varphi)\int \frac{dk}{2\pi} h^j_{k,m}(\rho)\exp(-ikz), \tag{7}$$

$$\psi(\rho,\varphi,z)=\sum_m \exp(im\varphi)\int \frac{dk}{2\pi} \psi_{k,m}(\rho)\exp(-ikz), \tag{8}$$

where the index j assumes the values ρ,φ,z.

In terms of the Fourier amplitudes $h^j_{k,m}$ and $\psi_{k,m}$ Eqs. (5),(6) transform to the following set of one-dimensional equations:

$$\frac{\partial^2 h^\rho_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial h^\rho_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2+1}{\rho^2}\right)h^\rho_{k,m}-\frac{2im}{\rho^2}h^\varphi_{k,m} =-\frac{\Phi^\rho_{k,m}}{\lambda^2},\quad \rho\leqslant R, \tag{9}$$

$$\frac{\partial^2 h^\varphi_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial h^\varphi_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2+1}{\rho^2}\right)h^\varphi_{k,m}+\frac{2im}{\rho^2}h^\rho_{k,m} =-\frac{\Phi^\varphi_{k,m}}{\lambda^2},\quad \rho\leqslant R, \tag{10}$$

$$\frac{\partial^2 h^z_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial h^z_{k,m}}{\partial\rho}-\left(Q^2+\frac{m^2}{\rho^2}\right)h^z_{k,m}=-\frac{\Phi^j_{k,m}}{\lambda^2},\quad \rho\leqslant R, \tag{11}$$

$$\frac{\partial^2 \psi_{k,m}}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial \psi_{k,m}}{\partial\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)\psi_{k,m}=0,\quad \rho\geqslant R, \tag{12}$$

with the boundary conditions

$$\psi_{k,m}(\infty)=0,\quad \psi'_{k,m}(R)=h^\rho_{k,m}(R),$$

$$-ik\psi_{k,m}(R)=h^z_{k,m}(R),\quad \frac{im}{R}\psi_{k,m}(R)=h^\varphi_{k,m}(R). \tag{13}$$

In Eqs. (9)–(12) we introduced the value $Q^2=k^2+1/\lambda^2$ and Fourier amplitude of the source function (4) $\Phi^j_{k,m}$ defined in the same manner as the field components in Eq. (7).

The set of equations (9)–(12) is not equivalent to Eqs. (5),(6) since by its derivation the equality curl curl$\mathbf{h}=-\Delta\mathbf{h}$ was used that implies div $\mathbf{h}=0$. To be the solution to Eqs. (5),(6) the solution of Eqs. (9)–(12) should also satisfy Eq. (3).

Upon the transformation $f^\pm_{k,m}=h^\rho_{k,m}\pm ih^\varphi_{k,m}$ all Eqs. (9)–(12) may be solved separately in terms of the modified Bessel functions.[23] We obtain then the solutions regular at $\rho=0$:

$$\psi_{k,m}(\rho)=\Psi_{k,m}K_m(|k|\rho), \tag{14}$$

$$h^z_{k,m}(\rho)=C_{k,m}I_m(Q\rho), \tag{15}$$

$$\begin{pmatrix} h^\rho_{k,m}\\ ih^\varphi_{k,m}\end{pmatrix}=\frac{1}{2}\Bigg[\left[F^+_{k,m}-\int_\rho^R d\rho\rho\,\eta^+_{k,m}(\rho)K_{m+1}(Q\rho)\right]I_{m+1}(Q\rho)\pm\left[F^-_{k,m}-\int_\rho^R d\rho\rho\,\eta^-_{k,m}(\rho)K_{m-1}(Q\rho)\right]I_{m-1}(Q\rho)$$
$$-K_{m+1}(Q\rho)\int_0^\rho d\rho\rho\,\eta^+_{k,m}(\rho)I_{m+1}(Q\rho)\mp K_{m-1}(Q\rho)\int_0^\rho d\rho\rho\,\eta^-_{k,m}(\rho)I_{m-1}(Q\rho)\Bigg], \tag{16}$$

where $\eta^\pm_{k,m}=-\lambda^{-2}(\Phi^\rho_{k,m}\pm\Phi^\varphi_{k,m})$, I_ν and K_ν are the modified Bessel functions.[23] The coefficients in Eqs. (14)–(16) may be found with the help of Eqs. (3), (13) and read

