Lower bound for the Fermi-level density of states of a disordered *d*-wave superconductor in two dimensions

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We consider a disordered *d*-wave superconductor in two dimensions. Recently, we have shown in an exact calculation that for a lattice model with a Lorentzian distributed random chemical potential the quasiparticle density of states at the Fermi level is nonzero. As the exact result holds only for the special choice of the Lorentzian, we employ different methods to show that for a large class of distributions, including the Gaussian distribution, one can establish a nonzero lower bound for the Fermi-level density of states. The fact that the tails of the distributions are unimportant in deriving the lower bound shows that the exact result obtained before is generic. [S0163-1829(98)01617-8]

I. INTRODUCTION

Considerable evidence for *d*-wave superconductivity in the high-temperature cuprate superconductors has led to interest in studying the effect of disorder on *d*-wave paired systems. Unlike *s*-wave superconductors (SC's), where Anderson's theorem[1] predicts negligible effect of nonmagnetic impurities on thermodynamic properties, simple defects are expected to be pairbreaking in superconductors with gap nodes, and are in fact generally thought to induce finite density of quasiparticle states $N(0)$ at the Fermi level. As in disordered normal metals, one might expect properties of such systems to depend strongly on dimensionality. In fact, Nersesyan *et al.* (NTW) have shown[2] that the usual *t*-matrix approximation for impurity scattering, which is exact in the dilute limit in three dimensions (3D), breaks down for a strictly 2D *d*-wave superconductor. By mapping the problem onto a continuum model of Dirac fermions in a random gauge field, subsequently solved by bosonization methods, NTW claimed that the density of states of such systems must go to zero at the Fermi level as a power law, $N(E) \propto E^{\alpha}$. Later it was realized that for a realistic *d*-wave SC with four nodes on the Fermi surface their result might not be applicable.[3] Although the real materials in question are quasi-2D, it is of considerable importance to establish the effect of disorder in the strictly 2D case because the existence of a 2D-3D crossover at low energies could invalidate the standard picture of low-temperature thermodynamics in a *d*-wave superconductor developed under the assumption of a finite residual density of states $N(0)$.

Recently,[4] we have shown that for a lattice model of a disordered *d*-wave superconductor in two spatial dimensions, one can obtain an exact result for the density of states (DOS) $N(E)$, provided that the disorder is modeled by a Lorentzian distribution of the chemical potential. The result was a finite DOS at the Fermi level $N(0)/N_0 \propto \gamma \ln 4\Delta_0/\gamma$ with N_0 the normal DOS at the Fermi level, Δ_0 the maximum value of the superconducting order parameter over a circular Fermi surface, and γ the width of the Lorentzian distribution. We also quoted rigorous lower bounds for $N(0)$ for a large class of disorder distributions which we obtained using methods developed in a different context. These results and the underlying methods were questioned in a Comment by Nersesyan and Tsvelik.[6] They claimed that our result for a Lorentzian distribution, while simple to obtain and exact for all energies, is nongeneric. Their claim was based on the observation that an expansion of the resulting DOS for small disorder strength is inconsistent with a straightforward perturbative calculation of the DOS using standard diagram techniques with Gaussian disorder. In a Reply[7] we pointed out that one cannot expect such a comparison to make sense since the perturbation series based on a Lorentzian distribution cannot be defined due to the divergence of all moments. In the light of this controversy, our proof of lower positive bounds for the DOS in the case of more general disorder distributions acquires a special importance.

In this paper we therefore present in some detail the derivation of the nonzero lower bound for the DOS at the Fermi level which in a different context was first given in Ref. 5. We stress that since our results are lower bounds, no arguments about the dependence of the DOS on disorder strength can be made. It suffices for our purposes to show that a lower bound exists, and that its existence does not depend on the specifics of the tails of the distribution, i.e., power-law decay, exponential decay or compact support of the distribution will all give a nonzero lower bound for the DOS.

The paper is organized as follows. First, we formulate the problem and give a general outline of the proof. Second, we show preliminary calculations which will be used in the proof. We then derive the nonzero lower bound for a certain class of Hamiltonians. Finally, we show that the Hamiltonian of interest belongs to this class. We conclude with final remarks about cases where the method fails to give a nonzero lower bound (e.g., *s*-wave superconductors).

