Quasiclassical approach to transport in the vortex state and the Hall effect

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We derive generalized quasiclassical transport equations that include the terms responsible for the Hall effect in the vortex state of a clean type-II superconductor, and calculate the conductivity tensor for an s-wave superconductor in the high-field regime. We find that below the superconducting transition the contribution to the transverse conductivity due to dynamical fluctuations of the order parameter is compensated by the modification of the quasiparticle contribution. In this regime the nonlinear behavior of the Hall angle is governed by the change in the effective quasiparticle scattering rate due to the reduction in the density of states at the Fermi level. The connection with experimental results is discussed. [S0163-1829(98)06117-7]

I. INTRODUCTION

In recent years a significant body of work has been devoted to the better understanding of the Hall effect in the mixed state of type-II superconductors, which has remained a theoretical puzzle for almost thirty years.[1,2] The phenomenological[3,4] theories predict that the Hall angle in the flux-flow regime is either identical to that in the normal state[3] or constant,[4] and the underlying microscopic basis for recent generalizations[2] is not well understood. Theories that make use of the time-dependent Ginzburg-Landau equations (TDGL) find that the Hall conductivity is not modified in the superconducting state.[1] These predictions are at variance with the strongly nonlinear behavior (as a function of magnetic field) found in experiments performed on both low-T_c materials[5,6] and the high-T_c cuprates.[5,7] For dirty superconductors ($l \ll \xi_0$, where l is the mean-free path and ξ_0 is the superconducting coherence length), transport coefficients can be determined from microscopic theory by a straightforward expansion in powers of the order parameter, Δ. The results of such a calculation for the transverse resistivity[8,9] explain qualitatively the sharp increase in the Hall angle below the transition observed in experiment (although, to our knowledge, no systematic comparison has been made), and provide the physical basis for a generalized TDGL approach, in which the relaxation rate is assumed to be complex, rather than purely real, to allow for a modification of the transverse transport coefficients.[1,10,11] The small parameter in the expansion of the microscopic equations is proportional to both the order parameter and the mean-free path, therefore, it is not small in the clean ($l \gg \xi_0$) limit. In this regime a straightforward expansion is not possible; the TDGL equations are not applicable,[12,13] and so an alternative approach is needed to determine the transverse transport coefficients.

In this work we develop an approach to calculate the transport coefficients, including the Hall effect, of clean type-II superconductors in the vortex state and present the results of a calculation of the Hall conductivity of a clean s-wave superconductor in the mixed state near the upper critical field, H_{c2}. The method is based on the quasiclassical approximation to the microscopic theory, due originally, in the context of superconductivity, to Eilenberger[14] and Larkin and Ovchinnikov,[15] which we generalize to include the terms responsible for the Hall effect in a charged superfluid. We solve the equations of this quasiclassical theory to obtain the longitudinal and transverse resistivities in the mixed state. We choose to consider an s-wave superconductor, as both the normal state and superconducting properties of the low-T_c compounds are well known, and comparison between theory and experiment is fraught with less ambiguity; however the approach developed here can easily be generalized to consider superconductors with other than s-wave symmetry.

The microscopic Green's function contains all the information about the single-particle properties of the system. In particular, it oscillates on length scales of order of the inverse Fermi wave vector k_f^{-1}. However, when calculating transport coefficients, we are for the most part only interested in the long-wavelength response. It is then sufficient to determine the envelope of the Green's function rather than its detailed form. In the quasiclassical approach the rapid oscillations associated with the presence of the Fermi surface are integrated out of the basic equations and slower varying quantities such as external fields or the self-energy are expanded around their values at the Fermi surface. The resulting transportlike equations contain the microscopic physics relevant to the problem and are easier to solve. The basic premise of quasiclassical transport theory is that all macroscopic physical quantities vary slowly on a microscopic length scale, and that all the relevant momenta are small compared to the Fermi momentum p_f. This approximation has been applied successfully to study transport phenomena in superfluids[16] and superconductors[17] and to investigate the behavior of the unconventional superconductors.[18] Recently it has been used to analyze the most relevant contributions to the Hall effect in a dirty superconductor in the limit of isolated vortices[19] as well as to investigate the forces acting on a single vortex in the clean regime.[20]

In the next two sections we present a derivation of the generalized quasiclassical equations, which include all the terms contributing to the Hall effect in the mixed state of a clean type-II superconductor in the high-field regime. Section II introduces a general quasiclassical formalism and the basic ideas involved in the analysis of transverse transport in the quasiclassical approximation, illustrated by application to the simple case of a normal metal. We show how the standard Drude results for longitudinal and transverse conductivity are obtained within this quasiclassical approximation. In Sec. III we use the same approach to derive a generalization of the standard quasiclassical approximation for superconductors to include the terms responsible for the transverse conductivity and obtain linearized transportlike equations for a clean superconductor. To solve this system of equations near the upper critical field we employ the approximation of Brandt, Pesch, and Tewordt[21] (BPT), in which the normal part of the matrix propagator is replaced by its spatial average over a unit cell of the vortex lattice, while the exact spatial dependence of the order parameter is retained. Using an operator formalism, we are able to solve the leading-order equations for the distribution function in Sec. IV, and obtain the longitudinal and transverse conductivities within linear-response theory in Secs. V and VI, respectively. In the last section we summarize the results and compare them with the existing experimental data.

II. QUASICLASSICAL APPROACH TO TRANSPORT IN A NORMAL METAL

A. Mixed representation and the standard quasiclassical equations

Our starting point is the microscopic Dyson's equation

$$\left[-\frac{\partial}{\partial\tau}-\zeta(-i\nabla_{\mathbf{x}})-\int d^4y\,\Sigma(x,y)\right]G(y,x')=\delta(x-x') \tag{1}$$

for the Green's function

$$G(x,x')=-\langle T_\tau\psi(x)\psi^\dagger(x')\rangle. \tag{2}$$

Here $\psi(x)$ and $\psi^\dagger(x)$ are field creation and annihilation operators, which depend on the four-vector $x=(\mathbf{x},\tau)$, angular brackets denote the statistical average, and the operator T_τ arranges the field operators in ascending order of imaginary time τ. In Eq. (1) ζ is the single-particle energy operator, and Σ is the self-energy that may be due to interactions or impurity scattering, its exact form has to be determined from microscopic considerations. Dyson's equation can also be written in the form

$$G(x,x')\left[\frac{\partial}{\partial\tau'}-\zeta(+i\nabla_{\mathbf{x}'})\right]-\int d^4y\,\hat{G}(x,y)\hat{\Sigma}(y,x')$$
$$=\delta(x-x'). \tag{3}$$

The operators in this equation are understood to act on the Green's function on their left. It should be emphasized that Eqs. (1) and (3) contain the same physical information and only differ in the form of writing, i.e., the same function G satisfies both. We will use the terms right-hand and left-hand Dyson's equation for Eqs. (1) and (3), respectively.

The derivation of the quasiclassical equations given here follows the general approach of Rainer and Serene[16,22] and Eckern and Schmid.[23] First we consider the linear response of a metal to a constant uniform electric field described by a vector potential $\mathbf{A}(\tau)=\mathbf{A}\exp(i\omega_0\tau)$. To incorporate the vector potential into the microscopic equations we replace the momentum operator by its gauge-invariant counterpart $\zeta(-i\nabla_{\mathbf{x}})\rightarrow\zeta[-i\nabla_{\mathbf{x}}-e\mathbf{A}(\tau)]$, and expand this expression to obtain terms linear in the external field. To integrate out the rapid oscillations associated with the presence of the Fermi surface we first change variables from $\mathbf{x}$ and $\mathbf{x}'$ to center of mass and relative coordinates $\mathbf{R}=(\mathbf{x}+\mathbf{x}')/2$ and $\mathbf{r}=\mathbf{x}-\mathbf{x}'$, and carry out a Fourier transformation in the latter according to

$$G(\mathbf{p},\mathbf{R})=\int d^3\mathbf{r}\,G\left(\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right)\exp(-i\mathbf{pr}). \tag{4}$$

In a translationally invariant system the Green's function only depends on the relative coordinate. Therefore, dependence on the position of the center of mass $\mathbf{R}$ appears only in the presence of external fields. To treat the effect of slowly varying fields quasiclassically we expand in quantities varying on the length scale of the wavelength of these fields, which is equivalent to expanding in powers of $\nabla_{\mathbf{R}}$. If $A(\mathbf{x},-i\nabla_{\mathbf{x}})$ is a local operator that depends only on position and momentum and acts on the Green's function $G(x,x')$, then

$$\int d^3\mathbf{r}\exp(-i\mathbf{pr})A(\mathbf{x},-i\nabla_{\mathbf{x}})G(x,x')$$
$$=\int d^3\mathbf{r}\exp(-i\mathbf{pr})A\left(\mathbf{R}+\frac{\mathbf{r}}{2},-i\nabla_{\mathbf{r}}-\frac{i}{2}\nabla_{\mathbf{R}}\right)$$
$$\times G\left(\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right)$$
$$=\int d^3\mathbf{r}A\left(\mathbf{R}+\frac{i}{2}\nabla_{\mathbf{p}},\mathbf{p}-\frac{i}{2}\nabla_{\mathbf{R}}\right)G\left(\mathbf{R}+\frac{\mathbf{r}}{2},\mathbf{R}-\frac{\mathbf{r}}{2}\right)$$
$$\times\exp(-i\mathbf{pr})A\left(\mathbf{R}+\frac{i}{2}\nabla_{\mathbf{p}},\mathbf{p}-\frac{i}{2}\nabla_{\mathbf{R}}\right)G(\mathbf{p},\mathbf{R}). \tag{5}$$

The final expression can be written as $A\circ G$, where the "circle-product" is defined as[22,23]

$$A(\mathbf{p},\mathbf{R})\circ B(\mathbf{p},\mathbf{R})=\exp\left[\frac{i}{2}(\nabla_{\mathbf{p}_2}\nabla_{\mathbf{R}_1}-\nabla_{\mathbf{p}_1}\nabla_{\mathbf{R}_2})\right]$$
$$\times A(\mathbf{p}_1,\mathbf{R}_1)B(\mathbf{p}_2,\mathbf{R}_2)|_{\mathbf{R}_1=\mathbf{R}_2=\mathbf{R}}. \tag{6}$$

Using this definition, the right- and left-hand Dyson's equations can be written in terms of the mixed set of variables $\mathbf{p}$ and $\mathbf{R}$ as

$$\left(-\frac{\partial}{\partial\tau}-\zeta[\mathbf{p}-e\mathbf{A}(\tau)]\right)\circ G(\mathbf{p},\mathbf{R};\tau,\tau')$$
$$-\int d\tau_1\Sigma(\mathbf{p},\mathbf{R};\tau,\tau_1)\circ G(\mathbf{p},\mathbf{R};\tau_1,\tau')=\delta(\tau-\tau'), \tag{7}$$

$$G(\mathbf{p},\mathbf{R};\tau,\tau')\circ\left(\frac{\partial}{\partial\tau'}-\zeta[\mathbf{p}-e\mathbf{A}(\tau')]\right)$$
$$-\int d\tau_1 G(\mathbf{p},\mathbf{R};\tau,\tau_1)\circ\Sigma(\mathbf{p},\mathbf{R};\tau_1,\tau')=\delta(\tau-\tau'). \tag{8}$$

Direct expansion of Eqs. (7) and (8) in powers of the spatial gradient is not possible since in the definition Eq. (6) of the circle-product this gradient is coupled to derivatives with respect to momentum, and the Green's function varies rapidly with momentum near p_f. To avoid this problem we make a transformation from the set of variables $(\mathbf{p},\mathbf{R})$ to the set $(s,\zeta,\mathbf{R})$, where s is a parametrization of the Fermi surface, and integrate the quantum-mechanical equations over the quasiparticle energy ζ before expanding. The integrated Green's function

$$g(s,\mathbf{R};\tau,\tau')=\frac{1}{\pi}\int d\zeta G(\mathbf{p},\mathbf{R};\tau,\tau') \tag{9}$$

only depends on the components of momentum parallel to the Fermi surface and the remaining dependence on $\mathbf{p}$ and $\mathbf{R}$ is slow. We now transform Eqs. (7) and (8) for the full microscopic Green's function G into equations for the quasiclassical propagator g. This quasiclassical propagator will play the role of a distribution function in the resulting transportlike equation.

Let us first compare terms of zeroth order in the gradient expansion of Eqs. (7)and (8). Since the imaginary time τ varies between 0 and $1/T$, where T is temperature, the first term in the equation gives, after integration, a contribution of order Tg. If we assume that the self-energy varies slowly for momenta close to the Fermi momentum $|\mathbf{p}|\approx p_f$, we can approximate

$$\int d\zeta\Sigma(\mathbf{p},\ldots)G(\mathbf{p},\ldots)\approx\Sigma(\mathbf{p}_f,\ldots)\int d\zeta G(\mathbf{p},\ldots)$$
$$\equiv\sigma(s,\ldots)g(s,\ldots). \tag{10}$$

On the other hand, the term involving ζ gives a much larger contribution since the integration region includes $\zeta\sim\epsilon_f$. Because of this term and the δ function on the right-hand side, the equations cannot be integrated directly. Instead, we subtract Eq. (8) from Eq. (7) to obtain a homogeneous form before integrating term by term and expanding in the gradients. The zeroth order term involving ζG then cancels. Expanding to first order we obtain

$$\int d\zeta\{G\circ\zeta[\mathbf{p}-e\mathbf{A}(\tau')]-\zeta[\mathbf{p}-e\mathbf{A}(\tau)]\circ G\}$$
$$\approx -e\mathbf{v}[\mathbf{A}(\tau')-\mathbf{A}(\tau)]g+i\mathbf{v}\nabla_{\mathbf{R}}g, \tag{11}$$

where the Fermi velocity is defined as

$$\mathbf{v}=\frac{\partial\zeta}{\partial\mathbf{p}}(\mathbf{p}_f). \tag{12}$$

If the spatial dependence of the distribution function is determined by the wave vector $\mathbf{q}$ of an external field, the product $\mathbf{vq}$ is not necessarily small compared to the temperature and the self-energy, so that this term has to be retained in the leading-order equation. Since the small parameters in the expansion are of order $1/(k_f\lambda)$, where λ is a typical wavelength of the electric field, for the terms involving the external vector potential, or, if the self-energy is due to impurity scattering, $1/(k_f l)$, ζ must always be expanded to one order higher in small quantities than other terms in order to obtain a contribution of similar order. It should also be emphasized that, since there are several small parameters in the problem, it may be necessary to expand terms to different order in gradients to account for all the contributions to a particular physical quantity.

It is convenient to Fourier decompose the integrated Green's function into Matsubara frequencies

$$g(s,\mathbf{R};\tau,\tau')=T\sum_{n,n'}g(s,\mathbf{R};\omega_n,\omega_{n'})\exp(-i\omega_n\tau+i\omega_{n'}\tau'), \tag{13}$$

where ω_n are the fermionic frequencies $\omega_n=(2n+1)\pi T$. Then the basic transport equation of the quasiclassical formalism becomes

$$[i\omega_n-i\omega_{n'}+i\mathbf{v}(s)\nabla_{\mathbf{R}}]g(s,\mathbf{R};\omega_n,\omega_{n'})$$
$$-T\sum_{\omega_k}[\sigma(s,\mathbf{R};\omega_n,\omega_k)g(s,\mathbf{R};\omega_k,\omega_{n'})$$
$$-g(s,\mathbf{R};\omega_n,\omega_k)\sigma(s,\mathbf{R};\omega_k,\omega_{n'})]+e\mathbf{vA}[g(s,\mathbf{R};\omega_n$$
$$-\omega_0,\omega_{n'})-g(s,\mathbf{R};\omega_n,\omega_{n'}+\omega_0)]=0. \tag{14}$$

The exact form of the self-energy, σ, is determined from microscopic theory. In principle, all higher-order terms in the spatial gradient can be included in this equation consistently using the definition of the circle product.

It should be noted that, in the absence of a perturbing potential, or impurity scattering leading to the appearance of the self-energy, the Green's function is independent of the coordinate $\mathbf{R}$ and is diagonal in frequency space, and therefore Eq. (14) is trivially satisfied by any function g. This is not surprising since in subtracting the right-hand Dyson's equation from the left-hand equation the information about a particular solution of the inhomogeneous equation has been lost. The particular solution describes the unperturbed noninteracting electron gas, and is obtained by integrating the function $G_0=[i\omega_n-\zeta(\mathbf{p})]^{-1}$ over the quasiparticle energy to find the quasiclassical distribution function of a normal metal, $g_0=-i\,\mathrm{sgn}(\omega_n)$. This function serves as input for any perturbative approach to transport in a metal.

