Quantum disordered systems with a direction

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Models of disorder with a direction (constant imaginary vector potential) are considered. These non-Hermitian models can appear as a result of computation for models of statistical physics using a transfer-matrix technique, or they can describe nonequilibrium processes. Eigenenergies of non-Hermitian Hamiltonians are not necessarily real, and a joint probability density function of complex eigenvalues can characterize basic properties of the systems. This function is studied using the supersymmetry technique, and a supermatrix σ model is derived. The σ model differs from that already known by a new term. The zero-dimensional version of the σ model turns out to be the same as the one obtained recently for ensembles of random weakly non-Hermitian or asymmetric real matrices. Using a new parametrization for the supermatrix Q, the density of complex eigenvalues is calculated in zero dimension for both the unitary and orthogonal ensembles. The function is drastically different in these two cases. It is everywhere smooth for the unitary ensemble but has a δ-functional contribution for the orthogonal one. This anomalous part means that a finite portion of eigenvalues remains real at any degree of the non-Hermiticity. All details of the calculations are presented.
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I. INTRODUCTION

The physics of disordered metals and semiconductors has been attracting considerable attention for several decades. Various interesting phenomena were discovered experimentally and given theoretical explanations. Rather simple models of a particle moving in a random potential can be used to describe such different effects as Anderson localization,[1] mesoscopic fluctuations,[2,3] the integer quantum Hall effect,[4] and many others.

Although the phenomena can occur at a weak disorder, a simple perturbation theory in the disorder potential is not sufficient for their quantitative description. A proper theory is often based on summing certain classes of diagrams (cooperons and diffusons),[5,6] but in more complicated cases one has to use essentially nonperturbative methods like a supersymmetry technique[7] based on mapping of the disorder models onto a supermatrix σ model (for a recent review, see Ref. 8 and references therein). A disordered physical system can include a magnetic field, magnetic and spin-orbit impurities, etc. However, these additional interactions are included into the calculational schemes without considerable difficulty.

Presently diagrammatic expansions and the supersymmetry technique give the possibility of obtaining explicit results for most of the disorder problems. In addition, the supersymmetry method was applied for calculations with random matrices,[9] which resulted in application of the method in nuclear physics and quantum chaos, where the random matrix theory had been the basic computational tool (for a review see, e.g. Refs. 10–14). Recently, a supermatrix σ model was derived for ballistic billiards averaging over either rare impurities[15] or energy.[16] Thus the way of studying all these interesting problems appears quite clear, although in some cases one can encounter certain technical difficulties.

The systems mentioned above are described by quantum-mechanical Hermitian Hamiltonians. After averaging over disorder, the systems involved are invariant with respect to inversion of coordinates. Sometimes, in order to describe the decay width of eigenstates, non-Hermitian Hamiltonians are used. This approach is popular in the study of quantum dots coupled to leads. Of course, the Hamiltonian of the whole system of the dot with the leads is Hermitian, but it is often convenient to exclude the leads from the consideration by integrating out degrees of freedom related to them. As a result of such an integration one comes to an effective non-Hermitian Hamiltonian of the dot containing imaginary energies.[17] This type of non-Hermiticity can be easily included into the scheme of the supersymmetry technique as well as into diagrammatic expansions, and many results have been obtained explicitly.[8]

In a recent publication[18] Hatano and Nelson considered another type of non-Hermitian Hamiltonian with disorder, namely, Hamiltonians with a constant ''imaginary vector potential.'' In other words, the Hamiltonians contain not only the second-order derivative over the space coordinate, but also the first-order derivative with a real coefficient. The model appears as a result of mapping of flux lines in a $(d+1)$-dimensional superconductor to the world lines of d-dimensional bosons. Columnar defects produced experimentally by energetic heavy ion radiation[19] in order to pin the flux lines lead to a random potential in the boson system, whereas the component of the magnetic field perpendicular to the defects results in a constant imaginary vector potential.[20]

Already, qualitative arguments[18] indicate that the presence of the imaginary vector potential can lead to new effects. In particular, a one-dimensional chain of the bosons has to undergo a localization-delocalization transition; this result was also checked by a numerical computation. In ''conventional'' (without the first-order derivative) disordered systems, transitions in one dimension do not occur, and therefore the model with a direction belongs to a class of

systems that has not yet been studied, to our knowledge. It is argued that the localized states should have real eigenenergies, whereas eigenenergies of the extended eigenstates may have a nonzero imaginary part.