II. FORMULATION OF THE PROBLEM AND OUTLINE OF THE PROOF

The problem is defined[4] by the Bogliubov–de Gennes Hamiltonian

$$H=-(\nabla^2+\mu)\sigma_3+\hat{\Delta}^d\sigma_1, \tag{1}$$

μ being the random chemical potential, distributed according to a symmetric distribution function $P(\mu_r)$ (σ_i are the Pauli matrices, σ_0 is the 2×2 identity matrix). The kinetic energy operator $-\nabla^2$ is taken to act as $\nabla^2\Psi(r)=\Psi(r+2e_1)+\Psi(r-2e_1)+\Psi(r+2e_2)+\Psi(r-2e_2)$ on a function $\Psi(r)$ of the sites r of a 2D square lattice Λ spanned by the unit vectors e_1 and e_2. The diagonal term of $-\nabla^2$ can be absorbed in the nonrandom part of the chemical potential μ_0. For the particle-hole symmetric pure system we consider we have $\mu_0=0$. Note that this function involves displacements of two lattice sites rather than one, as would be the case in the simplest tight-binding representation of the lattice kinetic energy. For a system of fermions in the thermodynamic limit, the bare kinetic energy will then have a band representation quite similar to the usual tight-binding form, with no particular distinguishing features near the Fermi level. In the above definition of the kinetic energy, we have taken the hopping matrix element as our unit of energy. The bilocal lattice operator $\hat{\Delta}^d\equiv\Delta_{r,r'}$ is taken be $\hat{\Delta}^d\Psi(r)=\Delta[\Psi(r+e_1)+\Psi(r-e_1)\pm\Psi(r+e_2)\pm\Psi(r-e_2)]$. The Matsubara Green function $G(iE)=(iE\sigma_0-H)^{-1}$ determines the density of states in the usual way, namely, by

$$N(E)=\frac{-1}{2\pi}\,\mathrm{Im}\,\mathrm{Tr}_2\langle G_{r,r}(iE\to E+i\epsilon)\rangle \tag{2}$$

where the trace Tr_2 refers to the 2×2 structure of the Hamiltonian, corresponding to quasiparticles and quasiholes of the superconductor. $\langle\cdots\rangle$ denotes the disorder average, which consists of integration over the disorder variable (the chemical potential) at every site of the lattice, with a measure given by $P(\mu_r)d\mu_r$.

To derive a lower bound of the average DOS we first write the Green function as

$$G(iE)=\frac{i(iE\sigma_0+H)}{2E}[(\tilde{H}-iE\sigma_0)^{-1}-(\tilde{H}+iE\sigma_0)^{-1}], \tag{3}$$

where the new Hamiltonian $\tilde{H}=HD\sigma_3=-(\nabla^2+\mu)D\sigma_0-i\Delta D\sigma_2$ has been introduced for formal reasons (cf. Ref. 4 and below). The matrix D is diagonal with matrix elements $D_{r,r'}=(-1)^{r_1+r_2}\delta_{r,r'}$ (r_1,r_2 are the two components of the 2D r vector). Equation (3) holds for any distribution of randomness in the Hamiltonian, before averaging. We are interested in the DOS at the Fermi level, i.e., at zero energy. This means that after the analytic continuation $iE\to E+i\epsilon$ we will set $E=0$. Consequently, $iE\to i\epsilon$ (with positive ϵ), and the local DOS at the Fermi level at lattice site r reads

$$N_r(0)=\frac{-1}{2\pi}\,\mathrm{Tr}_2\,\mathrm{Im}\,G_{rr}(i\epsilon)=\frac{-i}{4\pi}\,\mathrm{Tr}_2[(\tilde{H}-i\epsilon\sigma_0)^{-1}_{rr}-(\tilde{H}+i\epsilon\sigma_0)^{-1}_{rr}]=\frac{\epsilon}{2\pi}\,\mathrm{Tr}_2(\tilde{H}^2+\epsilon^2\sigma_0)^{-1}_{rr}. \tag{4}$$

From the analytic properties of G it follows that N_r is nonnegative (either positive or zero).