B. Semiclassical treatment of the magnetic field and the Lorentz force

Equation (14) is sufficient to analyze longitudinal transport in a normal metal but it has to be generalized to determine the Hall conductivity. If the vector potential $\mathcal{A}(\mathbf{R})$ describing the magnetic field is taken to be of order "*small*," the field itself, $\mathbf{H}=\nabla\times\mathcal{A}$, becomes of order "(small)"^{2} and the Lorentz force, which is proportional to both electric and magnetic fields, disappears from the perturbative expansion of the quasiclassical equations. This observation led Rainer[22] to point out that in order to analyze the Hall effect in a normal metal, the vector potential $\mathcal{A}(\mathbf{R})$ must be considered as a leading-order quantity and should be included in the equations semiclassically rather than being treated perturbatively. Now the quasiparticle energy ζ depends on the generalized momentum $\mathbf{p}-e\mathbf{A}-e\mathcal{A}(\mathbf{R})$. This replacement is exact. The semiclassical approximation, which is applicable in the long-wavelength limit where the quasiclassical approach is appropriate, treats the momentum operator as a c number. Therefore in the transformations described in Eq. (5) the momentum **p** and the coordinate **R** are no longer independent variables, rather, they are coupled by the presence of the vector potential, which depends upon the coordinates. As a result, the gradient expansion of the integrated Green's function cannot be carried out independently in the Fermi-surface parametrization s and the spatial variable **R**. For a general transformation of variables from the set $(\mathbf{p},\mathbf{R})$ to the set $(\zeta,s_i,\mathbf{R})$

$$\frac{\partial}{\partial R_\alpha}=\frac{\partial}{\partial R_\alpha}+\frac{\partial \zeta}{\partial R_\alpha}\frac{\partial}{\partial \zeta}+\frac{\partial s_i}{\partial R_\alpha}\frac{\partial}{\partial s_i}, \tag{15}$$

$$\frac{\partial}{\partial p_\alpha}=\frac{\partial \zeta}{\partial p_\alpha}\frac{\partial}{\partial \zeta}+\frac{\partial s_i}{\partial p_\alpha}\frac{\partial}{\partial s_i}, \tag{16}$$

where the derivatives on the right-hand side are computed at constant $\zeta,s,\mathbf{R}$ rather than $\mathbf{p},\mathbf{R}$. Using the explicit semiclassical **R** dependence of ζ and s,

$$\frac{\partial \zeta}{\partial R_\alpha}=-e\frac{\partial \zeta}{\partial p_\beta}\frac{\partial \mathcal{A}_\beta}{\partial R_\alpha}=-ev_\beta\frac{\partial \mathcal{A}_\beta}{\partial R_\alpha}, \tag{17}$$

$$\frac{\partial s_i}{\partial R_\alpha}=-e\frac{\partial s_i}{\partial p_\beta}\frac{\partial \mathcal{A}_\beta}{\partial R_\alpha}, \tag{18}$$

we obtain from the expansion of the terms involving ζ,

$$\begin{aligned}&\int d\zeta(-\zeta[\mathbf{p}-e\mathbf{A}(\tau)-e\mathcal{A}(\mathbf{R})]\circ G\\&\quad+G\circ\zeta[\mathbf{p}-e\mathbf{A}(\tau')-e\mathcal{A}(\mathbf{R})])\\&\approx\int d\zeta\bigg(-\zeta[\mathbf{p}-e\mathbf{A}(\tau)-e\mathcal{A}(\mathbf{R})]G\\&\quad+\zeta[\mathbf{p}-e\mathbf{A}(\tau')-e\mathcal{A}(\mathbf{R})]G+i\left[\frac{\partial \zeta}{\partial \mathbf{p}}\frac{\partial G}{\partial \mathbf{R}}-\frac{\partial \zeta}{\partial \mathbf{R}}\frac{\partial G}{\partial \mathbf{p}}\right]\bigg)\\&\to-e\mathbf{v}[\mathbf{A}(\tau')-\mathbf{A}(\tau)]g+i\mathbf{v}\nabla_{\mathbf{R}}g+ie(\mathbf{v}\times\mathbf{H})\frac{\partial g}{\partial \mathbf{p}_\parallel},\end{aligned} \tag{19}$$

where $\mathbf{p}_\parallel$ denotes the component of the momentum **p** parallel to the Fermi surface. In the last line of Eq. (19) we have used the result

$$\begin{aligned}\left[\frac{\partial \zeta}{\partial \mathbf{p}}\frac{\partial G}{\partial \mathbf{R}}-\frac{\partial \zeta}{\partial \mathbf{R}}\frac{\partial G}{\partial \mathbf{p}}\right]&=\frac{\partial \zeta}{\partial p_\alpha}\frac{\partial G}{\partial R_\alpha}+\left[\frac{\partial \zeta}{\partial p_\alpha}\frac{\partial s_i}{\partial R_\alpha}-\frac{\partial \zeta}{\partial R_\alpha}\frac{\partial s_i}{\partial p_\alpha}\right]\frac{\partial G}{\partial s_i}\\&=v_\alpha\frac{\partial G}{\partial R_\alpha}+ev_\alpha\left[\frac{\partial \mathcal{A}_\alpha}{\partial R_\beta}-\frac{\partial \mathcal{A}_\beta}{\partial R_\alpha}\right]\frac{\partial s_i}{\partial p_\beta}\frac{\partial G}{\partial s_i}\\&=\mathbf{v}\nabla_{\mathbf{R}}G+e(\mathbf{v}\times\mathbf{H})\frac{\partial G}{\partial \mathbf{p}_\parallel}.\end{aligned} \tag{20}$$

The new term is the familiar Lorentz force driving term of the classical Boltzmann transport equation. Here it appears on taking into account correctly the semiclassical dependence of the momentum on the external field. The basic quasiclassical equation (14) now takes the form

$$\begin{aligned}&\left[i\mathbf{v}(s)\nabla_{\mathbf{R}}+ie(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial \mathbf{p}_\parallel}\right]g(s,\mathbf{R};\omega_n,\omega_{n'})\\&\quad+[i\omega_n-i\omega_{n'}]g(s,\mathbf{R};\omega_n,\omega_{n'})\\&\quad-T\sum_{\omega_k}[\sigma(s,\mathbf{R};\omega_n,\omega_k)g(s,\mathbf{R};\omega_k,\omega_{n'})\\&\quad-g(s,\mathbf{R};\omega_n,\omega_k)\sigma(s,\mathbf{R};\omega_k,\omega_{n'})]+e\mathbf{v}\mathbf{A}[g(s,\mathbf{R};\omega_n\\&\quad-\omega_0,\omega_{n'})-g(s,\mathbf{R};\omega_n,\omega_{n'}+\omega_0)]=0.\end{aligned} \tag{21}$$

C. Linear response

In general, Eq. (21) is a nonlinear equation. To calculate transport coefficients it is sufficient to keep only the terms linear in the external perturbation—in this case in the electric field—and determine the Green's function g within linear response. We separate the propagator into a leading term and a part linear in the perturbing potential

$$g=g_0(\omega_n)\frac{1}{T}\delta_{\omega_n,\omega_{n'}}+g^{(1)}(s,\mathbf{R};\omega_n,\omega_0)\frac{1}{T}\delta_{\omega_n,\omega_{n'}+\omega_0}. \tag{22}$$

If the self-energy is due to elastic impurity scattering, it can be separated in a similar way into σ_0 and $\sigma^{(1)}$. As noted the equation for the leading-order terms g_0 and σ_0 is satisfied trivially; the terms of linear order are given by

$$\begin{aligned}&[i\omega_0+\sigma_0(-)-\sigma_0+i\mathbf{v}(s)\nabla_{\mathbf{R}}]g^{(1)}+ie(\mathbf{v}\times\mathbf{H})\frac{\partial g^{(1)}}{\partial \mathbf{p}_\parallel}\\&\quad=(e\mathbf{v}\mathbf{A}+\sigma^{(1)})[g_0(-)-g_0].\end{aligned} \tag{23}$$

Here we have used a short-hand notation $g_0=g_0(\omega_n)$ and $g_0(-)=g_0(\omega_n-\omega_0)$. This equation is the basis for the analysis of transport in a normal metal. It has to be solved together with the self-consistency condition relating the change in the self-energy to the modification of the Green's function $g^{(1)}$.

Since, throughout this work, we will be concerned with the electrical conductivity, we have to define the current in terms of the distribution function. It is well known[24] that, if in the microscopic equation for the current density

$$\mathbf{j}(\mathbf{x})=\frac{e}{m}T\sum_{\omega_n}\int d^3\mathbf{p}\mathbf{p}G^{(1)}(\mathbf{p},\mathbf{R}\to\mathbf{x};\omega_n)-\frac{Ne^2}{m}\mathbf{A}, \tag{24}$$

the integration over energy is carried out before summing over frequencies, the contribution from the high-energy regions (far above and below the Fermi surface) exactly cancels the diamagnetic term in Eq. (24). Then the quasiclassical expression for the current becomes[24,25]

$$\mathbf{j}(\mathbf{R})=\pi N(0)eT\sum_{\omega}\int d^2s\mathbf{v}(s)g^{(1)}(s,\mathbf{R};\omega), \tag{25}$$

where $N(0)$ is the density of states at the Fermi surface. The problem of calculating the transport coefficients of a normal metal is now fully defined.

D. Conductivity of a normal metal

As an example of the usefulness of the quasiclassical method we will use it to determine the conductivity tensor of a normal metal in a magnetic field. We consider an experimental arrangement with constant electric and magnetic fields $\mathbf{E}=E\hat{\mathbf{x}}$ and $\mathbf{H}=H\hat{\mathbf{z}}$. We also assume a spherical Fermi surface

$$\mathbf{v}=v(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) \tag{26}$$

and include the effect of isotropic impurity scattering in the Born approximation, so that the self energy is given by

$$\sigma=\frac{1}{2\tau}\int d^2sg, \tag{27}$$

where τ is the quasiparticle lifetime. The unperturbed Green's function is given by $g_0=-i\,\mathrm{sgn}(\omega_n)$, and therefore $\sigma_0=-i\,\mathrm{sgn}(\omega_n)/2\tau$.

First consider the longitudinal dc conductivity. In the absence of a magnetic field Eq. (23), becomes

$$\left\{i\omega_0+\frac{i}{2\tau}[\mathrm{sgn}(\omega_n)-\mathrm{sgn}(\omega_n-\omega_0)]\right\}g^{(1)}$$
$$=-i(e\mathbf{vA}+\sigma^{(1)})[\mathrm{sgn}(\omega_n)-\mathrm{sgn}(\omega_n-\omega_0)]. \tag{28}$$

Since the driving term in Eq. (28) is proportional to $\mathbf{vA}$, it is evident that the angular dependence of $g^{(1)}$ is given by that dot product, and there is no correction to the self energy since the angular average of $g^{(1)}$ vanishes. Then it is obvious from Eq. (28) that $g^{(1)}=0$ when ω_n and $\omega_n-\omega_0$ have the same sign. Otherwise, in the intermediate frequency region where $\omega_0>\omega_n>0$,

$$g^{(1)}=-\frac{2e\mathbf{v}(s)\mathbf{A}}{\omega_0+1/\tau}. \tag{29}$$

Integrating over the Fermi surface, carrying out the summation in the definition of current density, and analytically continuing to the real external frequency according to $i\omega_0\to\overline{\omega}+i\delta$, in the dc-limit ($\overline{\omega}\to 0$) we recover from this solution the standard Drude theory result for the current

$$\mathbf{j}=\frac{1}{3}N(0)e^2v^2\tau\mathbf{E}=\sigma_n\mathbf{E}. \tag{30}$$

We now turn on the magnetic field. Writing the expression for the Lorentz force in spherical coordinates it is easy to check that

$$e(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}_\parallel}=-\omega_c\frac{\partial}{\partial\phi}, \tag{31}$$

and the linearized transport equation becomes

$$\left(i\omega_0+\frac{i}{2\tau}[\mathrm{sgn}(\omega_n)-\mathrm{sgn}(\omega_n-\omega_0)]-i\omega_c\frac{\partial}{\partial\phi}\right)g^{(1)}$$
$$=-ie\mathbf{v}(s)\mathbf{A}[\mathrm{sgn}(\omega_n)-\mathrm{sgn}(\omega_n-\omega_0)]. \tag{32}$$

Again, the response function is nonzero in the intermediate region only. In the regime $\omega_c\tau\ll 1$ it is sufficient to solve the equation perturbatively, namely,

$$g_H^{(1)}=-\frac{2e\mathbf{v}(s)\mathbf{A}}{\omega_0+1/\tau}+\delta g, \tag{33}$$

$$\delta g=\frac{2e^2}{(\omega_0+1/\tau)^2}\omega_c vA\sin\theta\sin\phi. \tag{34}$$

The transverse current obtained from the correction δg is, as expected,

$$j_y=-\sigma_n\omega_c\tau E. \tag{35}$$

We have therefore reproduced the results of the Drude theory using the quasiclassical formalism.

III. QUASICLASSICAL EQUATIONS FOR A SUPERCONDUCTOR

In this section we generalize the approach developed in Sec. II to derive a set of quasiclassical equations that can be used to analyze both longitudinal and transverse transport in superconductors.

A. Gorkov equations

Gorkov's equations[27] for a matrix Green's function $\hat{G}$ replace Dyson's equations in a fully microscopic approach to a superconductor. The diagonal elements of the matrix Green's function

$$\hat{G}=\begin{pmatrix} G & -F \\ F^\dagger & \overline{G}\end{pmatrix} \tag{36}$$

are the particle and hole propagators,

$$\overline{G}(x,x')=G(x',x); \tag{37}$$

for singlet pairing, the off-diagonal elements are related to the probability amplitudes for the destruction or creation of a Cooper pair by

$$(i\hat{\sigma}_y)_{\alpha\beta}F(x,x')=-\langle T_\tau\psi_\alpha(x)\psi_\beta(x')\rangle, \tag{38}$$

$$(i\hat{\sigma}_y)_{\alpha\beta}F^\dagger(x,x')=\langle T_\tau\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')\rangle, \tag{39}$$

where $\hat{\sigma}_y$ is the Pauli matrix. Then the right- and left-hand Gorkov equations are

$$\left[-\frac{\partial}{\partial\tau}\hat{\sigma}_z-\zeta(-i\nabla_{\mathbf{x}}\hat{\sigma}_z)+\hat{\Delta}(x)\right]\hat{G}(x,x')$$
$$-\int d^4y\hat{\Sigma}(x,y)\hat{G}(y,x')=\delta(x-x')\;1 \tag{40}$$

$$\hat{G}(x,x')\left[\frac{\partial}{\partial\tau'}\hat{\sigma}_z-\zeta(+i\nabla_{\mathbf{x}'}\hat{\sigma}_z)+\hat{\Delta}(x')\right]$$
$$-\int d^4y\hat{G}(x,y)\hat{\Sigma}(y,x')=\delta(x-x')\;1. \tag{41}$$

The matrix order parameter

$$\hat{\Delta}=\begin{pmatrix} 0 & \Delta \\ -\Delta^{\star} & 0 \end{pmatrix} \tag{42}$$

is related to the off-diagonal elements of the Green's function by

$$\Delta(x)=\mathrm{g}F(x+0,x), \tag{43}$$

$$\Delta^{\star}(x)=\mathrm{g}F^{\dagger}(x+0,x), \tag{44}$$

where g is the coupling constant.

B. Quasiclassical approximation

The general approach to the derivation of the quasiclassical equations for superconductors is exactly the same as that of Sec. II. We introduce the vector potentials $\mathbf{A}(\tau)$ and $\mathcal{A}(\mathbf{x})$ of an electric and magnetic field into the energy operator, transform the equations to a set of ''mixed'' variables $\mathbf{p}$ and $\mathbf{R}$ by performing a Fourier transform in the relative coordinate, and expand in gradients with respect to the center-of-mass coordinate, after integration over the quasiparticle energy.