$$\Psi_{k,m}=\left[\frac{K_{m+1}(QR)}{I_{m+1}(QR)}\int_0^R d\rho\rho\,\eta^+_{k,m}(\rho)I_{m+1}(Q\rho)+\frac{K_{m-1}(QR)}{I_{m-1}(QR)}\int_0^R d\rho\rho\,\eta^-_{k,m}(\rho)I_{m-1}(Q\rho)-\frac{2R}{Q\lambda^2}K_m(QR)\boldsymbol{\Phi}^\rho_{k,m}(R)\right]$$
$$\times\left[\frac{2k^2}{Q}\frac{K_m(|k|R)}{I_m(QR)}+|k|\left(\frac{K_{m+1}(|k|R)}{I_{m+1}(QR)}+\frac{K_{m-1}(|k|R)}{I_{m-1}(QR)}\right)\right]^{-1}, \tag{17}$$

$$F^\pm_{k,m}=-|k|\Psi_{k,m}\frac{K_{m\pm1}(|k|R)}{I_{m\pm1}(QR)}+\frac{K_{m\pm1}(QR)}{I_{m\pm1}(QR)}\int_0^R d\rho\rho\,\eta^\pm_{k,m}(\rho)I_{m\pm1}(Q\rho), \tag{18}$$

$$C_{k,m}=-ik\Psi_{k,m}\frac{K_m(|k|R)}{I_m(QR)}. \tag{19}$$

The solution (14)–(16) with coefficients (17)–(19) satisfies Eqs. (9)–(12) and Eq. (3) and, hence, Eqs. (5),(6). It is valid so far for an arbitrary form of the vortex, described by the function (4).

Although being rather complicated, the obtained solution may nevertheless be used for calculation of the physical properties of the vortices of definite configurations and critical parameters of the superconductors. We derive here first some general formulas for the physical characteristics of the arbitrary vortex.

The self-energy of the vortex[17,19] takes a form

$$F=\frac{1}{8\pi}\int_{\rho\leqslant R}dV[\mathbf{h}^2+\lambda^2(\mathrm{curl}\mathbf{h})^2]+\frac{1}{8\pi}\int_{\rho\geqslant R}dV(\nabla\psi)^2$$
$$=\frac{1}{8\pi}\int_{\rho\leqslant R}dV\mathbf{h}\mathbf{\Phi}-\frac{R}{8\pi}\sum_m\int dk\,\psi_{k,m}(R)\Phi^{\rho}_{-k,-m}(R). \quad (20)$$

One can see that the last surface term in Eq. (20) vanishes if the vortex does not cross the surface of the sample $[\Phi^{\rho}_{k,m}(R)=0]$, that is known for flux lines of various forms lying completely inside the SC sample,[14,15,24] though the field at the surface and outside the cylinder is not equal to zero. That takes place by virtue of the fact that the contribution from the outer space and from the surface term [following from the first integral in Eq. (20)] exactly compensate each other. Compensation of this sort was found by Brandt in a calculation of the energy of an arbitrary distorted vortex lattice near a flat surface.[25]

The magnetic moment projection on the field direction (see Fig. 1) is

$$M_y=\frac{1}{2c}\int dV[\boldsymbol{\rho}\times\mathbf{j}]_y$$
$$=-\frac{R^2}{2}h^{\varphi}_{0,1}(R)-\frac{1}{2i}\int_0^R d\rho\rho(h^{\rho}_{0,1}-ih^{\varphi}_{0,1}). \quad (21)$$

The magnetic flux flowing through the vortex is written

$$\Phi=\int_{-\infty}^{\infty}dz\int_0^R d\rho[h^{\varphi}(\rho,\varphi=0,z)-h^{\varphi}(\rho,\varphi=\pi,z)]$$
$$=2\sum_{m=1}^{\infty}\int_0^R d\rho h^{\varphi}_{0,2m-1}(\rho). \quad (22)$$

In the derivation of Eqs. (20–22) the symmetry relations were used that follow from the general structure of the solution (14–19):

$$\Psi^*_{k,m}=-\Psi_{k,m}=\Psi_{k,-m},(F^+_{k,m})^*=-F^+_{k,m}=F^-_{k,-m},$$

$$(h^{\rho}_{k,m})^*=-h^{\rho}_{k,m}=h^{\rho}_{k,-m},(h^{\varphi}_{k,m})^*=h^{\varphi}_{k,m}=h^{\varphi}_{k,-m}.$$

The above general formulas are used in the following sections for the calculation of the critical parameters of the superconductor.