The average DOS $(1/|\Lambda|)\langle\Sigma_{r\in\Lambda}N_r\rangle$ can be estimated from below using the method worked out in Ref. 5. The central idea of the proof is to divide the lattice Λ into finite lattice blocks $\{S_j\}$. We then evaluate the average DOS on these lattice blocks and also the contribution from the interactions of the lattice blocks. On the lattice blocks a ''coarse graining'' method will be used by relating the disorder integration over all other sites to one at the ''center'' of S. On this ''center'' site the range of integration of the random variable $z_r=\mu_r D$ will be restrained to a finite interval $[-a,a]$. The cutoff a eliminates the contribution of the tails of the distribution. Since N_r is nonnegative, the tail contribution can only add to the result obtained by integrating over $[-a,a]$. Thus, *if* we are able to find a *nonzero* average DOS by integrating only over the finite interval $[-a,a]$ we have obtained a nonzero DOS without relying on tail contributions. This explains why distributions with power-law tails (e.g., Lorentzian distribution) lead qualitatively to the same results as, for example, the Gaussian distribution or distributions with exponential decay.

The proof rests on an identity [Eq. (14) in the next section] that is intimately connected with the fact that the local DOS N_r (before averaging) is nonnegative. It also relies on the infimum of the disorder distribution in the restricted range $[-a,a]$ being finite. This puts some limits on the applicability of the proof to compact distributions, but it always holds for unbounded distributions such as the Gaussian or the Lorentzian. The result can be summarized by the following statement: For any finite subregion S of the lattice Λ with boundary ∂S, defined by the lattice sites of $\Lambda\setminus S$ which are connected to S by the matrix elements of $\tilde{H}$, there exists a distribution dependent positive constant P_S, related to a restricted disorder distribution on S, with

$$\frac{1}{|\Lambda|}\left\langle\sum_{r\in\Lambda}N_r\right\rangle\geq P_S(1-|\partial S|/|S|). \tag{5}$$

Since the block size $|S|$ grows faster than the size of the boundary $|\partial S|$ the right-hand side is positive above a certain block size.

It should be noted that the method of this paper will not give a *nonzero* lower bound for the DOS for *every* Hamiltonian. We will determine the conditions for the lower bound to be nonzero and show that the Hamiltonian of interest (d-wave SC) fulfills these conditions. We will also show that for an isotropic s-wave SC with a local order parameter the method will only yield a (trivial) vanishing bound for the Fermi level DOS.

III. PRELIMINARY CONSIDERATIONS

As a first step we evaluate the integral $\int_{-\infty}^{\infty} N_r dz_r$. For this purpose the identity

$$I_S(\tilde{H}+I_S A I_S)^{-1} I_S = [(\tilde{H}^{-1})_S^{-1} + I_S A I_S]_S^{-1} \tag{6}$$

is useful. I_S is the projector onto the region S, and $(\cdots)_S^{-1} = I_S(I_S \cdots I_S)^{-1} I_S$ is the inverse on the region S. The proof of this identity is given in Appendix A. Choosing $S=\{r\}$, i.e., just a single lattice site, we note that the 2×2 matrix $[(\tilde{H}+z_r\sigma_0+i\epsilon\sigma_0)^{-1}]_{\{r\}}^{-1}$ is diagonal. Furthermore, it is proportional to the unit matrix σ_0 as a consequence of the definition of $\tilde{H}$, which involves only σ_0 and σ_2. (Terms of the inverse that are proportional to σ_2 are nonlocal and, consequently, projected out by $I_{\{r\}}$.) Therefore, we can write

$$[(\tilde{H}+z_r\sigma_0+i\epsilon\sigma_0)^{-1}]_{\{r\}}^{-1} = \begin{pmatrix} X_r+iY_r & 0 \\ 0 & X_r+iY_r \end{pmatrix}, \tag{7}$$

where $Y_r \propto \epsilon > 0$. For the special choice $I_{\{r\}} A I_{\{r\}} = -z_r\sigma_0$ in Eq. (6) we obtain

$$(\tilde{H}+i\epsilon\sigma_0)_{rr}^{-1} = \{[(\tilde{H}+z_r\sigma_0+i\epsilon\sigma_0)^{-1}]_{\{r\}}^{-1} - z_r\sigma_0\}_{rr}^{-1}. \tag{8}$$