Expanding the circle product $\zeta(-i\nabla_{\mathbf{x}})\circ G$ to first order in gradients, we obtain

$$\int d^3\mathbf{r}\exp(-i\mathbf{pr})\zeta(-i\nabla_{\mathbf{x}}-\mathcal{A}(\mathbf{x})-\mathbf{A})G$$
$$\to\left(\zeta(\mathbf{p})-\frac{i}{2}\mathbf{v}[\nabla_{\mathbf{R}}-2ie\mathcal{A}(\mathbf{R})] -\mathbf{vA}-\frac{ie}{2}(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}}\right)G(\mathbf{p},\mathbf{R}). \tag{45}$$

On the other hand on expanding the operator $\zeta(+i\nabla_{\mathbf{x}'})$ the combination $\mathbf{p}+i\nabla_{\mathbf{R}}$ rather than $\mathbf{p}-i\nabla_{\mathbf{R}}$ appears after Fourier transform in $\mathbf{r}$. Consequently, the magnetic-field-dependent terms arising from the expansion of $\nabla_{\mathbf{R}}$ in Eq. (15) have the opposite sign, and

$$\int d^3\mathbf{r}\exp(-i\mathbf{pr})\zeta(+i\nabla_{\mathbf{x}'}-\mathcal{A}(\mathbf{x}')-\mathbf{A})G$$
$$\to\left(\zeta(\mathbf{p})+\frac{i}{2}\mathbf{v}[\nabla_{\mathbf{R}}+2ie\mathcal{A}(\mathbf{R})]-\mathbf{vA}(\tau') +\frac{ie}{2}(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}}\right)G(\mathbf{p},\mathbf{R}). \tag{46}$$

Subtracting Eq. (46) from Eq. (45) we regain the result of the Sec. II B. The vector potential $\mathcal{A}$ appears in the expansions in different gauge-invariant combinations. This can be easily understood if we remember that operator $\zeta(-i\nabla_{\mathbf{x}})$ acts on the annihilation operator ψ while operator $\zeta(+i\nabla_{\mathbf{x}'})$ acts on the creation operator $\psi^{\dagger}$. Then the time evolution of the operators describes the motion of particles and holes, respectively, and the appropriate gauge-invariant derivative is different in each case. To determine the transverse conductivity all contributions of the order of the cyclotron frequency $\omega_c=eH/mc$ have to be included in the equations. In a type-II superconductor in the vortex state, the coherence length ξ_0 sets the length scale for spatial change of the order parameter. Near the upper critical field H_{c2}, the magnetic length $\Lambda=(2eH)^{-1/2}\simeq\xi_0$. This immediately implies that the expansion of the operator $\zeta(\mathbf{p})$ has to be carried out not to first, but to second order in spatial derivatives. The second-order derivative of ζ with respect to momentum is, by definition, the inverse effective mass tensor, which in the simple case of a spherical Fermi surface becomes equal to the inverse effective mass m. In the expansion this term is coupled to square of the spatial gradient, so that its contribution

$$\frac{\partial^2\zeta}{\partial p_\alpha\partial p_\beta}\nabla_\alpha\nabla_\beta G\sim\frac{1}{m\Lambda^2}G\propto\omega_c G \tag{47}$$

is comparable to that of the Lorentz force term and has to be taken into account. Neglecting terms quadratic in the electric and magnetic fields and assuming a Fermi surface with the reflection symmetry $\zeta(\mathbf{p})=\zeta(-\mathbf{p})$, we obtain the expansion of the quasiparticle energy operator

$$\zeta(-i\nabla_{\mathbf{x}})\to\zeta(\mathbf{p})-\frac{i}{2}\mathbf{v}[\nabla-2ie\mathcal{A}(\mathbf{R})]-\mathbf{vA}(\tau) -\frac{ie}{2}[\mathbf{v}\times\mathbf{H}]\frac{\partial}{\partial\mathbf{p}}-\frac{1}{8m}[\nabla-2ie\mathcal{A}(\mathbf{R})]^2 +\frac{ie}{2m}\mathbf{A}[\nabla-2ie\mathcal{A}(\mathbf{R})], \tag{48}$$

$$\zeta(+i\nabla_{\mathbf{x}})\to\zeta(\mathbf{p})-\frac{i}{2}\mathbf{v}[\nabla+2ie\mathcal{A}(\mathbf{R})]+\mathbf{vA}(\tau) +\frac{ie}{2}[\mathbf{v}\times\mathbf{H}]\frac{\partial}{\partial\mathbf{p}}-\frac{1}{8m}[\nabla+2ie\mathcal{A}(\mathbf{R})]^2 -\frac{ie}{2m}\mathbf{A}[\nabla+2ie\mathcal{A}(\mathbf{R})], \tag{49}$$

and similar expressions for the operators $\zeta(+i\nabla_{\mathbf{x}'})$ and $\zeta(-i\nabla_{\mathbf{x}'})$.

Now consider the remaining terms in the expansion of the microscopic Eqs. (40) and (41). Here we are concerned with the change in the Hall conductivity of a superconductor relative to the normal-state value. This change involves the magnitude of the superconducting order parameter Δ, which appears in our analysis in the dimensionless combination $(\Lambda\Delta/v)$. It is then easily seen that linear terms in the gradient expansion of the order parameter have to be retained in the equation since a typical term in the expansion

$$\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\frac{\partial\hat{G}}{\partial\mathbf{p}}\propto\frac{\Delta}{\Lambda}\frac{1}{mv}\hat{G}\approx\omega_c\left(\frac{\Lambda\Delta}{v}\right)\hat{G}, \tag{50}$$

will contribute significantly to the change of transverse conductivity upon entering the superconducting state. Expanding to first order in the gradients we obtain from Eq. (6)

$$\hat{\Delta}(x)\hat{G}(x,x')-\hat{G}(x,x')\hat{\Delta}(x') \rightarrow \hat{\Delta}(\mathbf{R},\tau)\hat{G}(\mathbf{p},\mathbf{R})-\hat{G}(\mathbf{p},\mathbf{R})\hat{\Delta}(\mathbf{R},\tau') + \frac{i}{2}\left[\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\frac{\partial\hat{G}}{\partial\mathbf{p}}+\frac{\partial\hat{G}}{\partial\mathbf{p}}\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\Delta}}{\partial\mathbf{p}}\frac{\partial\hat{G}}{\partial\mathbf{R}}+\frac{\partial\hat{G}}{\partial\mathbf{R}}\frac{\partial\hat{\Delta}}{\partial\mathbf{p}}\right], \tag{51}$$

and, similarly,

$$\int d^4y[-\hat{\Sigma}(x,y)\hat{G}(y,x')+\hat{G}(x,y)\hat{\Sigma}(y,x')] \rightarrow \int d\tau_1\Bigg(\hat{\Sigma}(\mathbf{p},\mathbf{R};\tau,\tau_1)\hat{G}(\mathbf{p},\mathbf{R};\tau_1,\tau') - \hat{G}(\mathbf{p},\mathbf{R};\tau,\tau_1)\hat{\Sigma}(\mathbf{p},\mathbf{R},\tau_1,\tau')+\frac{i}{2}\left[\frac{\partial\hat{\Sigma}}{\partial\mathbf{R}}\frac{\partial\hat{G}}{\partial\mathbf{p}}+\frac{\partial\hat{G}}{\partial\mathbf{p}}\frac{\partial\hat{\Sigma}}{\partial\mathbf{R}}\right] - \frac{i}{2}\left[\frac{\partial\hat{\Sigma}}{\partial\mathbf{p}}\frac{\partial\hat{G}}{\partial\mathbf{R}}+\frac{\partial\hat{G}}{\partial\mathbf{R}}\frac{\partial\hat{\Sigma}}{\partial\mathbf{p}}\right]\Bigg). \tag{52}$$

Using the results of Eqs. (48), (49), (51), and (52), subtracting the left-hand Gorkov-Dyson equation from the right-hand equation, and integrating over the quasiparticle energy, we obtain the quasiclassical transport equation for a superconductor, which can be written using the matrix notation as follows:

$$\begin{aligned} &i\omega_n\hat{\sigma}_z\hat{g}-i\omega_{n'}\hat{g}\hat{\sigma}_z+\hat{\Delta}\hat{g}-\hat{g}\hat{\Delta}+i\mathbf{v}\nabla\hat{g}+e\mathbf{v}\mathcal{A}(\hat{\sigma}_z\hat{g}-\hat{g}\hat{\sigma}_z) \\ &-\frac{ie}{2m}\mathcal{A}(\hat{\sigma}_z\nabla\hat{g}+\nabla\hat{g}\hat{\sigma}_z)+\frac{ie}{2}(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}_\parallel}(\hat{\sigma}_z\hat{g}+\hat{g}\hat{\sigma}_z) \\ &+e\mathbf{v}\mathbf{A}[\hat{\sigma}_z\hat{g}(\omega_n-\omega_0,\omega_{n'})-\hat{g}(\omega_n,\omega_{n'}-\omega_0)\hat{\sigma}_z] \\ &-\frac{ie}{2m}\mathbf{A}[\hat{\sigma}_z\nabla\hat{g}(\omega_n-\omega_0,\omega_{n'})-\nabla\hat{g}(\omega_n,\omega_{n'}-\omega_0)\hat{\sigma}_z] \\ &-T\sum_{\omega_k}[\hat{\sigma}(s,\mathbf{R};\omega_n,\omega_k)\hat{g}(s,\mathbf{R};\omega_k,\omega_{n'}) \\ &-\hat{g}(s,\mathbf{R};\omega_n,\omega_k)\hat{\sigma}(s,\mathbf{R};\omega_k,\omega_{n'})]+\frac{i}{2}\left[\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}\right. \\ &\left.+\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\Delta}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}}{\partial\mathbf{R}}+\frac{\partial\hat{g}}{\partial\mathbf{R}}\frac{\partial\hat{\Delta}}{\partial\mathbf{p}_\parallel}\right]-\frac{i}{2}T\sum_{\omega_k}\left[\frac{\partial\hat{\sigma}}{\partial\mathbf{R}}\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}\right. \\ &\left.+\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\sigma}}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\sigma}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}}{\partial\mathbf{R}}+\frac{\partial\hat{g}}{\partial\mathbf{R}}\frac{\partial\hat{\sigma}}{\partial\mathbf{p}_\parallel}\right]=0, \end{aligned} \tag{53}$$

where the quasiclassical matrix propagator is, as usual,[14,15]

$$\hat{g}(s,\mathbf{R};\omega_n,\omega_{n'})=\int\frac{d\zeta_p}{\pi}\hat{G}(p,\mathbf{R};\omega_n,\omega_{n'})=\begin{pmatrix} g & -f \\ f^\dagger & \bar{g}\end{pmatrix} \tag{54}$$

and the order parameter is given by the self-consistency condition

$$\Delta(\mathbf{R})=gN(0)\pi\sum_n\int d^2sf(s,\mathbf{R};\omega_n,\omega_n). \tag{55}$$

Equations (53) and (55) are the generalization of the standard quasiclassical theory[14,15] to include terms giving rise to nonzero Hall conductivity.

Before we linearize Eq. (53) and solve it to find the longitudinal and Hall conductivities, several comments should be made. First, the vector potential of the magnetic field enters the quasiclassical equation explicitly in contrast to the case of a normal metal [cf. Eq. (21)]. This is readily understood if we notice that in the last term on the first line of Eq. (53) the matrix $\hat{\sigma}_z\hat{g}-\hat{g}\hat{\sigma}_z$ has only off-diagonal elements, so that the term involving $\mathbf{v}\mathcal{A}$ only appears in equations for the anomalous propagator. It would seem that the second term involving the vector potential $\mathcal{A}$ [the last term in the second line of Eq. (53)] is present even in a normal metal since the matrix $\hat{\sigma}_z\hat{g}+\hat{g}\hat{\sigma}_z$ has only diagonal elements, and, consequently, this term contributes only to the equations for the quasiparticle part of the matrix Green's function. However, in a normal metal in the presence of uniform electric and magnetic fields the response function is spatially uniform, and this term is irrelevant. In the superconducting state the spatial variation of the quasiparticle Green's function is due to the spatial dependence of the order parameter Δ in the vortex state, and this term describes the coupling of the current, induced by the spatial dependence of $\Delta(\mathbf{R})$, to the external field. Finally, the Lorentz force is accompanied by the matrix propagator in a combination $\hat{\sigma}_z\hat{g}+\hat{g}\hat{\sigma}_z$, and therefore the Lorentz force does not act directly on the Cooper pairs. This result is perhaps not too surprising as in a reference frame associated with the center of mass the electrons have opposite momenta, and hence there is no net force acting on a pair.

C. Linear response

We now use the approach given in Sec. II C to linearize the basic equation in the external field. If we decompose the propagator $\hat{g}$, self-energy $\hat{\sigma}$, and order parameter Δ into a leading-order term $\hat{g}_0$, $\hat{\sigma}_0$, and Δ_0, and a part (denoted by index 1) linear in the applied electric field, the equation for the Green's function of a superconductor in a magnetic field reads

$$i\omega_n(\hat{\sigma}_z\hat{g}_0-\hat{g}_0\hat{\sigma}_z)+e\mathbf{v}\mathcal{A}(\hat{\sigma}_z\hat{g}_0-\hat{g}_0\hat{\sigma}_z)-(\hat{\sigma}_0\hat{g}_0-\hat{g}_0\hat{\sigma}_0)+\hat{\Delta}_0\hat{g}_0-\hat{g}_0\hat{\Delta}_0+i\mathbf{v}\nabla\hat{g}_0+\frac{ie}{2}(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}_\parallel}(\hat{\sigma}_z\hat{g}_0+\hat{g}_0\hat{\sigma}_z)$$
$$-\frac{ie}{2m}\mathcal{A}(\hat{\sigma}_z\nabla\hat{g}_0+\nabla\hat{g}_0\hat{\sigma}_z)+\frac{i}{2}\left[\frac{\partial\hat{\Delta}_0}{\partial\mathbf{R}}\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\Delta}_0}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\Delta}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}_0}{\partial\mathbf{R}}+\frac{\partial\hat{g}_0}{\partial\mathbf{R}}\frac{\partial\hat{\Delta}_0}{\partial\mathbf{p}_\parallel}\right]$$
$$-\frac{i}{2}\left[\frac{\partial\hat{\sigma}_0}{\partial\mathbf{R}}\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\sigma}_0}{\partial\mathbf{R}}\right]+\frac{i}{2}\left[\frac{\partial\hat{\sigma}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}_0}{\partial\mathbf{R}}+\frac{\partial\hat{g}_0}{\partial\mathbf{R}}\frac{\partial\hat{\sigma}_0}{\partial\mathbf{p}_\parallel}\right]=0, \tag{56}$$

while the equation for the response function $g^{(1)}$ is given by

$$i\mathbf{v}\nabla\hat{g}^{(1)}+i\omega_n\hat{\sigma}_z\hat{g}^{(1)}-i(\omega_n-\omega_0)\hat{g}^{(1)}\hat{\sigma}_z+\hat{\Delta}^{(1)}\hat{g}_0(-)-\hat{g}_0\hat{\Delta}^{(1)}+\hat{\Delta}\hat{g}^{(1)}-\hat{g}^{(1)}\hat{\Delta}-[\hat{\sigma}_0\hat{g}^{(1)}-\hat{g}^{(1)}\hat{\sigma}_0(-)]$$
$$-[\hat{\sigma}^{(1)}\hat{g}_0(-)-\hat{g}_0\hat{\sigma}^{(1)}]+e\mathbf{v}\mathcal{A}(\hat{\sigma}_z\hat{g}^{(1)}-\hat{g}^{(1)}\hat{\sigma}_z)+e\mathbf{v}\mathbf{A}(\hat{\sigma}_z\hat{g}_0(-)-\hat{g}_0\hat{\sigma}_z)+\frac{ie}{2}(\mathbf{v}\times\mathbf{H})\frac{\partial}{\partial\mathbf{p}_\parallel}(\hat{\sigma}_z\hat{g}^{(1)}+\hat{g}^{(1)}\hat{\sigma}_z)$$
$$-\frac{ie}{2m}\mathcal{A}(\hat{\sigma}_z\nabla\hat{g}^{(1)}+\nabla\hat{g}^{(1)}\hat{\sigma}_z)-\frac{ie}{2m}\mathbf{A}[\hat{\sigma}_z\nabla\hat{g}_0(-)-\nabla\hat{g}_0\hat{\sigma}_z]+\frac{i}{2}\left[\frac{\partial\hat{\Delta}_0}{\partial\mathbf{R}}\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\Delta}_0}{\partial\mathbf{R}}\right]$$
$$+\frac{i}{2}\left[\frac{\partial\hat{\Delta}^{(1)}}{\partial\mathbf{R}}\frac{\partial\hat{g}_0(-)}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\Delta}^{(1)}}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\Delta}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{R}}+\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{R}}\frac{\partial\hat{\Delta}_0}{\partial\mathbf{p}_\parallel}\right]$$
$$-\frac{i}{2}\left[\frac{\partial\hat{\Delta}^{(1)}\partial\hat{g}_0(-)}{\partial\mathbf{p}_\parallel\;\partial\mathbf{R}}+\frac{\partial\hat{g}}{\partial\mathbf{R}}\frac{\partial\hat{\Delta}^{(1)}}{\partial\mathbf{p}_\parallel}\right]-\frac{i}{2}\left[\frac{\partial\hat{\sigma}_0}{\partial\mathbf{R}}\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\sigma}_0(-)}{\partial\mathbf{R}}\right]-\frac{i}{2}\left[\frac{\partial\hat{\sigma}^{(1)}}{\partial\mathbf{R}}\frac{\partial\hat{g}_0(-)}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\sigma}^{(1)}}{\partial\mathbf{R}}\right]$$
$$+\frac{i}{2}\left[\frac{\partial\hat{\sigma}_0}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{R}}+\frac{\partial\hat{g}^{(1)}}{\partial\mathbf{R}}\frac{\partial\hat{\sigma}_0(-)}{\partial\mathbf{p}_\parallel}\right]+\frac{i}{2}\left[\frac{\partial\hat{\sigma}^{(1)}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{g}_0(-)}{\partial\mathbf{R}}+\frac{\partial\hat{g}_0}{\partial\mathbf{R}}\frac{\partial\hat{\sigma}^{(1)}}{\partial\mathbf{p}_\parallel}\right]=0. \tag{57}$$

The rest of this work will be devoted to solving these two equations to determine the transverse electrical conductivity of a type-II superconductor in the vortex state.

To calculate the response of a superconductor it will be convenient to modify the definition of current given in Eq. (25). Since the diagonal elements of the matrix propagator are related by Eq. (37), it is easy to check that the current can be written as

$$\mathbf{j}(\mathbf{R})=\frac{1}{2}\pi eN(0)\sum_{\omega}\int d^2s\mathbf{v}(s)(g_1-\overline{g_1}), \tag{58}$$

where g_1 and $\overline{g_1}$ are the diagonal elements of the response function $\hat{g}^{(1)}$. In the standard quasiclassical approach the distribution function g also satisfies a ''normalization condition''[14,15]

$$\sum_{\omega_k}\hat{g}(\omega_n,\omega_k)\hat{g}(\omega_k,\omega_{n'})=-\delta_{\omega_n,\omega_{n'}}. \tag{59}$$

In particular, using this condition for the leading-order distribution function, which is diagonal in frequency [see Eq. (57)], we find

$$\hat{g}_0^2(\omega_n)=-1. \tag{60}$$

However, it has to be emphasized that this normalization condition holds if and only if the gradient of the function g can be written as a commutator of an operator with the distribution function, as is evident from the original derivation.[14,15] It does not apply when terms responsible for the Hall effect are taken into account, since they have the form of an anticommutator of a matrix operator with the Green's function. Nevertheless, this normalization condition will prove useful in determining the quasiclassical Green's function of a superconductor in a high magnetic field at zeroth order.