III. CRITICAL FIELDS AND CURRENTS OF SUPERCONDUCTING CYLINDER IN A TRANSVERSE FIELD

A. First critical field of a SC cylinder in a transverse magnetic field

To evaluate the first critical field H_{c1} we should study the case of a vortex assuming a stable position in the center of the sample.[17–19] The vortex directed along the cylinder diameter parallel to the applied field $\mathbf{H_0}$ apparently leads to a local minimum of the Gibbs free energy of the system

$$G=F-\Delta W_H,\quad \Delta W_H=\mathbf{MH_0}, \quad (23)$$

where F is the vortex self-energy (20) and $\mathbf{M}$ is the magnetic moment of the sample due to presence of the vortex (the constant Meissner contribution is omitted). Formula (23) follows from the general expression for the Gibbs energy[26,27] of the system if form and position of the vortex are fixed and then the external field is slowly switched on.

The vortex lying on the cylinder diameter is shown in Fig. 2. The corresponding rhs $\mathbf{\Phi}$ in Eq. (1) is written in this case in the form

$$\Phi^{\rho}_{k,m}=\frac{\Phi_0}{i\pi\rho}\theta(R-\rho)\sin\frac{m\pi}{2},\quad \Phi^{\varphi}=\Phi^z=0. \quad (24)$$

Upon the substitution of Eq. (24) in the solution (14–19), one can find the approximate expressions for the self-energy of the vortex (20)

$$F=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 2R\left[\ln\frac{4R}{e\xi}+\frac{9}{4\pi}-\frac{R}{\lambda}\right],\quad \xi\ll R\ll\lambda, \quad (25)$$

for the case of the thin sample, and

$$F=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 2R\left[\ln\kappa-\gamma+\frac{\lambda}{R}\ln\frac{R}{\lambda}\right],\quad R\gg\lambda, \quad (26)$$

for thick samples where $\gamma=0.577(2\ldots)$ is the Euler constant[23].

The magnetic flux flowing through the vortex (22) is in this case

$$\Phi=\Phi_0\frac{R^2}{2\lambda^2}\ln\frac{\lambda}{R}, \quad (27)$$

for $\xi\ll R\ll\lambda$, and

$$\Phi=\Phi_0\left(1-\frac{8\lambda^2}{\pi R^2}\right), \quad (28)$$

for a macroscopic cylinder of radius $R\gg\lambda$.

The magnetic moment (21) induced by the vortex is

$$M=\frac{\Phi_0 R}{2}\left[\frac{L_0(R/\lambda)}{I_0(R/\lambda)}I_1(R/\lambda)-L_1(R/\lambda)\right], \quad (29)$$

where L_ν is the modified Struve function.[23] That gives in limiting cases

$$M=\frac{\Phi_0 R^3}{6\pi\lambda^2}\left(1-\frac{7\lambda^2}{40R^2}\right),\quad R\ll\lambda \quad (30)$$

and

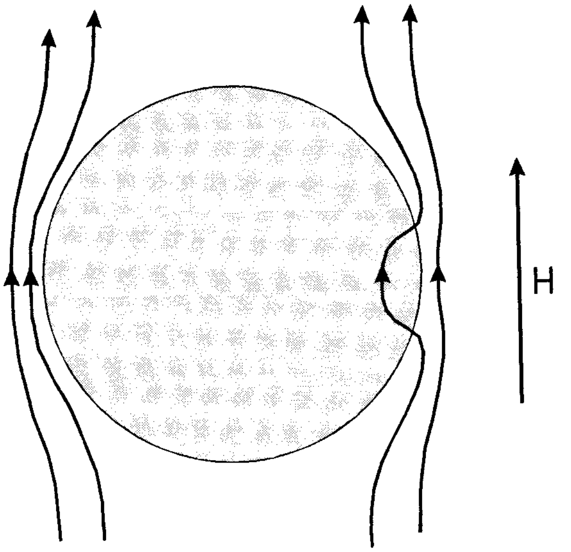

FIG. 3. First vortex loop nucleation at the surface of the superconducting cylinder exposed to an external magnetic field.

$$M=\frac{\Phi_0 R}{\pi}\left(1-\frac{\lambda}{R}\right),\quad R\gg\lambda. \tag{31}$$