The importance of the investigation of such systems becomes even more evident if one recalls that, e.g., the equation for heat transfer with convection has a term with a first-order derivative. One can imagine a situation when quantum hopping of a particle from site to site on a lattice has a different probability depending on direction. The presence of the first-order derivative in the Hamiltonian corresponds to introducing a certain direction. The nonequivalence of the directions can be provided by coupling to another subsystem, with broken inversion symmetry playing the role of a reservoir; this reservoir may be out of equilibrium. The classical analog of the disordered models with a direction (so-called directed percolation) has been discussed in the literature.[21]

Another problem, where one comes to a stochastic equation containing first-order derivatives, is the problem of turbulence in flow dynamics. It is generally believed that the most important features of the turbulence can be described by the so-called noisy Burgers equation,[22] which is a nonlinear equation with a white noise random force. Besides its application in flow dynamics, this equation is used as a toy model by field theorists due to a striking analogy between the constant flux states in turbulence and some anomalies in quantum field theories.[23] The Burgers equation is equivalent to the Kardar-Parisi-Zhang equation introduced to describe the crystal growth.[24] The nonlinear Burgers equation can be reduced through a Hopf-Cole transformation to a linear $(d+1)$-dimensional equation with a random potential, and time playing the role of the additional dimension. This equation has a first-order time derivative and there has already been an attempt to solve it using the replica method.[25] The noisy Burgers equation can also be reduced to a quantum spin model with a non-Hermitian Hamiltonian.[26] Recently, some interesting results have been obtained for the Burgers equation using an ''instanton'' approximation.[27]

Independently of the study of stochastic models with a direction, considerable attention has been paid in the last decade to an investigation of models of random real asymmetric and complex non-Hermitian matrices. Eigenvalues of such matrices are, generally speaking, complex, and so these models are quite different from models of random real symmetric or Hermitian matrices. Starting from the first work in this direction,[28] a number of publications[29–32,14] contained discussion of properties of these models. Complex random matrices appeared in studies of dissipative quantum maps,[30,14] while real asymmetric random matrices have found applications in neural network dynamics.[33,34] Many interesting aspects of non-Hermitian matrices were discussed in the Refs. 35 and 36. Very recently, a regime of weak non-Hermiticity was found for complex random matrices.[37] In this regime, an explicit formula for the density of complex eigenvalues was obtained by mapping the problem onto a zero-dimensional supermatrix σ model.

Although one may guess that models with non-Hermitian or real asymmetric matrices should be related to disordered systems with non-Hermitian Hamiltonians, no convincing arguments have been given as yet. In fact, generally this is not true because, e.g., the models of open quantum dots described by non-Hermitian Hamiltonians can hardly correspond to the models of random non-Hermitian matrices discussed in the literature.[28,32,14,37] However, as will be shown later, such a correspondence does exist in some limiting cases for disorder models with a direction.

The goal of the present paper is to develop a method that would allow one to make analytical calculations for the disordered problems with a direction. This goal is achieved by modifying the supersymmetry technique in a way so as to include in the nonlinear supermatrix σ model terms corresponding to the imaginary vector potential. Although a proper σ model for the physical real vector potential was derived long ago,[7] changing to the imaginary one is far from trivial and, as a result, a completely new term in the σ model appears. If the time-reversal invariance is broken, the zero-dimensional version of the σ model turns out to be exactly the same as the one obtained in Ref. 37 for the model of weakly non-Hermitian random matrices.