IV. TYPE-II SUPERCONDUCTOR IN A HIGH MAGNETIC FIELD

A. Model

We consider a clean type-II superconductor in a magnetic field H close to the upper critical field H_{c2}. Again we consider a spherical Fermi surface, and impurity scattering is treated in the Born approximation. The condition for a superconductor to be in the clean regime is $l \gg \xi_0$. In fields not too far below the upper critical field the magnetic length $\Lambda \simeq \xi_0$, so that in the clean regime $l \gg \Lambda$. In type-II superconductors the spatial variations of the internal field become less pronounced as the superfluid density decreases with increased applied uniform magnetic field. As a result, near H_{c2} internal fields can be assumed spatially uniform and equal to the applied field and the vortex lattice can be modeled by an order parameter of the same form as the periodic Abrikosov solution[26]

$$\Delta(\mathbf{R})=\sum_{k_y} C_{k_y} e^{ik_y y}\exp[-(x-\Lambda^2 k_y)^2/2\Lambda^2]$$

$$=\sum_{k_y} C_{k_y} e^{ik_y y}\Phi_0(x-\Lambda^2 k_y), \qquad (61)$$

where $\Phi_0(x)$ is the lowest energy eigenfunction of the linearized Ginzburg-Landau equation (i.e., the eigenfunction of a harmonic oscillator with the Cooper pair mass $M=2m$ and frequency ω_c). The vector potential of the magnetic field has been chosen in an asymmetric gauge $\mathcal{A}(\mathbf{R})=(0,Hx,0)$. The periodicity of the coefficients C_{k_y} determines the type of vortex lattice. Here, we do not consider a specific periodicity, the only assumption made is that there are flux lines in the system; this solution, therefore, can serve as a model for a rigid line liquid as well.

B. Quasiclassical equations in the absence of an electric field and the BPT approximation

First we consider the leading-order Eqs. (56) and neglect terms of order ω_c. Then the elements are

$$i\mathbf{v}\nabla g+\Delta f^\dagger-\Delta^\star f=\frac{1}{2\tau}\Delta\langle f^\dagger\rangle-\frac{1}{2\tau}\Delta^\star\langle f\rangle, \qquad (62)$$

$$[2\omega_n+\mathbf{v}(\nabla-2ie\mathcal{A})]f=2i\Delta g+\frac{i}{\tau}\langle f\rangle g-\frac{i}{\tau}\langle g\rangle f, \quad (63)$$

$$[2\omega_n-\mathbf{v}(\nabla+2ie\mathcal{A})]f^\dagger=2i\Delta^\star g+\frac{i}{\tau}\langle f^\dagger\rangle g-\frac{i}{\tau}\langle g\rangle f^\dagger. \qquad (64)$$

Here angular brackets denote an average over the Fermi surface. The normalization condition, Eq. (59), can be used in this case so that

$$g^2-ff^\dagger=-1, \qquad (65)$$

$$g+\overline{g}=0. \qquad (66)$$

To solve these equations we employ the approach due to Brandt, Pesch, and Tewordt,[21] which was first used in the framework of the quasiclassical approximation by Pesch.[28,29] In this method the diagonal elements g and $\overline{g}$ of the matrix propagator are approximated by their spatial averages, while the exact spatial form of $\Delta(\mathbf{R})$ is retained in determining the off-diagonal functions f and $f^\dagger$. The crucial observation is that the diagonal part of the Green's function is periodic in the center-of-mass coordinate $\mathbf{R}$ with the same periodicity as the order parameter. Performing a Fourier decomposition of the full Green's function in the vectors $\mathbf{K}$ of the reciprocal flux-line lattice, these authors[21] showed that the Fourier components of the Green's function with $\mathbf{K}\neq 0$ are exponentially small [by a factor $\exp(-\Lambda^2K^2)$] compared to the component with $\mathbf{K}=0$. This component is, of course, the spatial average of the Green's function over a unit cell of the vortex lattice, which suggests the above approximation.

The diagonal part of the distribution function depends on the amplitude of the order parameter, but not on its phase. The length scale for the suppression of the mean-field order-parameter amplitude by a single vortex is the coherence length ξ_0, therefore near the upper critical field the order parameter is globally suppressed in the bulk of the superconductor. Consequently, spatial variations of the amplitude $|\Delta|^2$ can be ignored for fields close to H_{c2}. On the other hand, as the phase of the order parameter changes by 2π around a single vortex, the rapid spatial variation of phase in the vortex state must be taken into account to determine the off-diagonal elements of the quasiclassical propagator. After averaging over a single unit cell, the remaining spatial dependence of the amplitude $|\Delta|^2$ is determined by the nonuniformity of the electromagnetic fields; the relevant length scale is the London penetration depth λ_L. Therefore, the BPT approximation works very well for superconductors in the London limit $\kappa=\lambda_L/\xi_0\gg 1$; even for materials with moderate values of κ it remains valid over a wide field range below H_{c2}. Numerical results obtained by Brandt[30] indicate that the BPT approximation works extremely well as long as the parameter $(\Lambda\Delta/v)\leqslant 0.3$. Since the field dependence of the magnetic length is slow, $\Lambda\approx\xi_0(H_{c2}/H)^{1/2}$, this means that the approximation can be used over almost the entire region of linear magnetization, where the order parameter is suppressed.

In all of the following g stands for the spatially averaged distribution function. To determine the functions g, f, and $f^\dagger$, we solve Eqs. (63) and (64) for the off-diagonal elements of the matrix distribution function in terms of g, and apply the spatially averaged normalization condition of Eq. (65) to determine the diagonal part self-consistently. We introduce the impurity renormalized frequency

$$\widetilde{\omega}_n=\omega_n+\frac{i}{2\tau}\langle g(\widetilde{\omega}_n)\rangle \qquad (67)$$

and rewrite the equations for the off-diagonal part of the distribution function as

$$f=[2\widetilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\left(2ig\Delta+\frac{i}{\tau}\langle f\rangle g\right), \qquad (68)$$

$$f^\dagger=[2\widetilde{\omega}_n-\mathbf{v}(\nabla+2ie\mathcal{A})]^{-1}\left(2ig\Delta^\star+\frac{i}{\tau}\langle f^\dagger\rangle g\right). \qquad (69)$$

To proceed with this program we need to know the result of acting with the operator $[2\widetilde{\omega}_n\pm\mathbf{v}(\nabla\mp 2ie\mathcal{A})]^{-1}$ on the order parameter.

C. Operator formalism

Since the order parameter given in Eq. (61) is a superposition of the lowest-energy eigenfuctions of a harmonic oscillator centered at different vortex cores, we introduce the raising and lowering operators

$$a=\frac{\Lambda}{\sqrt{2}}[\nabla_x+i(\nabla_y-2ieHx)], \qquad (70)$$

$$a^\dagger=-\frac{\Lambda}{\sqrt{2}}[\nabla_x-i(\nabla_y-2ieHx)]. \qquad (71)$$

These operators obey the usual bosonic commutation relations $[a,a^\dagger]=1$. We now interpret the Abrikosov solution as the ground state of this ensemble of oscillators $\Delta=|0\rangle$. The higher eigenstates of the system are generated by the standard formula

$$a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle. \tag{72}$$

This operation excites oscillator states centered on each vortex line so that

$$|n\rangle=\sum_{k_y} C_{k_y}e^{ik_y y}\Phi_n(x-\Lambda^2 k_y). \tag{73}$$

Similarly we can introduce conjugate operators corresponding to $\Delta^{\star}=\langle 0|$, the raising and lowering operators for these states are now defined as $b=(a)^{\star}$ and $b^{\dagger}=(a^{\dagger})^{\star}$. Wide use of bosonic operators for the description of the vortex lattice has been hampered by the fact that, even though the wavefunctions corresponding to different oscillator states centered on the same vortex line are orthogonal, functions centered on different flux lines overlap, so that different excited states as defined above are not orthogonal and the equations are nonlocal (see, for example, Ref. 31). What makes this approach successful when combined with the BPT approximation is that this set of states is orthogonal in the sense of a spatial average

$$\int d^3R\langle m|n\rangle=\Delta^2\delta_{m,n}, \tag{74}$$

where Δ is the spatial average of the order parameter. This condition is obeyed since the phase factor $\exp(ik_y y)$ ensures that only functions centered on the same site contribute to the integral. Therefore if we are only concerned with spatial averages of physical quantities, the excited states of the order parameter can be treated as states of a harmonic oscillator.

To evaluate the result of acting with the gradient operator $\mathbf{v}(\nabla-2ie\mathcal{A})$ on the order parameter Δ we rewrite it in terms of the raising and lowering operators a and $a^{\dagger}$

$$\mathbf{v}(\nabla-2ie\mathcal{A})=\frac{v\sin\theta}{\sqrt{2}\Lambda}[ae^{-i\phi}-a^{\dagger}e^{i\phi}]. \tag{75}$$

Then the result of the action of the operator $[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}$ on any mode $|m\rangle$ of the order parameter can be evaluated exactly. The technical details are given in Appendix A. Here we give only the final result,

$$[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}|m\rangle =\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{m}D_m^{m_1m_2}e^{i(m_2-m_1)\phi}|m+m_2-m_1\rangle, \tag{76}$$

where

$$D_m^{m_1m_2}=\frac{\sqrt{m!}\sqrt{(m-m_1+m_2)!}}{(m-m_1)!m_1!m_2!}(-1)^{m_1}\left(-\frac{i}{\sqrt{2}}\right)^{m_1+m_2}\times[\mathrm{sgn}(\omega_n)]^{m_1+m_2+1}W^{(m_1+m_2)}(u_n), \tag{77}$$

$$u_n=\frac{2i\tilde{\omega}_n\Lambda\,\mathrm{sgn}(\omega_n)}{v\sin\theta}, \tag{78}$$

$$W(u)=e^{-u^2}\mathrm{erfc}(-iu), \tag{79}$$

and $W^{(m)}$ is the mth derivative of the function W. Equation (77) is the main result of the operator formalism developed here; it allows further progress towards a solution of the quasiclassical equations to be made.

D. Type-II superconductor in high magnetic field

Guided by the work of Eilenberger[31] and Pesch,[28] we make an ansatz solving Eq. (68) for an s-wave superconductor. This ansatz makes use of the fact that the term dependent on impurity scattering in the right-hand side of the equation renormalizes the amplitude of the order parameter

$$f=2igD^{-1}(\tilde{\omega}_n)[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\Delta. \tag{80}$$

Since the order parameter Δ in this equation is the "ground state" of the Abrikosov vortex lattice $|0\rangle$, the form of the function f can be obtained immediately from Eqs. (76) and (A10)

$$f(s)=2ig(s)D^{-1}(\tilde{\omega}_n)\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(-\frac{i}{\sqrt{2}}\right)^m\times e^{im\phi}[\mathrm{sgn}(\omega_n)]^{m+1}W^{(m)}(u_n)|m\rangle. \tag{81}$$

Substituting this expression into Eq. (68), we find for the impurity renormalization of the order parameter

$$D(\tilde{\omega}_n)=1-i\sqrt{\pi}\frac{\Lambda}{2l}\mathrm{sgn}(\omega_n)\int_0^{\pi}d\theta g(\theta;\tilde{\omega}_n)W(u_n). \tag{82}$$

Using the corresponding Eq. (69) we obtain $f^{\dagger}(s)$

$$f^{\dagger}(s)=2ig(s)D^{-1}(\tilde{\omega}_n)\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(\frac{i}{\sqrt{2}}\right)^m\times e^{-im\phi}[\mathrm{sgn}(\omega_n)]^{m+1}W^{(m)}(u_n)\langle m|. \tag{83}$$

Then we can use the normalization condition, Eq. (65), to determine g (see Appendix A for details)

$$g=-i\,\mathrm{sgn}(\omega_n)P(\theta,\tilde{\omega}_n), \tag{84}$$

where

$$P(\theta,\tilde{\omega}_n)=\left[1-i\sqrt{\pi}\left(\frac{2\Lambda\Delta}{Dv\sin\theta}\right)^2W'(u_n)\right]^{-1/2}, \tag{85}$$

and the sign has been chosen to give the correct expression in the normal state. Equations (67) and (81)–(85) provide a complete self-consistent solution of the quasiclassical equations for an s-wave superconductor in a magnetic field. A Green's function very similar to that given in Eqs. (84) and (85) was obtained in the work of Pesch[28] by a different method. As in the microscopic theory, the order parameter is determined from the self-consistency condition given by Eq. (55), which is, in this case,

$$1 = i\pi\sqrt{\pi}gN(0)\frac{\Lambda}{v}\sum_n \int_0^\pi d\theta g(\tilde{\omega}_n)D^{-1}(\tilde{\omega}_n) \times \text{sgn}(\omega_n)W(u_n). \quad (86)$$

For the general case of finite mean-free path and applied magnetic field, a closed form solution of the self-consistent expressions cannot be easily found. However, with minor simplifications it is possible to obtain analytical results from this solution. Even though the dimensionless parameter $(\Lambda\Delta/v)^2$ in the Green's function given by Eq. (84) is small in the region where the BPT approximation is valid, it appears with the weight $(\sin\theta)^{-2}$, so that a straightforward expansion is impossible. We will see, in fact, that the density of states is a nonanalytic function of this parameter. However, while the full functional dependence of the Green's function on $(\Lambda\Delta/v)^2$ has to be retained, terms of higher order in this small quantity can be neglected in this functional form, provided that they do not result in more singular behavior. Both the impurity renormalization of the frequency $\tilde{\omega}_n$ and the renormalization of the order parameter depend on the weighted angular average of the Green's function g, which is nonsingular as a function of the order parameter in the vortex state. This is related to the gapless character of the quasiparticle spectrum. Therefore, in determining the function P to leading order in $(\Lambda\Delta/v)^2$, the Green's function in the definition of impurity renormalization of the order parameter D [Eq. (82)] can be replaced by its normal-state value. Similarly, the renormalized frequency in the argument of the function W' can be replaced by $\omega_n + \text{sgn}(\omega_n)/2\tau$. The resulting expression for the renormalization function is identical to that obtained by Helfand and Werthamer.[32] With these approximations, Eqs. (81)–(85) describe a closed-form solution. In the clean limit near the upper critical field of interest, here expressions for the quasiclassical propagator can be simplified even further. Since in this regime $l \gg \Lambda$, and the renormalization of the order parameter is $D = 1 + O(\Lambda/l)$, to leading order $D \approx 1$.

The anomalous Green's functions f and $f^\dagger$ are given as a Fourier series in the azimuthal angle ϕ, with the mth component of the series coupling to the mth excited state (or mode) of the order parameter Δ. Therefore in the presence of an external perturbation the mode with $m=0$ will couple to a scalar potential, the mode with $m=1$ to a transverse potential, etc. The function P given in Eq. (85) is related to the angular-dependent density of states. If the Green's function is analytically continued into the upper half plane by letting $i\omega_n \to \omega + i\delta$ then the density of states

$$N(\omega,\theta) = -N(0)\text{Im}g(\omega,\theta) = N(0)\text{Re}P(\omega,\theta) \quad (87)$$

is strongly angular dependent. For quasiparticles traveling parallel to the magnetic field, $N(\omega,\theta)$ is gapped and BCS-like, while in all other directions it is gapless. The total density of states $N_s(\omega)$, obtained by angular integration of the imaginary part of the function g, is gapless,[21] while the residual density of states at the Fermi surface $N_s(0)$ is a nonanalytic function of the order parameter[21,28]

$$N_s \approx N(0)\left[1 - 4\left(\frac{\Lambda\Delta}{v}\right)^2 \ln\left(\frac{\sqrt{2}v}{\Lambda\Delta}\right) + 2\left(\frac{\Lambda\Delta}{v}\right)^2\right]. \quad (88)$$

The Green's function obtained here also reproduces the BCS Green's function if the limit $H \to 0$ is taken, which suggests that it can be used to interpolate between the high-field and the low-field regimes. We now have a closed form expression for the matrix propagator near the upper critical field up to the order $(\Lambda\Delta/v)^4$, which we will use to determine the linear response of a superconductor to an electric field.

V. LONGITUDINAL CONDUCTIVITY

We begin by considering the longitudinal conductivity in the vortex state in the BPT approximation. We are concerned here with the transport coefficients in the clean limit, and will neglect all contributions to conductivity of relative order (Λ/l) compared to the most significant modifications upon entering the superconducting state. We again omit terms of order of cyclotron frequency.