The first critical field H_{c1} is determined from Eq. (23) as the field at which the vortex energy G equals zero. For the macroscopic sample one finds the result only slightly different from the critical field $H_{c1}^0/2$ of the bulk material (or the bulk cylinder parallel to field) multiplied by demagnetizing factor 1/2:[17–19]

$$H_{c1}^{\perp}=\frac{H_{c1}^0}{2}\left(1+\frac{\lambda}{R}-\frac{\gamma-(\lambda/R)\ln R/\lambda}{\ln\kappa}\right),\ R\gg\lambda, \tag{32}$$

where $H_{c1}^0=(\Phi_0/4\pi\lambda^2)\ln\kappa$. Let us note, though, that the correction to the bulk value H_{c1}^0 is not exponentially small because of the power tails in the field distribution typical of the vortex crossing a surface.[27,28]

For the thin sample one finds

$$H_{c1}^{\perp}=H_{c1}^0\frac{3\lambda^2}{R^2}\frac{\ln 4R/e\xi+9/4\pi-R/\lambda}{\ln\kappa},\ R\ll\lambda. \tag{33}$$

B. Vortex loop nucleation on the SC cylinder surface in a transverse magnetic field

Magnetic-flux entry in the type-II SC sample exposed to a transverse magnetic field starts with the small vortex loop nucleation at the surface as shown in Fig. 3. This problem was so far considered for the case of a flat surface and thin films.[1,9,11,29–32] To evaluate the critical field of the first vortex penetration into the SC cylinder using the Gibbs energy (23) we should specify the form of the loop and then estimate the energy (20) and moment (21) of the loop. The result should not depend essentially on the form of the loop.

The calculation of the magnetic moment may be performed for relatively general assumptions. A general expression for the magnetic moment of an arbitrary loop is derived in the Appendix. An arbitrary smooth loop may be described by the rhs in Eq. (1) represented by Fourier amplitudes as

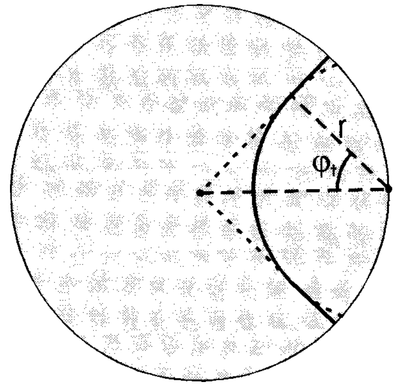

FIG. 4. The specific form of the vortex loop used for the free-energy calculation.

$$\Phi_{k,m}^{\rho}=-\frac{i}{\pi\rho}\Phi_0\sin(m\chi(\rho))\theta(\rho-R+r),\ \ \Phi_{k,m}^{z}=0,$$
$$\Phi_{k,m}^{\varphi}=\frac{\Phi_0}{\pi}\frac{d\chi(\rho)}{d\rho}\cos(m\chi(\rho))\theta(\rho-R+r), \tag{34}$$

where some smooth function $\chi(\rho)$ gives the specific form of a loop penetrating the sample to the depth of r.

Here we consider a small vortex loop with a characteristic length $\simeq r$ and the same depth of penetration in the sample. If $r\ll\lambda,R$ then to the accuracy of $(r/R)^2$ the magnetic moment is

$$M\simeq\frac{\Phi_0 a}{2\pi}\frac{r^2}{\lambda}\frac{I_1(R/\lambda)}{I_0(R/\lambda)}\rightarrow\frac{\Phi_0 a}{2\pi}\frac{r^2}{\lambda}\begin{cases}1, & R\gg\lambda\\ R/2\lambda, & R\ll\lambda\end{cases}, \tag{35}$$

where a is a factor of order of unity depending on the specific form of the vortex loop (for the details, see the Appendix).

To avoid divergencies during the energy calculation some more specified form for the trial vortex loop is taken close to a semicircle of radius r centered at the surface point ($\rho=R$, $\varphi=0$) but perpendicular to the surface as is shown in Fig. 4. The Fourier amplitudes of the corresponding source function (4) look like

$$\Phi_{k,m}^{\rho}=\frac{\Phi_0}{i\pi\rho}[\theta(R-\rho)\theta(\rho-\sqrt{R^2-\rho^2})\sin m\phi_t+\theta(\rho-R+r)\theta(\sqrt{R^2-r^2}-\rho)\sin m\chi(\rho)],$$
$$\Phi_{k,m}^{\varphi}=\frac{\Phi_0}{\pi\rho}\frac{R\cos\chi(\rho)-\rho}{R\sin\chi(\rho)}\theta(\rho-R+r)\times\theta(\sqrt{R^2-r^2}-\rho)\cos(m\chi(\rho)),\ \ \Phi_{k,m}^{z}=0, \tag{36}$$

where $\sin\phi_t=r/R$ and $\cos\chi(\rho)=(R^2+\rho^2-r^2)/2R\rho$. Then one finds for both thick ($R\gg\lambda$) and thin ($R\ll\lambda$) samples

$$F=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\pi r\ln\frac{r}{\xi},\ \ \xi\ll r\ll R,\lambda, \tag{37}$$

where ξ appears as cutoff parameter for the logarithmic divergency usual in the London approximation. This result coincides with that for flat surface[29,30,32] since in the limit $r\ll R,\lambda$ the curvature of the surface plays no role.