The supermatrix σ model derived below is valid in any dimension, and can be a proper tool for studying the localization-delocalization transitions in one and two dimensions proposed in Ref. 18. However, although one can use standard computational schemes,[8] the presence of new terms in the σ model make calculations with the known parametrizations of the supermatrix Q more difficult. Therefore, a new parametrization is suggested and corresponding Jacobians are calculated. To avoid ''overloading'' only the zero-dimensional case is considered in this paper. For the unitary ensemble the result of Ref. 37 for the joint probability density of complex eigenvalues of weakly non-Hermitian random matrices is reproduced. The density function is a smooth function of the imaginary part of the eigenvalues, which shows that the probability of real eigenvalues is zero.

In contrast, the density function for the orthogonal ensemble obtained below contains a δ function, which shows that the fraction of states with real eigenvalues is finite. This is a very unusual and interesting result. The entire function of the density of complex eigenvalues is obtained for the first time, to our knowledge. In the limit of strong non-Hermiticity the joint probability functions for the both unitary and orthogonal ensembles correspond to the ''elliptic law.'' [28,29]

The main results of this paper have been presented in a short form elsewhere.[38] The paper is organized as follows: In Sec. II models of disorder with a direction are introduced, and their basic properties are discussed. Section III contains the derivation of the supermatrix σ model. In Section IV a joint probability density of complex eigenvalues is calculated for systems in a limited volume with broken time-reversal symmetry (unitary ensemble). This is done by calculation of integrals over a supermatrix Q for the unitary ensemble. A new parametrization for the supermatrices Q is introduced. In Sec. V similar calculations are carried out for the orthogonal ensemble. The result for the density of complex eigenvalues proves to be qualitatively different from that for the unitary ensemble. Section VI contains a discussion of the results obtained, and a comparison with some other works. In the Appendixes the Jacobians corresponding to the new parametrizations for the supermatrix Q are derived.

II. THE MODEL AND ITS BASIC PROPERTIES

The initial classical model of vortices in a $(d+1)$-dimensional superconductor with line defects considered in Refs. 18 and 20 contains an interaction between the vortices. In the corresponding quantum model of d-dimensional bosons, this describes an interaction between the bosons. The interaction is, in principle, very important. Its short-range part does not allow bosons to condense at one localized state. At the same time, if it is strong enough there can be only one boson in a localized state, and the problem maps onto the model of noninteracting fermions. Of course, this is not true for extended states, for which one should use the model of interacting bosons.

It is clear that one should first understand when one-particle states are localized and when they are not. Therefore, as in Refs. 18 and 20, it is reasonable to start with a d-dimensional Hamiltonian H of noninteracting particles including a constant imaginary vector potential $i\mathbf{h}$ and a random potential of impurities $U(\mathbf{r})$,

$$H=H_0+U(\mathbf{r}), \quad H_0=\frac{(\hat{\mathbf{p}}+i\mathbf{h})^2}{2m}, \tag{2.1}$$

where $\hat{\mathbf{p}}=-i\nabla$, and m is the mass of a particle (boson or fermion).

The random potential $U(\mathbf{r})$ is assumed to be distributed according to the Gaussian δ-correlated law

$$\langle U(\mathbf{r})\rangle=0, \quad \langle U(\mathbf{r})U(\mathbf{r}')\rangle=\frac{1}{2\pi\nu\tau}\delta(\mathbf{r}-\mathbf{r}'), \tag{2.2}$$

where τ is the mean free time, and ν is the density of states of the Hamiltonian H_0 at $\mathbf{h}=0$ at the energy ϵ involved. It is assumed that the disorder is weak, and h is small. As mentioned in Sec. I, the potential $U(\mathbf{r})$ corresponds to the potential of the line defects and $\mathbf{h}$ to the component of the magnetic field perpendicular to the line defects for the model of the vortices. At the same time, the Hamiltonian H, Eq. (2.1), can describe other systems as well. So we may study properties of the Hamiltonian H without recalling each time where it comes from. Some of possible applications of Eq. (2.1) were listed in Sec. I. The directed quantum hopping appears to be a new interesting possibility. The Hamiltonian H_L of a lattice version of Eq. (2.1) can be written as follows:

$$H_L=-\frac{t}{2}\sum_{\mathbf{r}}\sum_{\nu=1}^{d}(e^{\mathbf{he}_\nu}c^+_{\mathbf{r}+\mathbf{e}_\nu}c_{\mathbf{r}}+e^{-\mathbf{he}_\nu}c^+_{\mathbf{r}}c_{\mathbf{r}+\mathbf{e}_\nu}) + \sum_{\mathbf{r}}U(\mathbf{r})c^+_{\mathbf{r}}c_{\mathbf{r}}, \tag{2.3}$$

where c^+ and c are creation and annihilation operators, and $\{\mathbf{e}_\nu\}$'s are the unit lattice vectors.

Although Eq. (2.3) was used in Ref. 18 only for numerical calculations, it has a clear physical application; that is, it describes quantum hopping of a particle from site to site in the presence of a random potential. However, the hopping probability along $\mathbf{h}$ is higher than in the opposite direction. In other words, the Hamiltonian H_L describes a directed hopping in a random potential. The systems with the Hamiltonians H and H_L, Eqs. (2.1) and (2.3), are not invariant with respect to inversion of the coordinates even after averaging over impurities. At the same time, they are time, reversal invariant, and therefore essentially different from systems with real magnetic fields.

If necessary the Hamiltonians H and H_L can be generalized to include the vector potential $\mathbf{A}$ corresponding to a physical magnetic field. This can be done by the standard replacement

$$\hat{\mathbf{p}}\rightarrow\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A} \tag{2.4}$$

in Eq. (2.1). Proper changes can also be done in Eq. (2.3).

Of course, the vortex model of Ref. 18 corresponds to Eq. (2.1) with $\mathbf{A}=0$, but already the hopping model can be considered in an arbitrary magnetic field. Changing the magnetic field (or, more precisely, the vector potential $\mathbf{A}$) results in a crossover between ensembles with different symmetries. In analogy with ''conventional'' (nondirected) disordered systems, these ensembles will be called orthogonal and unitary.

Although the Hamiltonians H and H_L, Eqs. (2.1) and (2.3), are not Hermitian, this fact does not contradict fundamental laws of nature. In the problem of the vortices in superconductors these Hamiltonians appear after a reduction of a $(d+1)$-dimensional classical problem to a d-dimensional quantum one using the transfer-matrix technique, which is a formal trick. As concerns the directed hopping model the vector $\mathbf{h}$ can appear as a result of a coupling with another system (reservoir) that is not necessarily in equilibrium. The latter system can be subjected, e.g., to an electric field, there can be nondecaying currents in it, etc. Integrating out degrees of freedom related to the reservoir one obtains an effective Hamiltonian that does not need to be Hermitian.

In other words, non-Hermitian Hamiltonians appear at intermediate steps of calculations, and manipulations with them should be considered merely as formal computational tricks. The corresponding wave functions and eigenenergies are only formal objects as well. Of course, one should understand how to relate initial physical observables to quantities calculated with the non-Hermitian Hamiltonians.

It is relevant to mention that a classical directed model that can be considered the counterpart of the directed quantum problem was introduced a long ago.[21] This is the model of a directed percolation that can describe, e.g., spreading of infection or fire in a forest affected by wind. According to the results of Ref. 21, the critical behavior near the percolation transition in the model of the directed percolation is different from that in an isotropic model. The analysis of Ref. 21 was based on a diagrammatic expansion. The bare Green functions $G^{(0)}(\mathbf{p})$ used in the expansion had the form

$$G^{(0)}(\mathbf{p})=\frac{1}{\mathbf{p}^2+i\mathbf{ap}+r}, \tag{2.5}$$

with a constant vector $\mathbf{a}$. Comparing Eq. (2.5) with Eq. (2.1), we see that $G^{(0)}$ is the Green function of the Hamiltonian H_0, which demonstrates that both models are really closely related to each other.