A. The response function

Since the electrical current given in Eq. (58) depends on $g_e = g_1 - \overline{g_1}$, we write the linearized quasiclassical Eq. (57) for the spatial average of this combination. Then the equations for the linear, in the applied electric field, averaged diagonal elements of the distribution function, and the equations for the anomalous functions are

$$g_e = g_1 - \overline{g_1} = \frac{2e\mathbf{vA}[g - g(-)]}{i\tilde{\omega}_0} + (i\tilde{\omega}_0)^{-1}[\overline{\Delta_1^\star f} - \overline{\Delta_1^\star f(-)}] + (i\tilde{\omega}_0)^{-1}[\overline{\Delta_1 f^\dagger} - \overline{\Delta_1 f^\dagger(-)}] + (2i\tilde{\omega}_0\tau)^{-1}([\overline{\langle f_1^\dagger\rangle f} - \overline{\langle f_1^\dagger\rangle f(-)}] + [\overline{\langle f_1\rangle f^\dagger} - \overline{\langle f_1\rangle f^\dagger(-)}]) - (2i\tilde{\omega}_0\tau)^{-1}([\overline{f_1^\dagger\langle f\rangle} - \overline{f_1^\dagger\langle f(-)\rangle}] + [\overline{f_1\langle f^\dagger\rangle} - \overline{f_1\langle f^\dagger(-)\rangle}]), \quad (89)$$

$$[2\tilde{\Omega}_n + \mathbf{v}(\nabla - 2ie\mathcal{A})]f_1 = ie\mathbf{vA}[f + f(-)] + i\Delta(g_1 - \overline{g_1}) + i\Delta_1[g + g(-)] + i(2\tau)^{-1}\{\langle f_1\rangle[g + g(-)] - \langle f\rangle\overline{g_1} + \langle f(-)\rangle g_1\}, \quad (90)$$

$$[2\tilde{\Omega}_n - \mathbf{v}(\nabla + 2ie\mathcal{A})]f_1^\dagger = ie\mathbf{vA}[f^\dagger + f^\dagger(-)] + i\Delta^\star(g_1 - \overline{g_1}) + i\Delta_1^\star[g + g(-)] + i(2\tau)^{-1}\{\langle f_1^\dagger\rangle[g + g(-)] - \langle f^\dagger(-)\rangle\overline{g_1} + \langle f^\dagger\rangle g_1\}. \quad (91)$$

The notation used here is identical to that of the previous section, and the frequency $\tilde{\Omega}_n$ is defined as $\tilde{\Omega}_n = \tilde{\omega}_n + \tilde{\omega}-$. It is possible to identify the different contributions to the right-hand side of Eq. (89). The first term is the quasiparticle contribution to the current, this term determines the response function in the normal state, and, with the modified Green's function, describes the contribution of quasiparticles to the current in superconductors. The other terms on the right-hand side exist only in the superconducting state. The first two of these involve the modification of the order parameter Δ, and can be associated with the motion of the vortex lattice under the influence of the applied electric field. The remaining terms mix the contributions of the quasiparticles and the Cooper pairs. It will be shown below that the most relevant contribution from these terms is due to the additional scattering of the quasiparticles by dynamical fluctuations of the order parameter, similar to the processes described in the dirty limit by the Thompson diagrams.[34]

The quasiparticle contribution to the response function $g_1-\overline{g_1}$ can be determined immediately since the unperturbed functions g and $g(-)$ are known from Eq. (84). To evaluate the other contributions to the response function, Eqs. (90) and (91) have to be solved for Δ_1 and $\langle f_1\rangle$, as well as for the conjugate quantities $\Delta_1^\star$ and $\langle f_1^\dagger\rangle$. As before, here we determine the complete functional dependence of the response function on the order parameter to order Δ^2, and neglect corrections that vanish faster than this as Δ decreases. Since both Δ_1 and $f^\dagger$ can be expanded in a complete set of functions $|m\rangle$ and $\langle m|$, which are normalized by Δ^2, see Eq. (74), it is sufficient to determine the expansion coefficients to zeroth order in Δ. Therefore in Eqs. (90) and (91) we can replace the functions g, g_1, and $\overline{g_1}$ by their normal state values. With these simplifications Eqs. (89)–(91) can be solved explicitly for Δ_1, f_1, and the ''daggered'' functions.

B. Quasiparticle contribution

Two different effects modify the quasiparticle contribution to the current relative to the current in a normal metal. First, the difference $g-g(-)$ is modified relative to its normal state form, and, second, as the impurity renormalization of the frequency $\widetilde{\omega}_0$ depends on the unperturbed Green's function it is also affected by the opening of the superconducting gap below the upper critical field. In the normal state, the difference $g-g(-)$ vanishes in the outside frequency region, for a type-II superconductor this difference is of order Δ^2. Therefore, in a calculation to lowest order in Δ^2, the renormalized frequency can be replaced by the bare frequency in this frequency range. On the other hand, in the intermediate frequency range, where the difference of the unperturbed Green's functions $g-g(-)$ is of order 1, it is important to keep the full dependence of the renormalized frequency on the order parameter. Further, as the contribution from the outside region is proportional to $\overline{\omega}$, but not τ, it is of order $(\Lambda\Delta/v)^2(\Lambda/l)$ and negligible compared to the contribution from the intermediate frequency range. This situation is not unusual when comparing different contributions to the conductivity. Two dimensionless quantities involving the frequency of the external electric field appear in our analysis. The first, $\overline{\omega}\tau$, usually comes from renormalization of the bosonic frequency ω_0 in the intermediate frequency range. The second, $(\Lambda\overline{\omega}/v)$, appears when the response functions are expanded in the external frequency since the argument u_n of these functions involves the frequency in the combination $(\Lambda\overline{\omega}/v)$, see Eq. (78). In the dc response only terms linear in $\overline{\omega}$ contribute to the absorptive part of the conductivity. Therefore, as the ratio of the two dimensionless parameters is of order (Λ/l), we keep terms of order $\overline{\omega}\tau$ while neglecting those of order $(\Lambda\overline{\omega}/v)$. In the quasiparticle contribution then the only relevant terms arise from the intermediate frequency range.

Since the quasiparticle spectrum is gapless in the high-field regime, the response function varies slowly over the scale $\omega\sim T$, and the frequency sums can be evaluated easily, see Appendix B. We find the quasiparticle contribution to the current

$$\mathbf{j}_{qp}=\frac{1}{4}N(0)e^2v^2\mathbf{A}\int_0^\pi \sin^3\theta d\theta \times\Bigg\{(P-1)+i\overline{\omega}\tau\Bigg[P+\langle(1-P)\rangle - \sqrt{\pi}W''\left(\frac{\Lambda}{l\sin\theta}\right)\left(\frac{2\Lambda\Delta}{v\sin\theta}\right)^2\left(\frac{\Lambda}{l\sin\theta}\right)P^3\Bigg]\Bigg\}. \tag{92}$$

Here all the functions are evaluated at $\omega=0$. For $\omega=0$ the argument of the function W' in Eq. (85) is purely imaginary, and the function P is purely real. It follows that the first term in Eq. (92) contributes to the nonabsorptive part of the conductivity; it is the remnant of the Meissner effect in a type-II superconductor in a magnetic field. The remaining terms contribute to the absorptive part, and the transport current can be written as

$$\mathbf{j}_{qp}=\frac{1}{4}N(0)e^2v^2\tau\mathbf{E}\int_0^\pi \sin^3\theta d\theta\Bigg[[P-1]+[1+\langle(1-P)\rangle] - \sqrt{\pi}W''\left(\frac{\Lambda}{l\sin\theta}\right)\left(\frac{2\Lambda\Delta}{v\sin\theta}\right)^2\left(\frac{\Lambda}{l\sin\theta}\right)P^3\Bigg]. \tag{93}$$

The first term in Eq. (93) is the direct modification of the quasiparticle current on entering the superconducting state

$$\mathbf{j}_{qp1}=\frac{1}{4}N(0)e^2v^2\tau\mathbf{E}\int_0^\pi \sin^3\theta d\theta\Bigg\{\Bigg[1-i\sqrt{\pi}\left(\frac{2\Lambda\Delta}{v\sin\theta}\right)^2 \times W'(i\Lambda/l\sin\theta)\Bigg]^{-1/2}-1\Bigg\} = \mathbf{j}'_{qp}-\sigma_n\mathbf{E}, \tag{94}$$

where the normal state conductivity σ_n was defined in Sec. II D. The contribution of small angles $\sin\theta\leqslant(\Lambda/l)$ to the angular integrals is of higher order in (Λ/l) and can be neglected. For larger angles the argument of the function W' can be set to zero since $\Lambda/l\ll 1$. Then the integration is easily carried out, expanding the resulting elliptic integrals for small values of the parameter $(\Lambda\Delta/v)$, we find that the correction to the conductivity from this term

$$\Delta\sigma_{xx}^{qp1}=-6\sigma_n\left(\frac{\Lambda\Delta}{v}\right)^2 \tag{95}$$

is negative. In the superconducting state in addition to the scattering of quasiparticles by impurities, quasiparticles are scattered by the vortex lattice. At a vortex core a quasiparticle can undergo Andreev scattering into a hole and a Cooper pair with no energy cost. This additional scattering process reduces the quasiparticle contribution to the current.

The second term in Eq. (93) arises from renormalization of the scattering time τ in the vortex state. It can be written as

$$\mathbf{j}_{qp2}=\sigma_n[1+(1-\langle P\rangle)]\mathbf{E}=\tfrac{1}{3}N(0)e^2v^2\tau_{\mathrm{eff}}\mathbf{E}, \quad (96)$$

where the scattering rate

$$\tau_{\mathrm{eff}}=\tau[1+(1-\langle P\rangle)]. \quad (97)$$

The quantity $N(0)\langle P\rangle$ evaluated at $\omega=0$ is the residual density of states in a superconductor N_s, see Eq. (88). Hence this term describes the effect of the change in the density of states on the scattering rate of the quasiparticles. Below the transition, as the superconducting gap opens, the residual density of states at the Fermi surface is suppressed compared to the density of states in the normal state; consequently, the effective scattering rate is smaller and the effective mean-free path is larger. The angular integral of the function P can be evaluated to leading order in (Λ/l) and expanded in $(\Lambda\Delta/v)$ in similar fashion to the integral analyzed above, we obtain, in agreement with the result of Eq. (88), the effective scattering time

$$\tau_{\mathrm{eff}}=\tau\left[1-4\left(\frac{\Lambda\Delta}{v}\right)^2\ln\left(\frac{\Lambda\Delta}{\sqrt{2}v}\right)-2\left(\frac{\Lambda\Delta}{v}\right)^2\right], \quad (98)$$

and the contribution to the longitudinal conductivity

$$\sigma_{xx}^{qp2}=\sigma_n\left[1+4\left(\frac{\Lambda\Delta}{v}\right)^2\ln\left(\frac{\sqrt{2}v}{\Lambda\Delta}\right)-2\left(\frac{\Lambda\Delta}{v}\right)^2\right]. \quad (99)$$

Since $(\Lambda\Delta/v)\ll 1$, the logarithmic term dominates near the transition and this contribution is enhanced relative to the normal-state value. The last term in Eq. (93) contributes at order $(\Lambda\Delta/v)^2(\Lambda/l)$.

C. Dynamical fluctuations of the order parameter

To compute the contribution of all the other terms in Eq. (89) to the current, we have to solve Eqs. (90) and (91) for the linear, in the electric field, correction to the order parameter and determine the functions f_1 and $f_1^\dagger$. As discussed above, the functions g_1 and $\overline{g_1}$ can be replaced by their normal-state values to the order to which we work. To use the operator formalism we need to evaluate the effect of acting with the differential operator, $[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}$, on the unperturbed function f. In the clean limit

$$f=2ig[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\Delta. \quad (100)$$

Then the two differential operators can be separated

$$[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}$$
$$=\tfrac{1}{2}(\tilde{\omega}_n-\tilde{\Omega}_n)^{-1}\{[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}-[2\tilde{\omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\}, \quad (101)$$

and Eq. (90) becomes

$$f_1=\frac{e\mathbf{v}\mathbf{A}[f-f(-)]}{i\tilde{\omega}_0}+i[g+g(-)][2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\Delta_1+i(2\tau)^{-1}[g+g(-)][2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\langle f_1\rangle+i(2\tau)^{-1}\frac{e\mathbf{v}\mathbf{A}[g-g(-)]}{i\tilde{\omega}_0}[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\times[\langle f\rangle+\langle f(-)\rangle]. \quad (102)$$

Since the self-consistency condition requires that

$$\Delta_1=\pi gN(0)T\sum_{\omega_n}\langle f_1\rangle, \quad (103)$$

and Eq. (102) is a standard Fredholm-type integral equation, it is clear that, since the function f given in Eq. (81) is a Fourier series in ϕ, the angular average of the product $f\cos\phi$ projects out only the component proportional to $\exp(i\phi)$, the first excited mode of the order parameter $|m=1\rangle$. This implies that the linear, in the electric field, change in the order parameter involves only the first excited state, and has the form $\Delta_1=C|1\rangle$, where C is to be determined from Eq. (102), similarly, $\Delta_1^\star=\overline{C}\langle 1|$. This result, which was anticipated in Sec. IV D, is in agreement with that of Caroli and Maki.[33] The contribution to the response due to the dynamical fluctuations of the order parameter is given by the second term in Eq. (89). Using the functions f and $f^\dagger$ from Eqs. (90) and (91) and the orthogonality condition given in Eq. (74), we find that

$$\overline{\Delta_1 f^\dagger}=-\frac{\sqrt{2\pi}\Lambda\Delta^2}{v\sin\theta}CgW'e^{-i\phi}, \quad (104)$$

$$\overline{\Delta_1^\star f}=\frac{\sqrt{2\pi}\Lambda\Delta^2}{v\sin\theta}\overline{C}gW'e^{i\phi}, \quad (105)$$

and the contribution to the longitudinal and transverse electrical current is

$$\mathbf{j}_x^{fl}=\frac{\pi}{4}\sqrt{2\pi}eN(0)(\overline{C}-C)\Lambda\Delta^2\int_0^\pi d\theta\sin\theta T\times\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}, \quad (106)$$

$$\mathbf{j}_y^{fl}=i\frac{\pi}{4}\sqrt{2\pi}eN(0)(\overline{C}+C)\Lambda\Delta^2\int_0^\pi d\theta\sin\theta T\times\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}. \quad (107)$$

We see that the ''odd'' part of the dynamical fluctuations of the order parameter gives rise to a contribution to the longitudinal resistivity, while the ''even'' part contributes to the Hall current. Both of these contributions are proportional to the ''vertex'' function

$$V(\omega_0)=T\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}, \quad (108)$$

describing the coupling between the electric field and the excited mode of the order parameter.

Equation (102) is solved in Appendix C, we find that the last two terms result in small, in (Λ/l), contributions, and the amplitude of the fluctuations C is given by

$$C=\frac{\sqrt{2}}{4}evA\left(T\sum_n \int_0^\pi \sin\theta d\theta \frac{gW'-g(-)W'(-)}{i\widetilde{\omega}_0}\right)\times\left[T\sum_n \int_0^\pi d\theta\left(ig\,\mathrm{sgn}(\omega_n)W(u_n)-\frac{i}{2}[g+g(-)]\times\mathrm{sgn}(\Omega_n)[W(U_n)+\tfrac{1}{2}W''(U_n)]\right)\right]^{-1}. \tag{109}$$

The denominator on the right-hand side of Eq. (109) is the propagator of the first excited mode of the order parameter. In general, the zeroes of this propagator correspond to the spectrum of propagating modes of the order parameter. In our case the transverse perturbation due to the vector potential of the electric field couples to the first excited mode of the order parameter, which is damped, i.e., there is a finite-energy gap in the spectrum of these excitations at zero frequency. The response to a scalar potential is quite different: there is a propagating mode at zero frequency.[33] Since the dynamical fluctuations of the order parameter are driven by the electric field, the coupling to the excited mode in the numerator of Eq. (109) is also proportional to the vertex function defined in Eq. (108).

Evaluating the sums in Eq. (109) we find [Eq. (C6)]

$$C=-\overline{C}=\frac{ie\Lambda A\sqrt{2}}{1-i\overline{\omega\tau}}, \tag{110}$$

and therefore there is no contribution to the transverse current due to the fluctuation term, as expected. The contribution to the longitudinal current can be evaluated from Eq. (106), it is

$$\mathbf{j}_{fl}=-i\sqrt{\pi}N(0)e^2v^2\tau\left(\frac{\Lambda\Delta}{v}\right)^2\int_0^\pi \sin\theta W'\left(\frac{i\Lambda}{l\sin\theta}\right)\ \mathbf{E}, \tag{111}$$

and the contribution of the dynamical fluctuations of the order parameter to the longitudinal conductivity is

$$\sigma_{xx}^{fl}=4N(0)e^2v^2\tau\left(\frac{\Lambda\Delta}{v}\right)^2=12\sigma_n\left(\frac{\Lambda\Delta}{v}\right)^2. \tag{112}$$

D. Thompson contribution

We now consider the remaining terms in Eq. (89). To evaluate their contribution to the longitudinal current we need the explicit expressions for the angular averages of the unperturbed anomalous Green's function f and $f^\dagger$, and the linear, in the electric field, corrections f_1 and $f_1^\dagger$ to the distribution function. These are obtained from Eq. (102) and its daggered counterpart.