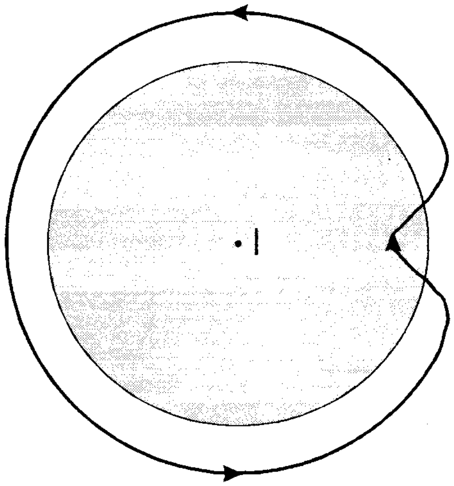

FIG. 5. First vortex loop nucleation at the surface of the current-carrying superconducting cylinder in the absence of an external field.

The factor a in the expression for magnetic moment (35) is equal $\pi/2$ for the above specific form of the loop (36) (see Appendix).

The Gibbs free energy of the loop (23) with F taken from Eq. (37) and M taken from Eq. (35) has a maximum as a function of loop size r forming a Bean-Livingston barrier against the vortex entry. If the field H_0 is high enough the maximum occurs at $r\approx\xi$, then the barrier vanishes and spontaneous vortex loop nucleation and further expansion becomes possible. Using the criterium $\partial G/\partial r(r=\xi)=0$ for the first vortex nucleation we find the corresponding entry field

$$H_p^{\perp}=\frac{H_c}{2\sqrt{2}}\frac{I_0(R/\lambda)}{I_1(R/\lambda)}, \tag{38}$$

where $H_c=\Phi_0/2\sqrt{2}\pi\lambda\xi$.[27] Thus, for the macroscopic cylinder with radius of curvature $R\gg\lambda$, the first penetration field $H_p^{\perp}$ is twice less than it may be obtained for the flat surface in the same approximation.[1] For small transverse size $R\ll\lambda$, it is λ/R times magnified as is reasonable for the small samples.[27]

C. Vortex loop nucleation on the surface of current-carrying SC sample

We proceed now with the case when a transport current is applied to the SC cylinder in the positive z direction and the external field is absent. The current self-field as well as the external one favors the nucleation of vortices at the cylinder surface. In Refs. 14 and 20 the idealized process of the entry of perfect ring into the SC cylinder was considered. In fact, even for the perfect surface case the vortex first nucleates as a small loop at some (arbitrary) location, since the entry process is stochastic in nature and is not correlated on the scales much more than λ.

We consider thus the same loop nucleation around the point $(\rho=R,\ \varphi=0)$ (Fig. 5) as in the previous section. The Gibbs free energy taking also into account the work done by the source of transport current is

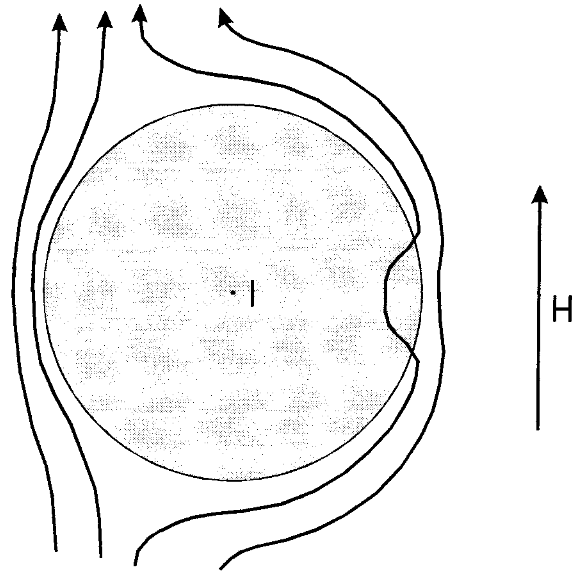

FIG. 6. First vortex loop nucleation at the surface of the current-carrying superconducting cylinder in an external transverse magnetic field.