This gives for the local DOS of Eq. (4)

$$\frac{1}{\pi} Y_r[(X_r-z_r)^2+Y_r^2]^{-1}. \tag{9}$$

The integration over z_r leads to

$$\int_{-\infty}^{\infty} N_r dz_r = 1. \tag{10}$$

This result will be used below. It is nontrivial as, for example, for an s-wave SC with a *local* order parameter term $\Delta\sigma_1$ the equivalent of Eq. (7) would have also off-diagonal entries. In fact, we have shown in Ref. 4 that the corresponding expression of the local DOS of the s-wave superconductor reads

$$N_r(0) = -\frac{1}{\pi}\frac{i\epsilon}{2\sqrt{\Delta^2+\epsilon^2}}[(-\nabla^2-\mu-i\sqrt{\Delta^2+\epsilon^2})_{rr}^{-1} - (-\nabla^2-\mu+i\sqrt{\Delta^2+\epsilon^2})_{rr}^{-1}], \tag{11}$$

i.e., it is proportional to $\epsilon/\sqrt{\Delta^2+\epsilon^2}$. This implies a vanishing DOS in the limit $\epsilon\to 0$ as long as the superconducting order parameter is nonzero. Of course, that is what is expected for a SC with a nonvanishing gap everywhere on the Fermi surface (Anderson's theorem).[1]

The local DOS Eq. (4) can be written in a differential form as

$$N_r = \frac{i}{4\pi}\frac{\partial}{\partial z_r}[\ln\det(\tilde{H}-i\epsilon\sigma_0) - \ln\det(\tilde{H}+i\epsilon\sigma_0)] = \frac{i}{4\pi}\frac{\partial}{\partial z_r}\{\ln\det[\mathbf{1}-2i\epsilon(\tilde{H}+i\epsilon\sigma_0)^{-1}]\}, \tag{12}$$

which follows from the fact that the differentiation with respect to z_r picks the r,r component of $\tilde{H}\pm i\epsilon\sigma_0$. We define a matrix A as

$$A := -2i\epsilon(\tilde{H}+i\epsilon\sigma_0)^{-1}. \tag{13}$$

Because the DOS is non-negative the function $i\ln\det(\mathbf{1}+A)$ is a nondecreasing function of z_r. As a consequence of the Eq. (10) we have the following integral:

$$\int_{-\infty}^{\infty} N_r dz_r = \frac{i}{4\pi}\ln\det(\mathbf{1}+A)\Big|_{z_r=-\infty}^{z_r=\infty} = 1. \tag{14}$$

IV. LOWER BOUND OF THE AVERAGE DOS

A. Coarse graining and elimination of the distribution tails

Now we consider the average local DOS summed over all lattice sites on a finite lattice block S, and restrict the range of integration over z_r to a finite region in order to eliminate the tail contributions. If we can find a nonzero lower bound for the DOS in this way, we have established it independently of the specific decay (e.g., power law or exponential) at large values of the disorder variable. First we choose a site $r_0 \in S$ for which we restrict the z_r integration to the interval $[-a,a]$. For the remaining integrations on S we define

$$z_r = z_{r_0} + \delta z_r \tag{15}$$

with $\delta z_r \in [0,\delta]$. The above choice of the range of integration on S is sufficient but not necessary, i.e., different choices can be made as long as the range of integration is finite and certain conditions discussed below are satisfied. In the case considered a must be chosen large enough to include all singularities of the Green function. This is the case if it satisfies the inequality

$$0 < a-\delta-4(1+\Delta) \tag{16}$$

as we will see below. Using the notation $\langle\cdots\rangle_S'$ for this restricted averaging on S we have

$$\langle N_r\rangle \geqslant \langle N_r\rangle_S' \tag{17}$$

because N_r is nonnegative. Then we can write with Eq. (12)

$$\left\langle \sum_{r\in S} N_r \right\rangle_S' = \frac{i}{4\pi}\left\langle \sum_{r\in S}\frac{\partial}{\partial z_r}\ln\det(\mathbf{1}+A)\right\rangle_S' = \frac{i}{4\pi}\left\langle \int_{-a}^{a} dz_{r_0} P(z_{r_0}) \times\left(\prod_{r\in S, r\neq r_0}\int_{z_{r_0}}^{z_{r_0}+\delta} dz_r P(z_r)\right) \times\frac{\partial}{\partial z_{r_0}}\ln\det(\mathbf{1}+A)\right\rangle_{\Lambda\setminus S}, \tag{18}$$

where $\langle\cdots\rangle_{\Lambda\setminus S}$ refers to the (unrestricted, i.e., $z_r\in[-\infty,\infty]$) averaging over z_r on all lattice sites on Λ except the ones on S.