Now, let us discuss, following Ref. 18, basic properties of eigenstates of the Hamiltonian H, Eq. (2.1). Due to the non-

Hermiticity of the Hamiltonian, one should distinguish between right $\phi_k(\mathbf{r})$ and left $\overline{\phi_k}(\mathbf{r})$ eigenfunctions. They obey the equations

$$H\phi_k(\mathbf{r})=\epsilon_k\phi_k(\mathbf{r}), \quad H^T\overline{\phi_k}(\mathbf{r})=\epsilon_k\overline{\phi_k}(\mathbf{r}), \tag{2.6}$$

where H^T is obtained by transposition of the Hamiltonian H. For spinless particles the operation of the transposition means simply changing of the sign of the space derivative. The functions $\overline{\phi_k}(\mathbf{r})$ are also considered conjugate to $\phi_k(\mathbf{r})$; for each eigenfunction one can construct its conjugate. The scalar product $(\phi_k,\phi_{k'})$ of two eigenfunctions $\phi_k(\mathbf{r})$ and $\phi_{k'}(\mathbf{r})$ is introduced as

$$(\phi_k,\phi_{k'})=\int \overline{\phi_k}(\mathbf{r})\phi_{k'}(\mathbf{r})d\mathbf{r}. \tag{2.7}$$

Using Eq. (2.7), one can prove in a standard way the orthogonality of eigenfunctions corresponding to different eigenenergies. Together with the normalization condition this can be written as

$$\int \overline{\phi_k}(\mathbf{r})\phi_{k'}(\mathbf{r})d\mathbf{r}=\delta_{kk'} \tag{2.8}$$

The eigenenergy ϵ_k in both Eqs. (2.6) is the same. Equation (2.8) enables us to reproduce basic properties of conventional (Hermitian) quantum mechanics, replacing everywhere complex conjugates $\phi_\kappa^*(\mathbf{r})$ of the functions $\phi_k(\mathbf{r})$ by the conjugates $\overline{\phi_k}(\mathbf{r})$. However, the eigenenergies ϵ_k in the non-Hermitian quantum mechanics are not necessarily real. They must be real only if the functions $\phi_k^*(\mathbf{r})$ and $\overline{\phi_k}(\mathbf{r})$ coincide. In order to obtain well-defined wave functions in the thermodynamic limit, it is convenient to impose periodic boundary conditions.

To understand better what the wave functions look like in different situations, it is instructive to consider a localized state with a localization center at a point x_0, and extended states in the absence of impurities (for simplicity we may restrict ourselves with the purely one-dimensional case). Assume that for $h=0$ the eigenfunctions $\phi_k^{(0)}$ and the eigenvalues $\epsilon_k^{(0)}$ are known. Then the functions

$$\phi_k(x)=e^{hx}\phi_k^{(0)}(x), \quad \overline{\phi_k}=e^{-hx}\phi_k^{(0)}(x) \tag{2.9}$$

are solutions of Eqs. (2.6) with the eigenenergy $\epsilon_k^{(0)}$.

At the same time, in order to satisfy the boundary conditions, the functions ϕ_k and $\overline{\phi_k}$ may not grow. If the function $\phi_k^{(0)}(x)$ is exponentially localized at a length l_c, the function $\phi_k(x)$ takes the form

$$\phi_k(x)=C\ \exp(h(x-x_0)-l_c^{-1}|x-x_0|). \tag{2.10}$$

The function $\phi_k(x)$, Eq. (2.10), and the corresponding function $\overline{\phi_k}(x)$ does not grow at $|x|\to\infty$ only if $|h|<l_c^{-1}$. The point $|h|=l_c^{-1}$ was identified[18] with a localization-delocalization transition.