Only one of these terms, the term involving the angular average of the functions f_1 and $f_1^\dagger$, gives a contribution to the conductivity at the order considered here. A typical term is given by

$$\frac{\overline{\langle f_1\rangle f^\dagger}-\overline{\langle f_1\rangle f^\dagger(-)}}{2i\widetilde{\omega}_0\tau}=i\pi evA\left(\frac{\Lambda\Delta}{v}\right)^2\frac{e^{-i\phi}}{\sin\theta}\frac{gW'-g(-)W'(-)}{\widetilde{\omega}_0\tau}\times\int_0^\pi\frac{d\theta'}{4}\left[\frac{gW'-g(-)W'(-)}{i\widetilde{\omega}_0}\sin\theta' -2\frac{\Lambda}{v}\frac{g+g(-)}{1-i\overline{\omega\tau}}\mathrm{sgn}(\Omega_n)\left(W(U_n)+\frac{1}{2}W''(U_n)\right)\right], \tag{113}$$

where the functions under the integral depend on the angle θ', and

$$U_n=\frac{2i\Lambda\widetilde{\Omega}_n\mathrm{sgn}(\Omega_n)}{v\sin\theta}, \tag{114}$$

in analogy to Eq. (77). As there is an additional factor of the scattering time in the denominator, it might be expected that this contribution is small. However, in the intermediate region the renormalized frequency $\widetilde{\omega}_0\sim 1/\tau$ and, since g and $g(-)$ have opposite signs, the contribution of $gW'-g(-)W'(-)$ is of order one. Therefore the first term contributes to the conductivity at the same order as the corrections found previously. On the other hand, as $g+g(-)=0$ in the intermediate region, the second term does not contribute to the current. In the outside region both terms give contributions to order (Λ/l) that can be neglected. The contribution to the current from Eq. (113) and the corresponding term involving $\langle f_1^\dagger\rangle f$ [which is obtained from Eq. (113) by replacing $e^{-i\phi}$ with $e^{i\phi}$] is

$$\mathbf{j}_{Th1}=\frac{\pi^2}{4}N(0)e^2v^2\mathbf{A}\left(\frac{\Lambda\Delta}{v}\right)^2\frac{1}{4i\widetilde{\omega}_0\tau}\frac{1}{i\widetilde{\omega}_0}\times\sum_{\omega_n>0}^{\omega_0}\left[\int_0^\pi \sin\theta d\theta[W'+W'(-)]^2\right]. \tag{115}$$

The Thompson-like contribution to the conductivity is given by

$$\sigma_{xx}^{Th}=-2N(0)e^2v^2\tau\left(\frac{\Lambda\Delta}{v}\right)^2=-6\sigma_n\left(\frac{\Lambda\Delta}{v}\right)^2. \tag{116}$$

In his original work Thompson[34] found that there is a contribution to the conductivity in the dirty $(l\ll\xi_0)$ limit due to scattering of quasiparticles by the dynamical fluctuations of the order parameter. The main contribution in the dirty limit arose from the outside region; the contribution of the intermediate region was smaller by a factor (l/ξ_0). The result obtained here for the clean limit is consistent with this picture. The term contributing to leading order is proportional to the angular average of f_1 and exists only in the presence of the excited mode of the order parameter as it depends on the angular average of f_1. As expected when $(l/\xi_0)\gg 1$, the relevant contribution comes from the intermediate region. In the presence of a transport current the vortex lattice moves, and individual vortices are deformed. As a result, additional scattering of quasiparticles by the vortices gives rise to a negative contribution to the conductivity given in Eq. (116).

E. Longitudinal conductivity

The longitudinal conductivity of a clean type-II superconductor in the mixed state is obtained by combining the results for the quasiparticle current from Eqs. (95) and (99), the current due to the dynamical fluctuations of the order parameter from Eq. (112) and the current due to the Thompson terms from Eq. (116). We notice that reduction in the quasiparticle contribution to the conductivity due to additional scattering off the ground state of the vortex lattice [Eq. (95)] and the excited modes of the order parameter (Thompson terms) is compensated to order $(\Lambda\Delta/v)^2$ by the increase in the current due to dynamical fluctuations of the order parameter. The conductivity then is given by Eq. (99),

$$\sigma_{xx}=\tfrac{1}{3}N(0)e^2v^2\tau_{\text{eff}}=\sigma_n\left\{1+2\left(\frac{\Lambda\Delta}{v}\right)^2\left[\ln\left(\frac{2v^2}{\Lambda^2\Delta^2}\right)-1\right]\right\}, \tag{117}$$

that is the modification of the longitudinal conductivity upon entering the superconducting state is determined solely by the increase in the effective mean-free path due to the suppression of the density of states at the Fermi level as the superconducting gap opens. The increase in the mean-free path is a nonanalytic function of the order parameter.

VI. HALL EFFECT

A. Stability of the leading-order solution

In determining the density of states and the longitudinal conductivity we have neglected terms of the order of cyclotron frequency not only in the linearized quasiclassical equations, but also in the leading-order equations (62)–(64). To investigate the behavior of the transverse conductivity the gradient terms in Eq. (56) have to be taken into account, and the solution for the propagator $\hat{g}$ at zeroth order in the electric field has to be obtained to order ω_c. Instead of attempting to solve Eq. (56) in full, we will show here that the solution obtained in Sec. IV is still valid when terms of order of the cyclotron frequency are included in the equations.

We saw in Sec. III that, as the matrix combination $\hat{\sigma}_z\hat{g}+\hat{g}\hat{\sigma}_z$ has only diagonal elements, the Lorentz force acts only on the quasiparticle (diagonal) part of the propagator. The function g given in Eq. (84) does not depend on the azimuthal angle ϕ, and, therefore, there is no correction to this function from the Lorentz force term. Next we observe that the term involving the gradient of the propagator in the third line of Eq. (56) is proportional to the same combination of matrices as the Lorentz term. Since in the BPT approximation the function g is replaced by its spatial average, this term vanishes. For an s-wave superconductor the order parameter Δ is constant at any point at the Fermi surface, and its derivative with respect to the components of momentum parallel to the Fermi surface vanishes, which means that the last term in the fourth line of Eq. (56) can be ignored. The momentum derivative of the self-energy due to impurity scattering vanishes for the same reason.

We now consider the remaining terms in Eq. (56). Omitting the subscript, since in this section we only consider functions at leading order, we write the first of these terms in the matrix form

$$\hat{M}=\frac{i}{2}\left[\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}+\frac{\partial\hat{g}}{\partial\mathbf{p}_\parallel}\frac{\partial\hat{\Delta}}{\partial\mathbf{R}}\right]. \tag{118}$$

The off-diagonal elements of this matrix are proportional to the trace of the quasiclassical propagator and vanish in accordance with the normalization condition. The contribution from the term $\hat{M}$ to the equation for the quasiparticle part of the distribution function is

$$M_{11}=\frac{i}{2}\left[\frac{\partial\Delta}{\partial\mathbf{R}}\frac{\partial f^\dagger}{\partial\mathbf{p}_\parallel}+\frac{\partial\Delta^\star}{\partial\mathbf{R}}\frac{\partial f}{\partial\mathbf{p}_\parallel}\right]. \tag{119}$$

To spatially average this term and determine its contribution to the diagonal part of the propagator, we need to recast the gradient operators in terms of raising and lowering operators a and $a^\dagger$ and the azimuthal and polar angles ϕ and θ. We find

$$\frac{\hat{\partial}}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_\parallel}=\frac{1}{\sqrt{2}p\Lambda}\left((ae^{-i\phi}-a^\dagger e^{i\phi})\cos\theta\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}(ae^{-i\phi}+a^\dagger e^{i\phi})\frac{\partial}{\partial\phi}\right). \tag{120}$$

Here the hat denotes the gauge-invariant gradient

$$\frac{\hat{\partial}}{\partial\mathbf{R}}=\frac{\partial}{\partial\mathbf{R}}\pm 2ie\mathcal{A}, \tag{121}$$

and the operator with the plus sign acts on $\Delta^\star$ while the operator with the minus sign acts on Δ. A direct check using the solution obtained in Sec. IV shows that the terms breaking gauge invariance vanish after spatial averaging. Consequently, the gradient can be replaced by its gauge invariant counterpart, as expected for an operator acting on the order parameter. For the ground state of the vortex lattice a direct check shows that this term does not result in any correction to the unperturbed propagator. The contribution from the term involving the spatial derivative of the self-energy vanishes in complete analogy to the term just discussed as their structure is identical.

Therefore the solution of the quasiclassical equations obtained in Sec. IV also satisfies the quasiclassical equations when terms of the order of cyclotron frequency are taken into account.

B. Linearized equations for the transverse response

We now consider the linearized Eq. (57). In the regime when $\omega_c\tau\ll 1$ terms of the order of cyclotron frequency can be included in the calculation of the response function perturbatively. We therefore solve for the linear, in the cyclotron frequency, corrections to the averaged response function $g_e=g_1-\overline{g_1}$ obtained in the calculation of the longitudinal conductivity in the preceding section.

Since the Hall conductivity in the normal state is proportional to the square of the scattering time τ, we can expect that the most relevant contributions to the transverse conductivity in the vortex state are also proportional to τ^2, other contributions to the Hall effect are smaller by a factor (Λ/l). Therefore we keep in the equations only terms that contribute to this order. If now $\delta\hat{g}$ is the part of the propagator linear in the cyclotron frequency, we arrive at the following equation for the function $\delta g_e = \delta g_1 - \delta\overline{g}_1$:

$$\delta g_e = \frac{i\omega_c}{i\widetilde{\omega}_0}\frac{\partial}{\partial\phi}(g_1-\overline{g}_1) + \frac{1}{i\widetilde{\omega}_0}\{\delta\Delta_1^\star[f-f(-)] + \delta\Delta_1[f^\dagger - f^\dagger(-)]\} + \frac{1}{2i\widetilde{\omega}_0\tau}\{[\langle\delta f_1^\dagger\rangle f - \langle\delta f_1^\dagger\rangle f(-)] + [\langle\delta f_1\rangle f^\dagger - \langle\delta f_1\rangle f^\dagger(-)]\} - \frac{i}{2i\widetilde{\omega}_0}\left(\frac{\partial\Delta_1^\star}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}\times[f+f(-)] + \frac{\partial\Delta_1}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}[f^\dagger+f^\dagger(-)] + 2\frac{\partial\Delta^\star}{\partial\mathbf{R}}\frac{\partial f_1^\dagger}{\partial\mathbf{p}_{||}} + 2\frac{\partial\Delta}{\partial\mathbf{R}}\frac{\partial f_1}{\partial\mathbf{p}_{||}}\right). \qquad (122)$$

Here $g_1-\overline{g}_1$ is given by Eq. (89). There are now two distinct contributions both to the term involving the fluctuation of the order parameter and to the Thompson term. One reason these terms contribute to the transverse response is that they give rise to additional scattering due to dynamical fluctuations of the order parameter induced by the electric field, as we saw in the previous section. When the quasiparticle trajectories are bent by the magnetic field, this additional scattering renormalizes the Hall conductivity. This effect is contained in the first term in Eq. (122), since the function $g_1-\overline{g}_1$ contains the contributions of the Thompson terms and the fluctuations of the order parameter induced by the electric field. The other contribution to these terms is due to fluctuations of the order parameter induced by the Lorentz force, these fluctuations result in corrections to the transverse conductivity and are contained in the terms involving $\delta\Delta_1$ and δf_1 in Eq. (122). Finally, the terms involving the gradient of the order parameter contribute to order τ^2 for the same reason that the Thompson term contributes to the longitudinal conductivity, namely, that there is an additional factor of the scattering time τ in the amplitude of the order parameter fluctuations C and in the functions f_1 and $f_1^\dagger$, so that the overall contribution is of order τ^2. This anomalous contribution to the transverse conductivity arises because the gradients of the order parameter created by the moving and deformed vortex lattice act as driving forces (analogous to the Magnus force) in the transportlike equations. The remaining terms in Eq. (57) contribute at higher order in (Λ/l).

The equations for the corrections to the off-diagonal elements of the matrix distribution function are

$$[2\widetilde{\Omega}_n + \mathbf{v}_f(\nabla - 2ie\mathcal{A})]\delta f = -\frac{e}{2m}\mathbf{A}(\nabla - 2ie\mathcal{A})[f-f(-)] + i\Delta\,\delta g_e + i\,\delta\Delta_1[g+g(-)] + \frac{i}{2}\frac{\partial\Delta}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}(g_1+\overline{g}_1) + \frac{i}{2}\frac{\partial\Delta_1}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}[g+g(-)] \qquad (123)$$

and

$$[2\widetilde{\Omega}_n - \mathbf{v}_f(\nabla + 2ie\mathcal{A})]\delta f^\dagger = -\frac{e}{2m}\mathbf{A}(\nabla + 2ie\mathcal{A})[f^\dagger - f^\dagger(-)] + i\Delta^\star\delta g_e + i\,\delta\Delta_1^\star[g+g(-)] + \frac{i}{2}\frac{\partial\Delta^\star}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}(g_1+\overline{g}_1) + \frac{i}{2}\frac{\partial\Delta_1^\star}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{||}}[g+g(-)]. \qquad (124)$$

In the normal state there is no angular dependence to the unperturbed function g, and also $g_1+\overline{g}_1=0$, so that the terms in the last line of each equation vanish. We can now solve Eqs. (123) and (124) to determine the contributions to the dynamical fluctuations of the order parameter induced by the Lorentz force. We can then evaluate the contributions to the transverse conductivity term by term.

C. Hall conductivity

The quasiparticle part of the response function is

$$\delta g_e^{qp} = -2evA\sin\theta\sin\phi\frac{i\omega_c}{i\widetilde{\omega}_0}\frac{[g-g(-)]}{i\widetilde{\omega}_0}. \qquad (125)$$

The contribution to the conductivity from the intermediate frequency range is readily evaluated; the correction to the transverse conductivity due to additional scattering off the vortex lattice

$$\Delta\sigma_{xy}^{qp1} = -6\sigma_n\omega_c\tau(\Lambda\Delta/v)^2 \qquad (126)$$

and the contribution to the Hall conductivity due to the modification of the scattering time

$$\sigma_{xy}^{qp2} = \tfrac{1}{3}N(0)e^2v^2\tau_{\rm eff}(\omega_c\tau_{\rm eff}) = \sigma_n\omega_c\tau\left\{1+4\left(\frac{\Lambda\Delta}{v}\right)^2\left[\ln\left(\frac{2v^2}{\Lambda^2\Delta^2}\right)-1\right]\right\}. \qquad (127)$$

In addition, there is a quasiparticle contribution to the Hall conductivity from the outer frequency range that is formally divergent,

$$\mathbf{j}_y^{an1} = -\tfrac{1}{4}N(0)e^2v^2A\frac{i\omega_c}{\overline{\omega}}\int_0^\pi \sin^3\theta\,d\theta\left[(P-1) + \left(\frac{\Lambda\overline{\omega}}{v\sin\theta}\right)\frac{\partial P}{\partial\omega}\right]. \qquad (128)$$

Since P' evaluated at zero frequency is purely imaginary, the second term describes a small correction to the Meissner-like term. The first term in this equation, on the other hand, has no physical meaning and must disappear from the final expression for the current.

It is in fact canceled by the contribution of the fluctuations of the order parameter

$$\mathbf{j}_y^{fl} = 2i\pi e\omega_c CN(0)\frac{\sqrt{2\pi}\Lambda\Delta^2}{v}\int d^2s\mathbf{v}\frac{\sin\phi}{\sin\theta}T \times\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{(i\tilde{\omega}_0)^2} \quad (129)$$

from the outer range, where the first term in the expansion of the vertex function

$$T\sum_{out}\frac{gW'-g(-)W'(-)}{(i\tilde{\omega}_0)^2} = -\frac{2i}{\overline{\omega}}T\sum_{\omega_n>0}\frac{\partial W'}{\partial\omega}+\ \ldots \quad (130)$$

is formally divergent. The remaining contribution from the outer range is obtained by expanding the coefficients C and $\overline{C}$ in the small quantity $\overline{\omega\tau}$, it is

$$\mathbf{j}_y^{fl2} = 6\sigma_n(\omega_c\tau)\left(\frac{\Lambda\Delta}{v}\right)^2 E. \quad (131)$$

This contribution is canceled by that of the intermediate frequency range in Eq. (129), so that there is no net contribution to the transverse conductivity due to the dynamical fluctuations of the order parameter driven by the electric field. This result is consistent with the predictions of time-dependent Ginzburg-Landau theory.[1] The terms considered so far in this section correspond directly to those contained in TDGL, which is an effective theory treating only the fluctuations of the order parameter, while the quasiparticle contribution is taken to be at the normal-state value. In the TDGL approach the Lorentz force has no effect on the dynamics of the order parameter, and there is no correction $\delta\Delta_1$ due to this force.