We now bound the right-hand side (RHS) of Eq. (18) from below by pulling out certain infima of the distribution. As a first step, we pull out the infimum of the distribution on z_{r_0}. This leads to

$$\geqslant \inf_{z_{r_0}\in[-a,a]} P(z_{r_0}) \frac{i}{4\pi} \left\langle \int_{-a}^{a} dz_{r_0} \left(\prod_{r\in S, r\neq r_0} \int_{z_{r_0}}^{z_{r_0}+\delta} dz_r P(z_r) \right) \frac{\partial}{\partial z_{r_0}} \ln \det(\mathbf{1}+A) \right\rangle_{\Lambda\setminus S}. \tag{19}$$

Furthermore, by pulling out the infimum of the integrand of $\Pi_{r\in S, r\neq r_0} \int_{z_{r_0}}^{z_{r_0}+\delta} dz_r P(z_r)$ we obtain

$$\geqslant \inf_{z_{r_0}\in[-a,a]} P(z_{r_0}) \frac{i}{4\pi} \left\langle \int_{-a}^{a} dz_{r_0} \inf_{z_r\in[z_0,z_0+\delta]} \frac{\partial}{\partial z_{r_0}} \ln \det(\mathbf{1}+A) \left[\prod_{r\in S, r\neq r_0} \int_{z_{r_0}}^{z_{r_0}+\delta} P(z_r) dz_r \right] \right\rangle_{\Lambda\setminus S}. \tag{20}$$

Pulling out the integral over z_r $(r_0\neq r\in S)$ eventually yields

$$\geqslant \inf_{z_{r_0}\in[-a,a]} P(z_{r_0}) \inf_{z_{r_0}\in[-a,a]} \left(\int_{z_{r_0}}^{z_{r_0}+\delta} P(z_r) dz_r \right)^{|S|-1} \frac{i}{4\pi} \left\langle \inf_{z_r\in[z_0,z_0+\delta]} \int_{-a}^{a} dz_{r_0} \frac{\partial}{\partial z_{r_0}} \ln \det(\mathbf{1}+A) \right\rangle_{\Lambda\setminus S}. \tag{21}$$

The disorder distribution on the block S is now taken care of by the coefficient

$$P_S := \inf_{z_{r_0}\in[-a,a]} P(z_{r_0}) \times \inf_{z_{r_0}\in[-a,a]} \left(\int_{z_{r_0}}^{z_{r_0}+\delta} P(z_r) dz_r \right)^{|S|-1} \tag{22}$$

which multiplies the remaining disorder average over $\Lambda\setminus S$. P_S is nonzero as long as the disorder distribution $P(z_r)$ is not vanishing in the restricted range of integration. This is certainly true for unbounded distributions, such as a Gaussian. However, compact distributions with a narrow range of disorder will fail to provide a nonzero lower bound. This will be discussed in more detail below.

B. General lower bound for the DOS

Combining Eqs. (18), (21), and (22) we obtain

$$\left\langle \sum_{r\in S} N_r \right\rangle_S' \geqslant \frac{i}{4\pi} P_S \left\langle \inf_{z_r\in[z_0,z_0+\delta]} [\ln \det(\mathbf{1}+A)|_{z_{r_0}=a} - \ln \det(\mathbf{1}+A)|_{z_{r_0}=-a}] \right\rangle_{\Lambda\setminus S}. \tag{23}$$

In the next step we isolate the lattice block S from the rest of the lattice Λ by sending z_r to $\pm\infty$ on the boundary ∂S of S. (Particles trying to occupy sites on the boundary will either be trapped or repelled by an infinitely strong barrier.) The boundary ∂S is defined by all sites of Λ which are not in S but connected with S by the matrix $\tilde{H}$, i.e., all sites $r\notin S$ with $|r-r'|=1,2$ for any $r'\in S$. (Observe that due to the definition of the Laplacian the ''boundary'' is actually two layers around the block S.) With the above definition of the boundary ∂S the matrix $(1-I_{\partial S})\tilde{H}(1-I_{\partial S})$ separates into one block matrix on S and another one on $\Lambda\setminus S\cup\partial S$

$$(1-I_{\partial S})\tilde{H}(1-I_{\partial S}) = I_S\tilde{H}I_S + I_{\Lambda\setminus S\cup\partial S}\tilde{H}I_{\Lambda\setminus S\cup\partial S}. \tag{24}$$