$$G=F-\Delta W_I, \tag{39}$$

where F is presented in Eq. (37). The current source contribution, ΔW_I, calculated in the spirit of publications[17,19] is the work done by the Lorentz driving force $\mathbf{f}=\Phi_0[\mathbf{j}\times\mathbf{dl}]/c$ exerted upon the flux-line element $\mathbf{dl}$ where $\mathbf{j}$ is the local density of transport current. Since in the problem of nucleation we consider only the small loops of size $r\ll R,\lambda$, the transport current density remains constant within the loop and equal to the surface value $j_s=(I/2\pi R\lambda)I_0(R/\lambda)/I_1(R/\lambda)$, I is the total transport current. Then the work done over the semicircle loop growing from the radius $\rho=0$ to $\rho=r$ reads as

$$\Delta W_I=\int_0^{\pi}d\varphi\int_0^{r}d\rho\rho\,\frac{j\Phi_0}{c}=\frac{j_s\Phi_0\pi r^2}{2c}. \tag{40}$$

Using the above criterium for vortex loop nucleation (see Sec. III B) $\partial G/\partial r(r=\xi)=0$ we find for the critical current density on the surface

$$j_{\rm cr}=j_L/\sqrt{2} \tag{41}$$

independently of sample radius, where the London critical value $j_L=cH_c/4\pi\lambda$. This result is twice less than the critical current for the perfect vortex ring entry into the cylinder obtained in the same approximation.[14,20]

D. Vortex loop nucleation in current-carrying SC cylinder in transverse field

Here we consider finally the case of a current-carrying SC cylinder subjected to a uniform transverse magnetic field as shown in Fig. 6. The current self-field imposed on the Meissner-response field makes the total field asymmetrical. In this case the equatorial line $(\rho=R,\ \varphi=0)$ where the two fields add directly is a weak spot for the first vortex loop nucleation. The process is governed by the Gibbs energy

$$G=F-\Delta W_H-\Delta W_I, \tag{42}$$

where F, ΔW_H, and ΔW_I are defined in Eqs. (37),(23),(40), respectively. Using the criterium for the first vortex nucleation (Sec. III B) $\partial G/\partial r(r=\xi)=0$ we find a critical condition connecting the external transverse field $H_\perp$ and current self-field at the surface $H_I=2I/cR$:

$$\alpha H_I+\beta H_\perp=H_c, \tag{43}$$

where the coefficients $\alpha=\sqrt{2}I_0(R/\lambda)/I_1(R/\lambda)$ and $\beta=2\sqrt{2}I_1(R/\lambda)/I_0(R/\lambda)$. For the macroscopic cylinder $(R\gg\lambda)\alpha\sim\beta/2\sim1$, and for the thin wire $(R<\lambda)2/\alpha\sim\beta\sim R/\lambda$. Thus, the critical condition for the first vortex loop entry in the thick cylinder is approximately

$$H=H_I+2H_\perp\simeq H_c, \tag{44}$$

where H is the maximal magnetic field at the cylinder surface achieved at the line $(\rho=R,\ \varphi=0)$.

In other words, the first vortex enters the perfect macroscopic current-carrying type-II SC cylindrical sample when the total magnetic field first attains at the surface the value of the thermodynamical magnetic field H_c. Thus, the breakdown of the nondissipative current-carrying state of the zero-field-cooled type-II SC subjected to a transverse magnetic field occurs in accordance with the generalized Silsbee's rule.[13] Let us note that this criterion turns out to be valid solely due to Bean-Livingston surface barrier.

IV. CONCLUSIONS

In this paper we have completed within the London approximation the study of magnetic-flux penetration in the current-carrying superconducting cylinder subjected to an external magnetic field of arbitrary direction. We have found an exact solution to the London and Maxwell equations for the magnetic vortex of the arbitrary flat shape. Then the energy and magnetic moment of the definite vortex configurations were calculated that allowed us to find the lower transverse critical field for the cylinder of arbitrary radius and critical field of the first vortex loop nucleation at the surface of a current-carrying SC cylinder subjected to a transverse magnetic field.

Now we are in a position to formulate the general criterion for the first flux-line penetration valid for an external field of arbitrary direction. Consider first the case of macroscopic sample $(R\gg\lambda)$. Unified together with the criterion for the first flux penetration in a longitudinal field[15] the above criterion (44) may be extended to the case of the external field of general direction as follows:

$$|\mathbf{H}_s+\mathbf{H}_\mathbf{I}||_{\max}\simeq H_c, \tag{45}$$

where $\mathbf{H}_s$ is a local value of the magnetic field at the surface, including external field and Meissner-response field.