In the present analysis, however, the equations for the quasiparticle propagators and the amplitude of the order parameter fluctuations are coupled, so that even though the Lorentz force does not appear explicitly in the equation for δf, it introduces changes in the diagonal part of the distribution function $g_1-\overline{g_1}$ and therefore brings about further modification $\delta\Delta_1$ of the order parameter. To find this contribution we have to solve Eqs. (123) and (124) for the changes in the order parameter $\delta\Delta_1 = \delta C|1\rangle$ and $\delta\Delta_1^\star = \delta\overline{C}\langle 1|$. The solution, given in Appendix C, follows the same steps as in the calculation of the longitudinal conductivity. We find that only the term involving δg_e contributes at the order to which we work, and

$$\delta C = \delta\overline{C} = ie\Lambda A\sqrt{2}(\overline{\omega}\tau)(\omega_c\tau), \quad (132)$$

and, since we saw in Eq. (107) that the ''even'' part of the dynamical fluctuations of the order parameter contributes to the transverse part of the conductivity

$$\sigma_{xy}^{fl} = 6\sigma_n(\omega_c\tau)\left(\frac{\Lambda\Delta}{v}\right)^2, \quad (133)$$

the dynamical fluctuations of the order parameter driven by the Lorentz force tend to increase the transverse conductivity.

Similarly, there are two parts to the Thompson terms: one is due to the longitudinal response, while the other is due to the linear in the Lorentz force corrections to the off-diagonal distribution functions. The contribution to the transverse conductivity due to the first of the Thompson terms is the longitudinal contribution multiplied by $\omega_c\tau$,

$$\sigma_{xy}^{Th1} = -6\sigma_n(\omega_c\tau)\left(\frac{\Lambda\Delta}{v}\right)^2. \quad (134)$$

To determine the contribution to the current from the second Thompson term, we use the angular averages $\langle\delta f\rangle$ and $\langle\delta f^\dagger\rangle$ given in Appendix C, to find that its contribution doubles that given in Eq. (134), so that the total contribution of the Thompson terms to the transverse conductivity is

$$\sigma_{xy}^{Th} = -12\sigma_n(\omega_c\tau)\left(\frac{\Lambda\Delta}{v}\right)^2. \quad (135)$$

In addition to the scattering of quasiparticles by the fluctuations of the order parameter induced by the applied electric field, which tends to reduce the current, quasiparticles also undergo additional scattering off the dynamical fluctuations driven by the Lorentz force, which again tends to reduce the transverse response. In this sense, the Lorentz force results in anisotropic scattering of the quasiparticles by the fluctuations of the order parameter.

The last two terms, the gradient terms in Eq. (122) have a structure identical to that of the terms considered in Sec. VI A. Their contribution to the current can be evaluated using the operator approach as shown in Appendix A. First, consider the term involving the gradient of the dynamical fluctuations of the order parameter Δ_1 and $\Delta_1^\star$. These fluctuations involve the first excited mode of the order parameter, which, when they are acted upon by the annihilation and creation operators in Eq. (A17), gives terms proportional to the ground state and the second excited state of the order parameter, respectively. Then spatial averaging projects out the same modes from the functions f and $f^\dagger$ given in Eqs. (81) and (83). In the second of these terms the gradient operator acts on the ground state of the vortex lattice, Δ, which the creation operator promotes to the first excited state. Spatial averaging now projects out the first excited component from the functions f_1 and $f_1^\dagger$. The contribution to the transverse current due to these terms, is found to be

$$\mathbf{j}_y^{gr} = -3\sigma_n\frac{\Delta^2\tau}{\epsilon_f}E. \quad (136)$$

Since

$$\frac{\Delta^2\tau}{\epsilon_f} = \frac{2\Delta^2\tau}{mv^2} = 2\left(\frac{\Lambda\Delta}{v}\right)^2\frac{\tau}{m\Lambda^2} = 4\omega_c\tau\left(\frac{\Lambda\Delta}{v}\right)^2, \quad (137)$$

the contribution to the transverse conductivity due to these terms is given by

$$\sigma_{xy}^{gr} = 12\sigma_n\omega_c\tau\left(\frac{\Lambda\Delta}{v}\right)^2. \quad (138)$$

The induced gradients of the order parameter enhance the transverse conductivity.

The total transverse conductivity is the sum of all the contributions considered here. We find that the modification of the quasiparticle Hall current due to additional scattering off the vortex lattice given in Eq. (126) is exactly compensated by the enhancement of the transverse current due to

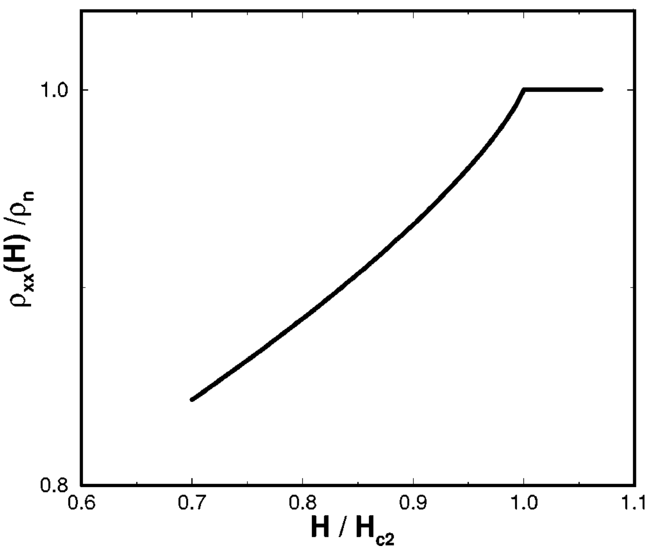

FIG. 1. Longitudinal resistivity as a function of the reduced magnetic field.

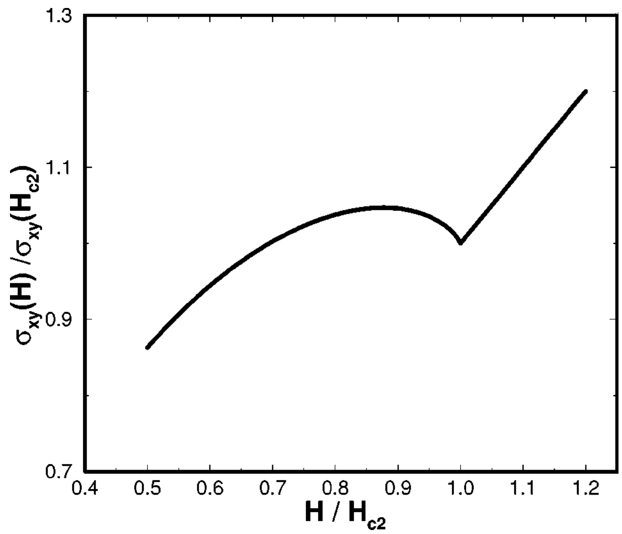

FIG. 2. Transverse conductivity as a function of the reduced magnetic field.

the Lorentz force driven fluctuations of the order parameter obtained in Eq. (133). The Thompson contribution due to additional scattering by the deformed and moving vortex lattice is given in Eq. (135) and is canceled by the enhancement of the transverse conductivity due to the forces generated by the gradient of the excited mode of the order parameter found in Eq. (138). As a result, the behavior of the transverse conductivity σ_{xy} is determined solely by the modification of the effective elastic scattering time $\tau_{\rm eff}$ and is given by Eq. (127),

$$\begin{aligned}\sigma_{xy}^{qp2}&=\tfrac{1}{3}N(0)e^2v^2\tau_{\rm eff}(\omega_c\tau_{\rm eff})\\&=\sigma_n\omega_c\tau\left\{1+4\left(\frac{\Lambda\Delta}{v}\right)^2\left[\ln\left(\frac{2v^2}{\Lambda^2\Delta^2}\right)-1\right]\right\}. \quad (139)\end{aligned}$$

For the dc conductivity this change is due to the decrease in the number of states at the Fermi surface available for scattering as the superconducting gap opens.

VII. CONCLUSIONS AND DISCUSSION

We now plot qualitatively the longitudinal resistivity (Fig. 1), the transverse conductivity (Fig. 2), and the Hall angle (Fig. 3) as functions of the applied magnetic field for Niobium. The order parameter, which is linear in the applied magnetic field in the high-field regime, is given by the expression due to Maki and Tsuzuki,[35]

$$\Delta^2=\frac{1}{\pi N(0)}\,\frac{H_{c2}-H}{\beta_A(2\kappa_2^2-1)}\left(H_{c2}-\frac{T}{2}\,\frac{dH_{c2}}{dT}\right), \quad (140)$$

and the values of the superconducting material parameters were taken from Refs. 36–38. The longitudinal resistivity in Fig. 1 has a pronounced increase in slope as a function of the magnetic field below the superconducting transition due to the logarithmic dependence in Eq. (117). The transverse conductivity shown in Fig. 2 is enhanced below the upper critical field and has negative curvature in the high-field region. The negative curvature arises from the competition between the enhancement due to the increase in the effective mean-free path and the linear decrease of the cyclotron frequency with the field; the Hall conductivity is substantially enhanced when compared to the linear decrease expected from the normal-state behavior. While the transverse conductivity is proportional to the square of the scattering time, the Hall angle

$$\tan\theta_H=\sigma_{xy}/\sigma_{xx}=\omega_c\tau_{\rm eff} \quad (141)$$

is only linearly dependent on the scattering time and the corresponding nonlinear dependence on magnetic field is weaker, as can be seen in Fig. 3. Finally, as the transverse resistivity

$$\rho_{xy}=\frac{\sigma_{xy}}{\sigma_{xx}^2+\sigma_{xy}^2}\approx\sigma_{xy}/\sigma_{xx}^2 \quad (142)$$

is independent of the effective scattering time, it remains linear in magnetic field upon entering the superconducting state with the same slope as in the normal metal. This behavior is to be contrasted with that of Bardeen-Stephen model,[3] where the resistivity is modified and is linear in the magnetic field, but the Hall angle obeys the same linear law as in the

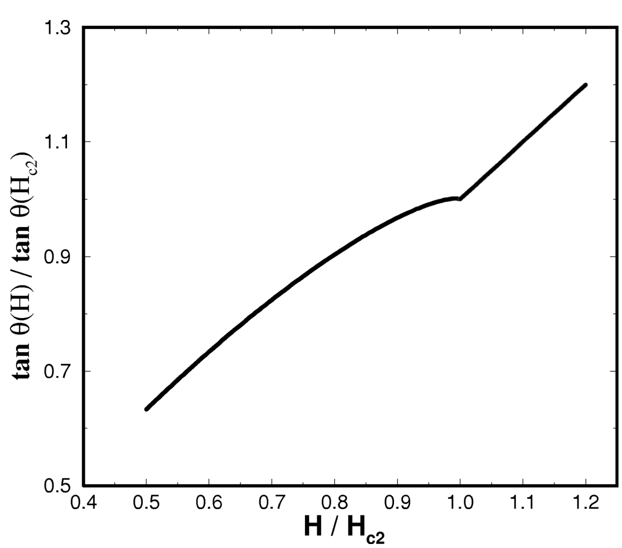

FIG. 3. Hall angle as a function of the reduced magnetic field.

normal state. The Nozieres-Vinen theory,[4] on the other hand, which predicts that the Hall angle should be constant in the flux-flow regime below H_{c2} at variance with the result of this work, also finds that the transverse resistivity is identical to that of the normal state, although the individual components of the conductivity tensor are quite different from those found here.

A comparison can be made with the experimental data of Fiory and Serin[6] on high purity Nb. These experiments find a transverse resistivity in the flux-flow regime that is linear in the applied magnetic field over a wide range of fields below H_{c2}. The Hall angle, however, flattens or even increases above its value at H_{c2} before decreasing at lower fields. These results are more suggestive of the behavior given here than the original interpretation given in terms of the Nozieres-Vinen theory. Also, the longitudinal resistivity found in Ref. 6 has a distinct increase in slope just below the upper critical field, which is consistent with the behavior discussed above. Detailed comparisons with the results of this work are difficult to make, since the authors of Ref. 6 used a high current density to reduce the pinning effects and achieve the flux-flow regime; as a result, the magnetoresistance is significant and the longitudinal resistivity in the normal state varies with magnetic field. We find the qualitative agreement with the experiment encouraging and suggest that more experimental work is needed to make a more detailed comparison with the theory. To conclude, we have presented here an approach to the calculation of the transport coefficients of a clean type-II superconductor in the vortex state in the high-field regime and used it to determine the Hall conductivity and the Hall angle of an s-wave superconductor in this regime. We find that the field dependence of the Hall conductivity in the high-field regime, which is nonanalytic, is entirely due to the change in the density of quasiparticle states at the Fermi level in the superconducting state. At the same time we find that the field dependence of the transverse resistivity below the upper critical field remains unchanged.

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APPENDIX A: THE OPERATOR FORMALISM

If $|m\rangle$ is the mth excited mode of the order parameter we have

$$[2\tilde{\omega}_n+\mathbf{v}_F(\nabla-2ie\mathcal{A})]^{-1}|m\rangle$$
$$=\mathrm{sgn}(\omega_n)\int_0^\infty \exp\{-[2\tilde{\omega}_n+\mathbf{v}_F(\nabla-2ie\mathcal{A})]\times\mathrm{sgn}(\omega_n)t\}dt|m\rangle$$
$$=\mathrm{sgn}(\omega_n)\int_0^\infty \exp[-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)t]\times\exp\left(-\frac{v\sin\theta}{\sqrt{2}\Lambda}[ae^{-i\phi}-a^\dagger e^{i\phi}]\mathrm{sgn}(\omega_n)t\right)dt|m\rangle. \tag{A1}$$

We use the operator identity

$$e^{A+B}=e^Ae^Be^{-1/2\,[A,B]}, \tag{A2}$$

where $[A,B]=AB-BA$ denotes a commutator, to separate the creation and annihilation operators and rewrite Eq. (A1) as

$$[2\tilde{\omega}_n+\mathbf{v}_F(\nabla-2ie\mathcal{A})]^{-1}|m\rangle$$
$$=\mathrm{sgn}(\omega_n)\int_0^\infty dt\exp\left(-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)t-\frac{v^2\sin^2\theta}{4\Lambda^2}t^2\right)\times\exp\left(\frac{vt\sin\theta}{\sqrt{2}\Lambda}\mathrm{sgn}(\omega_n)e^{i\phi}a^\dagger\right)\times\exp\left(-\frac{vt\sin\theta}{\sqrt{2}\Lambda}\mathrm{sgn}(\omega_n)e^{-i\phi}a\right)|m\rangle. \tag{A3}$$

We now write the exponentials as infinite series in powers of the arguments to find

$$[2\tilde{\omega}_n+\mathbf{v}_F(\nabla-2ie\mathcal{A})]^{-1}|m\rangle$$
$$=\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{m}\int_0^\infty dt\exp\left(-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)t-\frac{v^2\sin^2\theta}{4\Lambda^2}t^2\right)\times\frac{(-1)^{m_1}}{m_1!m_2!}e^{i(m_2-m_1)\phi}[\mathrm{sgn}(\omega_n)]^{m_1+m_2+1}\times\left[\frac{vt\sin\theta}{\sqrt{2}\Lambda}\right]^{m1+m2}(a^\dagger)^{m_2}(a)^{m_1}|m\rangle. \tag{A4}$$

In the integral, the parameter t can be replaced with a differential operator

$$t=\left(-\frac{1}{2}\mathrm{sgn}(\omega_n)\frac{\partial}{\partial\tilde{\omega}_n}\right) \tag{A5}$$

and the integral can be evaluated

$$\int_0^\infty dt\exp\left(-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)t-\frac{v^2\sin^2\theta}{4\Lambda^2}t^2\right)\left[\frac{vt\sin\theta}{\sqrt{2}\Lambda}\right]^{m_1+m_2}$$
$$=\left[\frac{v\sin\theta}{\sqrt{2}\Lambda}\right]^{m_1+m_2}\left(-\frac{1}{2}\mathrm{sgn}(\omega_n)\frac{\partial}{\partial\tilde{\omega}_n}\right)^{m_1+m_2}\times\int_0^\infty dt\exp\left(-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)t-\frac{v^2\sin^2\theta}{4\Lambda^2}t^2\right)$$
$$=\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\left(-\frac{i}{\sqrt{2}}\right)^{m_1+m_2}W^{(m_1+m_2)}(u_n), \tag{A6}$$

where $W(u)=e^{-u^2}\mathrm{erfc}(-iu)$, $W^{(m)}$ denotes the mth derivative and

$$u_n=\frac{2i\tilde{\omega}_n\Lambda\mathrm{sgn}(\omega_n)}{v\sin\theta}. \tag{A7}$$

The main result is

$$[2\tilde{\omega}_n+\mathbf{v}_f(\nabla-2ie\mathcal{A})]^{-1}|m\rangle$$
$$=\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m_2=0}^{\infty}\sum_{m_1=0}^{m}D_m^{m_1m_2}e^{i(m_2-m_1)\phi}|m+m_2-m_1\rangle, \tag{A8}$$

where

$$D_m^{m_1m_2}=\frac{\sqrt{m!}\sqrt{(m-m_1+m_2)!}}{(m-m_1)!m_1!m_2!}(-1)^{m_1}\times\left(-\frac{i}{\sqrt{2}}\right)^{m_1+m_2}[\mathrm{sgn}(\omega_n)]^{m_1+m_2+1}W^{(m_1+m_2)}(u_n). \tag{A9}$$

We make extensive use of two special cases of Eq. (A8):

$$(2\tilde{\omega}_n+\mathbf{v}[\nabla-2ie\mathcal{A})]^{-1}|0\rangle=\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\times\left(-\frac{i}{\sqrt{2}}\right)^{m}e^{im\phi}[\mathrm{sgn}(\omega_n)]^{m+1}W^{(m)}(u_n)|m\rangle \tag{A10}$$

and

$$(2\tilde{\omega}_n+\mathbf{v}[\nabla-2ie\mathcal{A})]^{-1}|1\rangle=\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(-\frac{i}{\sqrt{2}}\right)^{m}[\mathrm{sgn}(\omega_n)]^{m+1}\times[\sqrt{m+1}e^{im\phi}W^{(m)}(u_n)|m+1\rangle+\left(\frac{i}{\sqrt{2}}\right)[\mathrm{sgn}(\omega_n)]e^{i(m-1)\phi}W^{(m+1)}(u_n)|m\rangle]. \tag{A11}$$

Equations for the daggered quantities are obtained by replacing the phase $im\phi$ by its conjugate $-im\phi$, changing the sign of $(i/\sqrt{2})$, and using a bra vector instead of the ket vector.