Applying the identities of Appendix A, it follows that the inverse of $(1-I_{\partial S})\tilde{H}(1-I_{\partial S})$ separates into two block matrices. Consequently, $\lim_{z_r\to-\infty(r\in\partial S)} A = A_{\Lambda\setminus\partial S} := -2i\epsilon(\tilde{H}+i\epsilon\sigma_0)^{-1}_{\Lambda\setminus\partial S}$ also separates into two block matrices

$$A_{\Lambda\setminus\partial S} = A_S + A_{\Lambda\setminus S\cup\partial S}. \tag{25}$$

Performing the limiting process for the lower bound of the DOS we can use the fact that $i\ \ln\det(\mathbf{1}+A)$ is a nondecreasing function of z_r

$$\frac{\partial}{\partial z_r} \frac{i}{4\pi} \ln \det(\mathbf{1}+A) = N_r \geqslant 0. \tag{26}$$

This implies a lower bound for the RHS of Eq. (23) if we decrease the first term in Eq. (23) by taking $z_r\to-\infty$ and increase the second term by taking $z_r\to\infty$ (on the boundary of S). The result of this procedure is the lower bound

$$\frac{i}{4\pi} P_S \left\langle \inf_{z_r\in[z_0,z_0+\delta]} \left[\lim_{z_r\to-\infty(r\in\partial S)} \ln \det(\mathbf{1}+A)|_{z_{r_0}=a} - \lim_{z_r\to\infty(r\in\partial S)} \ln \det(\mathbf{1}+A)|_{z_{r_0}=-a} \right] \right\rangle_{\Lambda\setminus(S\cup\partial S)}. \tag{27}$$

Next, we rewrite the second (negative) term by applying successively Eq. (14) for all $r\in\partial S$. This yields

$$\lim_{z_r\to\infty(r\in\partial S)} i\ \ln\det(\mathbf{1}+A) = 4\pi|\partial S| + \lim_{z_r\to-\infty(r\in\partial S)} i\ \ln\det(\mathbf{1}+A). \tag{28}$$

We therefore have for the expression (27)

$$\frac{i}{4\pi}P_S\Big\langle \inf_{z_r\in[z_0,z_0+\delta]}\Big[\lim_{z_r\to-\infty(r\in\partial S)}\ln\det(\mathbf{1}+A)\big|_{z_{r_0}=a} - \lim_{z_r\to-\infty(r\in\partial S)}\ln\det(\mathbf{1}+A)\big|_{z_{r_0}=-a}\Big]\Big\rangle_{\Lambda\setminus(S\cup\partial S)} - P_S|\partial S|. \tag{29}$$

There is no contribution from the matrix A on $\Lambda\setminus S\cup\partial S$, since this matrix part does not depend on $z_{r_0}=\pm a$. Consequently, the difference of these contributions gives zero, and we find a lower bound of the form

$$\left\langle\sum_{r\in S}N_r\right\rangle_S' \geqslant \frac{i}{4\pi}P_S\inf_{z_r\in[z_0,z_0+\delta]}\left[\ln\det(\mathbf{1}+A_S)\big|_{z_{r_0}=a} - \ln\det(\mathbf{1}+A_S)\big|_{z_{r_0}=-a}\right] - P_S|\partial S|. \tag{30}$$

The right-hand side of Eq. (30) is a difference between a contribution from the block S (the logarithmic terms) and a boundary contribution (the $|\partial S|$ term). If the contribution of the block grows with its volume $|S|$ we find for sufficiently large lattice blocks a positive lower bound for the RHS of Eq. (30). We show below that this is indeed the case for the considered model of a d-wave SC.