In the region $|h|\geq l_c^{-1}$ the functions ϕ_k given by Eqs. (2.9) and (2.10) are no longer eigenfunctions because they do not satisfy the boundary conditions. To get an idea what the eigenfunctions look like in this region, we may neglect the disorder potential. Then the plane waves

$$\phi_k=L^{-1/2}e^{ikx}, \quad \overline{\phi_k}=L^{-1/2}e^{-ikx}, \tag{2.11}$$

where L is the length of the sample, are proper solutions of Eqs. (2.6) satisfying the boundary conditions. However, in this case the eigenvalue ϵ_k is complex,

$$\epsilon_k=\frac{(k+ih)^2}{2m}. \tag{2.12}$$

We see that the question about whether an eigenfunction in the presence of the imaginary vector potential is localized or extended is closely related in the thermodynamic limit to the question of whether the corresponding eigenenergy is real or complex. The arguments presented are qualitative but they were confirmed by numerical calculations.[18]

It is clear from the previous discussion that it is very important to understand when eigenenergies are real and when they become complex. A convenient function characterizing the system is the joint probability density of complex eigenenergies $P(\epsilon,y)$, defined as

$$P(\epsilon,y)=\frac{1}{V}\left\langle \sum_k \delta(\epsilon-\epsilon_k')\,\delta(y-\epsilon_k'')\right\rangle, \tag{2.13}$$

where ϵ_k' and ϵ_k'' are the real and imaginary parts of the eigenenergy ϵ_k, V is the volume, and the angular brackets stand for averaging over impurities. If all states are localized, such that $\epsilon_k''=0$, the function $P(\epsilon,y)$ equals

$$P(\epsilon,y)=\nu(\epsilon)\,\delta(y), \tag{2.14}$$

where $\nu(\epsilon)$ is the average density of states.

If all states are extended, the function $P(\epsilon,y)$ should be a smooth function of both variables. In some cases physical quantities can be expressed directly through the function $P(\epsilon,y)$, although other correlation functions are also of interest. The rest of this paper is devoted to a reduction of the function $P(\epsilon,y)$, which is the simplest nontrivial function characterizing the system, to a correlation function in a supersymmetric σ model and to some calculations with this model. To our knowledge, this is the first attempt at a quantitative analytical study of disordered directed quantum systems.

III. DERIVATION OF σ MODEL

According to the standard procedure of derivation of the supermatrix σ model,[7,8] one should express the physical quantity in terms of retarded G_ϵ^R and advanced G_ϵ^A Green functions of the Hamiltonian. Usually the average density of states that can be expressed through the average of one Green function is not an interesting quantity, because it does not distinguish between localized and extended states. The density of complex eigenvalues $P(\epsilon,y)$ is definitely more interesting, but how to express it in terms of integrals over supervectors, which is the first step of derivation of the σ model?

The problem is that it is not clear how to write the function $P(\epsilon,y)$ in terms of the functions G_ϵ^R and G_ϵ^A. However, even if this representation existed it would not help. The spectral expansion of the functions $G^{R,A}$,

$$G_{\epsilon}^{R,A}(\mathbf{r},\mathbf{r}')=\sum_k \frac{\phi_k(\mathbf{r})\overline{\phi_k}(\mathbf{r}')}{\epsilon-\epsilon_k\pm i\delta}, \tag{3.1}$$

contains the eigenenergies ϵ_k; some of them may be complex. But the very possibility to rewrite the Green functions in terms of convergent Gaussian integrals over the supervectors was based on the assumption that the eigenenergies were real.

Another possibility is based on the relation

$$\delta(a)\,\delta(b)=\frac{1}{\pi}\lim_{\gamma\to 0}\frac{\gamma^2}{(a^2+b^2+\gamma^2)^2}, \tag{3.2}$$

that holds for real a and b. With Eq. (3.2) the density function $P(\epsilon,y)$ can be rewritten as

$$P(\epsilon,y)=\frac{\gamma^2}{\pi V}\lim_{\gamma\to 0}\left\langle \sum_k \left[(\epsilon-\epsilon_k')^2+(y-\epsilon_k'')^2+\gamma^2\right]^{-2}\right\rangle. \tag{3.3}$$