To determine the quasiclassical Green's function g we need the spatial average $\overline{ff^{\dagger}}$. Using Eq. (A4) we have

$$\overline{ff^{\dagger}}=\int d^3\mathbf{R}ff^{\dagger} \tag{A12}$$

$$=-4g^2\Delta^2\int_0^{\infty}dt_1dt_2\exp\left(-2\tilde{\omega}_n\mathrm{sgn}(\omega_n)(t_1+t_2)-\frac{v^2\sin^2\theta}{4\Lambda^2}(t_1+t_2)^2\right)$$
$$=i\sqrt{\pi}g^2\left(\frac{2\Lambda\Delta}{v\sin\theta}\right)^2W'\left(\frac{2i\tilde{\omega}_n\Lambda\mathrm{sgn}(\omega_n)}{v\sin\theta}\right). \tag{A13}$$

Equation (84) obviously follows from the last line.

In the calculation of the transverse conductivity we will need to rewrite the gradient operators in terms of creation and annihilation operators. Since a gauge-invariant gradient can be written as

$$\frac{\hat{\partial}}{\partial x}=\frac{1}{\sqrt{2}\Lambda}[a-a^{\dagger}]=\frac{1}{\sqrt{2}\Lambda}[b-b^{\dagger}], \tag{A14}$$

$$\frac{\hat{\partial}}{\partial y}=-\frac{i}{\sqrt{2}\Lambda}[a+a^{\dagger}]=\frac{i}{\sqrt{2}\Lambda}[b+b^{\dagger}], \tag{A15}$$

and the momentum gradient in the direction parallel to the Fermi surface is

$$\frac{\partial}{\partial\mathbf{p}_{\parallel}}=\frac{1}{p}\hat{\mathbf{e}}_{\theta}\frac{\partial}{\partial\theta}+\frac{1}{p\sin\theta}\hat{\mathbf{e}}_{\phi}\frac{\partial}{\partial\phi}, \tag{A16}$$

where $\hat{\mathbf{e}}_{\theta}$ and $\hat{\mathbf{e}}_{\phi}$ are the unit vectors in θ and ϕ direction in the spherical coordinates, we obtain

$$\frac{\hat{\partial}}{\partial\mathbf{R}}\frac{\partial}{\partial\mathbf{p}_{\parallel}}=\frac{1}{\sqrt{2}p\Lambda}\left((ae^{-i\phi}-a^{\dagger}e^{i\phi})\cos\theta\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}(ae^{-i\phi}+a^{\dagger}e^{i\phi})\frac{\partial}{\partial\phi}\right)$$
$$=\frac{1}{\sqrt{2}p\Lambda}\left((be^{i\phi}-b^{\dagger}e^{-i\phi})\cos\theta\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}(be^{i\phi}+b^{\dagger}e^{-i\phi})\frac{\partial}{\partial\phi}\right). \tag{A17}$$

APPENDIX B: FREQUENCY SUMS

The sum of the values of a response function at Matsubara frequencies $i\omega_n=(2n+1)\pi iT$ in the upper half plane can be written as an integral,

$$T\sum_{n=0}^{\infty}K(i\omega_n)=\frac{1}{4\pi i}\int_{-\infty}^{+\infty}\tanh\left(\frac{\omega}{2T}\right)K(\omega). \tag{B1}$$

If the response function varies slowly over the scale $\omega\sim T$, the tangent can be replaced with a step function so that

$$T\sum_{n=0}^{\infty}K(i\omega_n)\approx\frac{1}{4\pi i}[\lim_{\omega\to\infty}F(\omega)+\lim_{\omega\to-\infty}F(\omega)-2F(0)], \tag{B2}$$

where $K(\omega)=[dF(\omega)/d\omega]$.

First consider the sum that appears in the quasiclassical contribution to the longitudinal current

$$S=\sum_{\omega_n}\frac{[g-g(-)]}{i\tilde{\omega}_0}. \tag{B3}$$

Since the frequency $\tilde{\omega}_0$ can be replaced by the bare frequency in the outer frequency range, but is renormalized in the intermediate range, we consider the sum separately in the two regions. In the outside region, transforming the sum in the lower half plane into a sum over the frequencies in the upper half plane,

$$S_{out}=-\frac{2}{\omega_0}\sum_{\omega_n>0}[P(+)-P]=-2i\sum_{\omega_n>0}\left(\frac{\partial P}{\partial\bar{\omega}}+\frac{\bar{\omega}}{2}\frac{\partial^2P}{\partial\omega^2}\right), \tag{B4}$$

after analytic continuation and expansion in $\overline{\omega}$. Using Eq. (B2) we obtain

$$S_{out}=\frac{1}{\pi}\left[(P-1)+\left(\frac{\Lambda\overline{\omega}}{v\sin\theta}\right)\frac{\partial P}{\partial\omega}\right], \tag{B5}$$

where the values of the functions are computed at $\omega=0$. To evaluate the sum in the intermediate frequency range to leading functional order in Δ^2, we write

$$S_{\rm int}=-i\sum_{\omega_n>0}^{\omega_0}\frac{P+P(-)}{i\omega_0+(i/2\tau)[\langle P\rangle+\langle P(-)\rangle]}$$
$$=-2i\sum_{\omega_n>0}^{\omega_0}P\frac{1}{i\omega_0+(i/2\tau)[\langle P\rangle+\langle P(-)\rangle]}. \tag{B6}$$

Adding and subtracting the contribution of a normal metal, so that the remaining sums are convergent at high frequency, we obtain after analytic continuation

$$S_{\rm int}=\frac{i\overline{\omega}\tau}{\pi}\{(P-1)+[1+\langle(1-P)\rangle]\}, \tag{B7}$$

and

$$S=\frac{1}{\pi}\left\{(P-1)+i\overline{\omega}\tau\left[P+\langle(1-P)\rangle-\sqrt{\pi}W''\left(\frac{i\Lambda}{l\sin\theta}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{2\Lambda\Delta}{v\sin\theta}\right)^2\left(\frac{\Lambda}{l\sin\theta}\right)P^3\right]\right\}. \tag{B8}$$

The vertex appearing in the calculation of the dynamical fluctuations of the order parameter is proportional to

$$\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}. \tag{B9}$$

Since the amplitude of the dynamical fluctuations only has to be evaluated to zeroth order in the superconducting order parameter, it is sufficient here to replace the Green's function in the renormalized frequency by its normal-state value. Since this sum is well behaved at high frequency, we easily obtain

$$V=\sum_{\omega_n}\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}$$
$$=-2i\left(\frac{1}{i\tilde{\omega}_0+i/\tau}-\frac{1}{i\tilde{\omega}_0}\right)\sum_{\omega_n>0}[PW'-P(+)W'(+)]. \tag{B10}$$

After analytic continuation we find

$$V=\frac{-2i}{1-i\overline{\omega}\tau}\frac{2\Lambda}{v\sin\theta}\sum_{\omega_n>0}\left[W''(u_n)+\left(\frac{\Lambda\overline{\omega}}{v\sin\theta}\right)W^{(3)}(u_n)\right]$$
$$=\frac{1}{\pi}\frac{1}{1-i\overline{\omega}\tau}\left[W'\left(\frac{i\Lambda}{l\sin\theta}\right)+\frac{\Lambda\overline{\omega}}{v\sin\theta}W''\left(\frac{i\Lambda}{l\sin\theta}\right)\right]. \tag{B11}$$

The sum in the fluctuation propagator in Eq. (109) is easily evaluated in a similar fashion

$$\sum_{\omega_n}\left(ig\,\mathrm{sgn}(\omega_n)W(u_n)-\frac{i}{2}[g+g(-)]\mathrm{sgn}(\omega_n)[W(U_n)\right.$$
$$\left.+\tfrac{1}{2}W''(U_n)]\right)$$
$$=\sum_{\omega_n>0}\{2PW-[P+P(+)]$$
$$\times[W(U_n^+)+\tfrac{1}{2}W''(U_n^+)]\}+\frac{1}{2}\sum_{\omega_n>0}^{\omega_0}[P-P(-)][W(U_n)$$
$$+\tfrac{1}{2}W''(U_n)]\mathrm{sgn}(\Omega_n), \tag{B12}$$

where $2U_n^+=u_n+u_n^+$. Since[39]

$$W(U_n)+\tfrac{1}{2}W''(U_n)=-U_nW'(U_n), \tag{B13}$$

the contribution of the last term to the final results is at least of order $\overline{\omega}^3$. The remaining terms give, after expansion in the external frequency,

$$T\sum_{\omega_n}\left(ig\,\mathrm{sgn}(\omega_n)W(u_n)-\frac{i}{2}[g+g(-)]\right.$$
$$\left.\times\mathrm{sgn}(\Omega_n[W(U_n)+\tfrac{1}{2}W''(U_n)])\right)$$
$$=-T\sum_{\omega_n>0}\left[W''(u_n)+\left(\frac{2\Lambda\overline{\omega}}{v\sin\theta}\right)[W'(u_n)+\tfrac{1}{2}W^{(3)}(u_n)]\right]$$
$$=\frac{1}{2\pi i}\frac{v\sin\theta}{2\Lambda}\left\{W'\left(\frac{i\Lambda}{l\sin\theta}\right)+\left(\frac{2\Lambda\overline{\omega}}{v\sin\theta}\right)\right.$$
$$\left.\times\left[W\left(\frac{i\Lambda}{l\sin\theta}\right)+\frac{1}{2}W''\left(\frac{i\Lambda}{l\sin\theta}\right)\right]\right\}. \tag{B14}$$

APPENDIX C: FLUCTUATIONS OF THE ORDER PARAMETER

Our starting point here is Eq. (102) for the linear correction to the anomalous propagator

$$f_1=\frac{e\mathbf{vA}[f-f(-)]}{i\tilde{\omega}_0}+i[g+g(-)]$$
$$\times[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\Delta_1+i(2\tau)^{-1}[g+g(-)]$$
$$\times[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\langle f_1\rangle+i(2\tau)^{-1}$$
$$\times\frac{e\mathbf{vA}[g-g(-)]}{i\tilde{\omega}_0}[2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}$$
$$\times[\langle f\rangle+\langle f(-)\rangle]. \tag{C1}$$

If we define the angular average of f_1 by

$$\langle f\rangle=S(\omega_n)|1\rangle, \tag{C2}$$

we find, after carrying out the angular integration and ignoring terms of order Λ/l

$$S=\sqrt{\pi}\frac{\Lambda}{v}\int_0^{\pi}\frac{d\theta}{4}\Bigg[2ievA\sin\theta\left(-\frac{i}{\sqrt{2}}\right)\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}+i[g+g(-)]2C\,\mathrm{sgn}(\Omega_n)[W(U_n)+\tfrac{1}{2}W''(U_n)]\Bigg]. \tag{C3}$$

We now use Eq. (103) to determine the amplitude of the excited mode of the order parameter $C=\pi gN(0)\Sigma_n S$

$$C\left(1-\pi gN(0)\sqrt{\pi}\frac{\Lambda}{v}\sum_n\int_0^{\pi}d\theta\frac{i}{2}[g+g(-)]\mathrm{sgn}(\Omega_n)\times[W(U_n)+\tfrac{1}{2}W''(U_n)]\right)=\pi gN(0)\sqrt{\pi}\frac{\Lambda}{v}\sum_n\int_0^{\pi}\frac{\sqrt{2}}{4}evA\sin\theta d\theta\times\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}. \tag{C4}$$

We can now use the gap equation to eliminate the need for a frequency cutoff and obtain

$$C\sum_n\int_0^{\pi}d\theta\left(ig\,\mathrm{sgn}(\omega_n)W(u_n)-\frac{i}{2}[g+g(-)]\mathrm{sgn}(\Omega_n)\times[W(U_n)+\tfrac{1}{2}W''(U_n)]\right)=\sum_n\int_0^{\pi}\frac{\sqrt{2}}{4}evA\sin\theta d\theta\frac{gW'-g(-)W'(-)}{i\tilde{\omega}_0}. \tag{C5}$$

It follows that to leading order

$$C=\frac{ie\Lambda A\sqrt{2}}{1-i\overline{\omega\tau}}. \tag{C6}$$

We also give the expression for the distribution function f_1. Neglecting the contributions of order (Λ/l), we use Eq. (A11) to compute

$$\begin{aligned}f_1&=e\mathbf{v}\mathbf{A}\frac{f-f(-)}{i\tilde{\omega}_0}+i[g+g(-)][2\tilde{\Omega}_n+\mathbf{v}(\nabla-2ie\mathcal{A})]^{-1}\Delta_1\\&=\frac{2i}{i\tilde{\omega}_0}\sqrt{\pi}e\Lambda A\cos\phi\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}e^{im\phi}\left(-\frac{i}{\sqrt{2}}\right)^m\\&\quad\times[gW^{(m)}\mathrm{sgn}^{m+1}(\omega_n)-g(-)W^{(m)}(-)\\&\quad\times\mathrm{sgn}^{m+1}(\omega_n-)]|m\rangle+i[g+g(-)]C\frac{\sqrt{\pi}\Lambda}{v\sin\theta}\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\\&\quad\times\left(-\frac{i}{\sqrt{2}}\right)^m\mathrm{sgn}^{m+1}(\Omega_n)(\sqrt{m+1}e^{im\phi}W^{(m)}(U_n)|m+1\rangle\\&\quad+\frac{i}{\sqrt{2}}\mathrm{sgn}(\Omega_n)e^{i(m-1)\phi}W^{(m+1)}(U_n)|m\rangle).\end{aligned} \tag{C7}$$

The evaluation of $\overline{C}$ is analogous to the calculation given above.

In the transverse response calculation our starting point here is Eq. (123) for the linear, in cyclotron frequency, correction to the anomalous propagator

$$\delta f=[2\tilde{\Omega}_n+\mathbf{v}_f(\nabla-2ie\mathcal{A})]^{-1}\left(-\frac{e}{2m}\mathbf{A}(\nabla-2ie\mathcal{A})\times[f-f(-)]+i\Delta\,\delta g_e+i\,\delta\Delta_1[g+g(-)]\right). \tag{C8}$$

The solution of this equation follows exactly the steps described in the previous section. First we solve for the coefficient δC in $\delta\Delta_1=\delta C|1\rangle$. As $\delta C=\Sigma_n\int d^2s\,\delta f$, the denominator of the expression for δC is the propagator for the first excited mode of the order parameter, as it was for C. To evaluate the contribution of each of the driving terms we notice that for our choice of **A**,

$$\mathbf{A}(\nabla-2ie\mathcal{A})=\mathbf{A}\nabla_x=A\frac{1}{\sqrt{2}\Lambda}[a-a^{\dagger}]. \tag{C9}$$

Explicit evaluation of this term using the expression for the function f from Eq. (81) shows that it contributes at order Λ/l compared to leading-order terms. The driving term $i\Delta\,\delta g_e$ only contributes in the intermediate frequency range since

$$\delta g_e=-2i\omega_c evA\sin\theta\sin\phi\frac{g-g(-)}{(i\tilde{\omega}_0)^2}, \tag{C10}$$

and, to the order in which we work, $g-g(-)$ vanishes in the outer range. Then, in analogy to the solution outlined above, we obtain

$$\delta C_1=\delta\overline{C}_1=ie\Lambda A\sqrt{2}(\overline{\omega\tau)}(\omega_c\tau), \tag{C11}$$

and, for the angular average of the function δf needed to calculate the Thompson contribution to the conductivity

$$\langle\delta f_1\rangle=\sqrt{2\pi}e\Lambda A\frac{\omega_c}{(i\tilde{\omega}_0)^2}\int_0^{\pi}\frac{d\theta}{4}(\sin\theta[g-g(-)]W'(U_n)+2i\,\delta C[g+g(-)]\mathrm{sgn}(\Omega_n)[W(U_n)+\tfrac{1}{2}W''(U_n)])|1\rangle \tag{C12}$$

$$\langle\delta f_1^{\dagger}\rangle=\sqrt{2\pi}e\Lambda A\frac{\omega_c}{(i\tilde{\omega}_0)^2}\int_0^{\pi}\frac{d\theta}{4}(\sin\theta[g-g(-)]W'(U_n)+2i\,\delta\overline{C}[g+g(-)]\mathrm{sgn}(\Omega_n)[W(U_n)+\tfrac{1}{2}W''(U_n)])\langle 1|. \tag{C13}$$

In the intermediate region $g+g(-)=0$, and only the first term in each function contributes to the conductivity to leading order in (Λ/l).

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