C. Lower bound for a 2D d-wave superconductor

The growth of the block contribution with the volume $|S|$ follows from the range of the disorder integration on S [Eqs. (15) and (16)]. To see this we define

$$H' = I_S\widetilde{H}I_S + z_{r_0}\sigma_0 I_S\,. \tag{31}$$

$I_S\widetilde{H}I_S$ and H' can be diagonalized by unitary transformations. An eigenvalue λ_j of $I_S\widetilde{H}I_S$ satisfies $-z_{r_0}+\min\lambda_j'\leqslant\lambda_j\leqslant -z_{r_0}+\max\lambda_j'$. This implies for the terms in Eq. (30), where $z_{r_0}=\pm a$

$$\mp a+\min\lambda_j'\leqslant\lambda_j\leqslant\mp a+\max\lambda_j'\,. \tag{32}$$

An upper bound of $\lambda_j'^2$ can be derived from the eigenvalues of $I_S(H+z_{r_0}D\sigma_3)I_S$ (see Appendix B). It yields $|\lambda_j'|\leqslant 4(1+\Delta)+\delta$, since the deterministic part of the Hamiltonian $-\nabla^2\sigma_3+\hat{\Delta}^d\sigma_1$ has an upper bound $4(1+\Delta)$, and the random part comes from δz_r $(0\leqslant\delta z_r\leqslant\delta)$. Thus we obtain

$$-a-4(1+\Delta)\leqslant\lambda_j\leqslant -a+4(1+\Delta)+\delta \quad (z_{r_0}=a), \tag{33}$$

$$a-4(1+\Delta)\leqslant\lambda_j\leqslant a+4(1+\Delta)+\delta \quad (z_{r_0}=-a). \tag{34}$$

The condition for a in Eq. (16) guarantees that for $z_{r_0}=a$ $(z_{r_0}=-a)$ all eigenvalues λ_j are negative (positive). Consequently, the argument of the logarithm for any eigenvalue λ_j, $1-2i\epsilon/(\lambda_j+i\epsilon)$, is $1+i\epsilon$ $(1-i\epsilon)$ for the first (second) term in Eq. (30). In order to deal with the branch cut of the complex logarithm we let $-\epsilon\to 2\pi-\epsilon$ for the second term in Eq. (30). Now we can safely let $\epsilon\to 0$ in both terms and obtain for

$$i\,\ln\det(\mathbf{1}+A_S)\big|_{z_{r_0}=a} - i\,\ln\det(\mathbf{1}+A_S)\big|_{z_{r_0}=-a} \tag{35}$$

a contribution of 2π for each of the $2|S|$ eigenvalues λ_j, i.e., a total of $4\pi|S|$. From Eqs. (17) and (30) it therefore follows that the DOS is given by

$$\left\langle\sum_{r\in S}N_r\right\rangle \geqslant P_S(|S|-|\partial S|). \tag{36}$$

The average DOS is the sum of the local average DOS, normalized by the lattice size $|\Lambda|$. Dividing the lattice Λ into identical blocks S we sum over all blocks and obtain after normalization

$$\frac{1}{|\Lambda|}\left\langle\sum_{r\in\Lambda}N_r\right\rangle \geqslant P_S(1-|\partial S|/|S|). \tag{37}$$

Since the lattice block size $|S|$ grows faster than the size of its boundary $|\partial S|$, there is a finite size which gives a positive bound on the RHS and therefore a positive lower bound on the DOS.

Equation (37) holds for our lattice model of a d-wave SC, given by the Hamiltonian Eq. (1) for all unbounded and symmetric disorder distribution that vanish at large disorder parameters z_r. In particular, the lower bound holds for both power law (e.g., Lorentzian) and exponential (e.g., Gaussian) distributions. It also holds for compact distributions of sufficient width, with the width being determined by the requirement that the factor P_S must be nonzero when a is chosen according to the condition Eq. (16) in order to let the DOS on S grow with $|S|$. This does not imply that narrow compact distributions will have a vanishing DOS at the Fermi level. However, to show the finiteness of the DOS for such distributions a more sophisticated method is required.

V. CONCLUSIONS

In conclusion, we have shown that for rather generic conditions a nonzero lower bound for the Fermi-level density of quasiparticle states exists. The bound does not depend on the specifics of the ''tails'' of the distribution as both Lorentzian and Gaussian distributions yield a nonzero lower bound. This proves that our exact result for the case of Lorentzian disorder[4] is generic.