Thus, the first vortex nucleation at the perfect surface of the bulk cylinder obeys a rather general modified Silsbee's rule. The physical sense of this rule is quite simple: the first vortex enters at a location, where the total (transport+shielding) current per unit length of the surface $I_s=cH/4\pi$ is the largest and the total current density j_t achieves the critical value

$$j_t\simeq j_L=\frac{cH_c}{4\pi\lambda}, \tag{46}$$

which is of order of the depairing current.[17–19,26] In other words, the first vortex entry occurs when (and where) the SC order parameter is locally supressed by the current.

The last criterion (46) is, in fact, more general than the Silsbee's rule (45) and holds also for thin samples $(R\ll\lambda)$. To check this, let us multiply Eq. (43) by $c/4\pi\lambda$ to get the London critical current value j_L on the right-hand side. Than the first term on the left-hand side

$$\frac{c}{4\pi\lambda}\alpha H_I\simeq\frac{c}{4\pi\lambda}\frac{2\lambda}{R}\frac{2\pi R^2 j_{\rm tr}}{cR}=j_{\rm tr} \tag{47}$$

presents the transport current density and the second term

$$\frac{c}{4\pi\lambda}\beta H_\perp\sim\frac{cR}{4\pi\lambda^2}H_\perp\simeq j_M \tag{48}$$

presents the current density contribution induced by the transverse magnetic field. Since the currents $j_{\rm tr}$ and j_M are in this case of the same direction, the sum $(j_{\rm tr}+j_M)$ is equal to the total current j_t which provides the validity of criterion (46). The latter is valid exactly for the parallel current-field configuration too, as may be seen from Ref. 15. That allows one to expect the above criterion (46) to hold for arbitrary field directions and an arbitrary transverse size of the sample.

Though this result is formulated for a pin-free SC it seems to be in qualitative agreement with numerical and magneto-optic observations of magnetic-flux penetration into the hard SC sample of a rectangular form performed by Schuster *et al.*[8]

The surface of a real sample is not perfect and contains (normally) imperfections of the size $\delta>\xi$. In this case one should use in the criterion for the first vortex penetration (Sec. III B) the cutoff length δ instead of ξ which gives

$$\alpha H_I+\beta H_\perp=(\xi/\delta)H_c,\quad \xi<\delta<\lambda \tag{49}$$

instead of Eq. (43). The modified value $(\xi/\delta)H_c$ substitutes the field H_c in Eqs. (44)–(46) too. When the defect size $\delta>\lambda$ the thermodynamical critical field in Eqs. (38), (43)–(46) should be substituted by the lower critical field H_{c1} as was done in Refs. 17–19 and many others. Such a roughness of the surface allows one to ignore completely the effect of surface on the flux entry since the Bean-Livingston barrier width is (at $H>H_{c1}$) of the order of λ. Let us note, that, thanks to the circular cross-section of the sample, there is no geomerical barrier effect in this case.[6,12] Thus, Silsbee's rule for the breakdown of the nondissipative current-carrying state of zero-field-cooled bulk samples turned out to be valid for the type-II superconducting cylinder solely due to the surface (Bean-Livingston) effect.

The critical conditions of the vortex nucleation at the surface of the thin wire of radius $R<\lambda$ Eq. (43) shows that the vortex entry is strongly hindered in this case. If the thin wire contains no inclusions or surface imperfections of the size of its radius it should be capable of carrying a maximal possible persistent current of the order of j_L in the external magnetic field $H<H_c$ of arbitrary direction as well as in zero field.[14] That means particularly that, in the absence of weak links,

the large internal share of the multifilamentary superconducting cables composed of the thin filaments with $R<\lambda$ may be in the vortex-free state. Some recent experimental observations are in favor of the above conclusion. Extremely high values of critical current $j\simeq j_L\simeq 10^9$ A/cm^2 in microbridges of the transverse size $\simeq\lambda$ were reported in Refs. 33 and 34. The growth of the current-carrying capability of the submicron multifilament cables with the decrease of the filament diameter was observed in Ref. 35.