This result applies to a class of Hamiltonians describing 2D superconductors with *nonlocal* order parameters, like extended s-wave, p-wave, and d-wave SC's. In contrast, for a *local* isotropic s-wave SC our method will yield a vanishing lower bound, in complete agreement with Anderson's theorem for nonmagnetic disorder in SC's with a finite order parameter everywhere on the Fermi surface. It should be noted that our results imply that the selfconsistent t-matrix approximation[8] gives qualitatively correct physics as long as only the DOS at the Fermi level is concerned (i.e., for thermodynamic properties). Whether this also holds for the dynamic (transport) properties is an interesting question to be resolved.

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APPENDIX A: PROJECTIONS OF THE GREEN FUNCTION

Consider a general square matrix H defined on a lattice Λ. R is a subset of Λ, and I_R is the projector on the region R which can be written as a diagonal matrix

$$I_{R,q,q'}=I_{R,q}\delta_{qq'} \quad \text{with} \quad I_{R,q}=\begin{cases}1 & \text{if } q\in R,\\ 0 & \text{otherwise.}\end{cases} \tag{A1}$$

If the inverse of H and $H+I_RCI_R$ exist then we find the identity

$$(H+I_RCI_R)^{-1}=H^{-1}-H^{-1}(\mathbf{1}+I_RCI_RH^{-1})_R^{-1}I_RCI_RH^{-1}, \tag{A2}$$

where

$$(\cdots)_R^{-1}=I_R(I_R\cdots I_R)^{-1}I_R \tag{A3}$$

is the inverse with respect to R. From Eq. (A2) follows immediately

$$(H+I_RCI_R)^{-1}=H^{-1}+H^{-1}\{(H^{-1})_R^{-1}[(H^{-1})_R^{-1}+I_RCI_R]_R^{-1}(H^{-1})_R^{-1}-(H^{-1})_R^{-1}\}H^{-1}, \tag{A4}$$

and on R follows

$$I_R(H+I_RCI_R)^{-1}I_R=[(H^{-1})_R^{-1}+I_RCI_R]_R^{-1} \tag{A5}$$

by means of Eq. (A3). If we choose $C=z_{r_0}\sigma_0$ and let $z_{r_0}\to\pm\infty$ we obtain with Eq. (A2)

$$\lim_{z_{r_0}\to\pm\infty}(H+I_RCI_R)^{-1}=H^{-1}-H^{-1}(H^{-1})_R^{-1}H^{-1}. \tag{A6}$$

All matrix elements on R are zero. Therefore, we can write this expression also as a projection onto $\Lambda\setminus R$ which can eventually be rewritten as the inverse on $\Lambda\setminus R$

$$\equiv(1-I_R)H^{-1}(1-I_R)-(1-I_R)H^{-1}I_R(H^{-1})_R^{-1}\times I_RH^{-1}(1-I_R)=(H)_{\Lambda\setminus R}^{-1}. \tag{A7}$$

We use the above identity in the text with the choice $R=\partial S$, the boundary of the block S.

APPENDIX B: ESTIMATION OF THE EIGENVALUES

H and $\widetilde{H}=HD\sigma_3$ are Hermitian matrices. Therefore, both matrices can be diagonalized by unitary transformations U and $\widetilde{U}$, respectively. There are eigenvalues λ_j and $\widetilde{\lambda}_j$ with

$$\lambda_j=(UHU^\dagger)_{jj} \tag{B1}$$

and

$$\widetilde{\lambda}_j=(\widetilde{U}\widetilde{H}\widetilde{U}^\dagger)_{jj}.$$

Then we have

$$\widetilde{\lambda}_j^2=[(\widetilde{U}\widetilde{H}\widetilde{U}^\dagger)_{jj}]^2=(\widetilde{U}\widetilde{H}\widetilde{U}^\dagger\widetilde{U}\widetilde{H}\widetilde{U}^\dagger)_{jj}=(\widetilde{U}HD\sigma_3HD\sigma_3\widetilde{U}^\dagger)_{jj}. \tag{B2}$$

Since H and $D\sigma_3$ commute and $(D\sigma_3)^2=\mathbf{1}$, we obtain for the RHS

$$(\widetilde{U}H^2\widetilde{U}^\dagger)_{jj}\leq\max\lambda_j^2. \tag{B3}$$

This estimation holds for any projection of H and $\widetilde{H}$ on a region S as long as the relation $\widetilde{H}=HD\sigma_3$ is valid on S. We apply the above inequality in our estimation of the eigenvalues of the projection of $\widetilde{H}$ on the lattice block S.

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