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APPENDIX: MAGNETIC MOMENT OF AN ARBITRARY FLAT VORTEX LOOP

The form of the vortex loop lying in the plane $z=0$ may be defined in polar coordinates by some vector function

$$\boldsymbol{\rho}=\mathbf{l}(\varphi) \tag{A1}$$

if this form is not too complicated. There are at least two solutions of this equation with respect to angle as is shown in Fig. 1: $\theta_+(\rho)$ and $\theta_-(\rho)$. Let us consider for simplicity the symmetrical loop for which $\theta_+=-\theta_-=\chi(\rho)$ is a monotonous function of ρ. Let us denote $R-r$ as the least value of radius ρ for which the solution $\chi(\rho)$ exists, then $\chi(R-r)=0$.

Making use of function $\chi(\rho)$ one can carry out integration in Eq. (4) and find

$$\begin{aligned}\Phi^{\rho}(\rho,\varphi,z)&=\frac{\Phi_0}{\rho}\delta(z)\,\theta(\rho-R+r)[\delta(\varphi-\chi(\rho))\\&\quad-\delta(\varphi+\chi(\rho))],\\ \Phi^{\varphi}(\rho,\varphi,z)&=\Phi_0\delta(z)\,\theta(\rho-R+r)\,\frac{d\chi}{d\rho}[\delta(\varphi-\chi(\rho))\\&\quad+\delta(\varphi+\chi(\rho))].\end{aligned} \tag{A2}$$

That gives for the Fourier components defined in Eq. (7) expressions (34). The amplitudes $\eta^{\pm}_{0,1}=-\lambda^{-2}(\Phi^{\rho}_{0,1}\pm\Phi^{\varphi}_{0,1})$ entering the magnetic moment expression (21) may be presented in a form convenient for integration

$$\eta^{\pm}_{0,1}=\mp\frac{i\Phi_0}{\pi\lambda^2}\rho^{\pm1}\frac{d}{d\rho}[\rho^{\mp1}\sin\,\chi(\rho)]. \tag{A3}$$

Then upon the substitution of the above amplitudes in the general expression (16),(21) one can obtain the magnetic moment of the arbitrary loop:

$$\begin{aligned}M&=\frac{\Phi_0}{2\pi\lambda I_0(R/\lambda)}\int_{R-r}^{R}d\rho\rho\,\sin\,\chi(\rho)\{I_1(\rho/\lambda)\\&\quad+(R/\lambda)I_1(R/\lambda)[K_0(R/\lambda)I_1(\rho/\lambda)\\&\quad+I_0(R/\lambda)K_1(\rho/\lambda)]\}\\&\quad-\frac{\Phi_0}{2\pi\lambda^3}\int_{R-r}^{R}d\rho\rho\int_{R-r}^{\rho}dss\,\sin\,\chi(s)[K_0(\rho/\lambda)I_1(s/\lambda)\\&\quad+I_0(\rho/\lambda)K_1(s/\lambda)].\end{aligned} \tag{A4}$$

Let us now consider a small loop with the length $l\simeq r\ll R$. That means that $\sin\chi(\rho)\simeq\chi(\rho)$ changes between 0 and r/R over the interval $(R-r,R)$. The last integral in Eq. (A4) is as small as $(r/R)^3$ and may be neglected. In the first integral the product of the length of interval of integration and $\sin\chi$ is as small as $(r/R)^2$ which allows us to evaluate all the Bessel functions at $\rho=R$. A specific form of the loop may result only in a factor of order of unity therefore we take here for the estimation the value $r/2R$ as the average of $\sin\chi$ over the interval of integration. Then we obtain to the accuracy of $(r/R)^2$ the magnetic moment of the small vortex loop (35).

For the definite form of the vortex loop described by Eq. (36) one finds to the accuracy of $(r/R)^2$

$$M=\frac{\Phi_0 I_1(R/\lambda)}{\pi\lambda I_0(R/\lambda)}\int_{R-r}^{R}d\rho\rho\,\sin\,\chi(\rho)\rightarrow\frac{\Phi_0 r^2}{4\lambda}\frac{I_1(R/\lambda)}{I_0(R/\lambda)}. \tag{A5}$$

For the vortex lying on the cylinder diameter [see Eq. (24)] $\chi=\pi/2$ and one can find from Eq. (A4) that its magnetic moment equals

$$\begin{aligned}M&=\frac{\Phi_0\lambda}{\pi I_0(R/\lambda)}\int_0^{R/\lambda}dxxI_1(x)\\&=\frac{\Phi_0 R}{2}\left[\frac{L_0(R/\lambda)}{I_0(R/\lambda)}I_1(R/\lambda)-L_1(R/\lambda)\right].\end{aligned} \tag{A6}$$

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