Using the orthogonality of the eigenfunctions ϕ_k, Eq. (3.3) can be also represented as

$$P(\epsilon,y)=\frac{1}{\pi V}\lim_{\gamma\to 0}\int B(\mathbf{r},\mathbf{r}')B(\mathbf{r}',\mathbf{r})d\mathbf{r}\,d\mathbf{r}', \tag{3.4}$$

where the function $B(\mathbf{r},\mathbf{r}')$ has the form

$$B(\mathbf{r},\mathbf{r}')=\sum_k \frac{\gamma\phi_k(\mathbf{r})\overline{\phi_k}(\mathbf{r}')}{(\epsilon-\epsilon_k')^2+(y-\epsilon_k'')^2+\gamma^2}. \tag{3.5}$$

The representation of the density function $P(\epsilon,y)$ by Eq. (3.4) is very convenient because it allows one to rewrite this function in terms of a Gaussian integral over supervectors.

In order to derive a proper expression, let us introduce a Hermitian operator $\hat{M}$

$$\hat{M}=\begin{pmatrix} H'-\epsilon & i(H''-y) \\ -i(H''-y) & -(H'-\epsilon)\end{pmatrix}, \tag{3.6}$$

where

$$H'=\frac{1}{2}(H+H^+),\quad H''=-\frac{i}{2}(H-H^+). \tag{3.7}$$

In Eq. (3.7), the symbol ''+'' means Hermitian conjugation. For real Hamiltonians this conjugation coincides with the transposition ''T.'' However, let us write formulas in a general form such that the Hamiltonian H may include magnetic interactions and be complex.

Instead of manipulating the non-Hermitian operator H, one can try to use the Hermitian operator $\hat{M}$. To follow the standard procedure of the supersymmetry technique one should find first the eigenstates of this operator. For the complex non-Hermitian operator H, one can write four equations for the eigenstates:

$$H\phi_k=\epsilon_k\phi_k,\quad H^T\overline{\phi_k}=\epsilon_k\overline{\phi_k}, \tag{3.8}$$

$$H^*\phi_k^*=\epsilon_k^*\phi_k^*,\quad H^+\overline{\phi_k^*}=\epsilon_k^*\overline{\phi_k^*}. \tag{3.9}$$

Equations (3.9) are merely complex conjugates of Eqs. (3.8).

Now let us introduce two sets of two-component vectors u_k and v_k,

$$u_k=\frac{1}{2}\begin{pmatrix}\phi_k+\overline{\phi_k^*}\\ \phi_k-\overline{\phi_k^*}\end{pmatrix},\quad v_k=\frac{1}{2}\begin{pmatrix}\phi_k-\overline{\phi_k^*}\\ \phi_k+\overline{\phi_k^*}\end{pmatrix}, \tag{3.10}$$

$$\overline{u_k}=\tfrac{1}{2}(\overline{\phi_k}+\phi_k^*\quad \overline{\phi_k}-\phi_k^*),\quad \overline{v_k}=\tfrac{1}{2}(\overline{\phi_k}-\phi_k^*\quad \overline{\phi_k}+\phi_k^*).$$

Using the orthogonality of the eigenfunctions ϕ_k, Eq. (2.8), one can prove the orthogonality of the vectors u_k and v_k:

$$\int \overline{u_k}(\mathbf{r})u_{k'}(\mathbf{r})d\mathbf{r}=\int \overline{v_k}(\mathbf{r})v_{k'}(\mathbf{r})d\mathbf{r}=\delta_{kk'}, \tag{3.11}$$

$$\int \overline{u_k}(\mathbf{r})v_{k'}(\mathbf{r})d\mathbf{r}=\int \overline{v_k}(\mathbf{r})u_{k'}(\mathbf{r})d\mathbf{r}=0.$$

It is not difficult to see that vectors $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ are eigenvectors of the matrix operator $\hat{M}$ satisfying the equations

$$\hat{M}u_k=M_ku_k,\quad \overline{M}v_k=M_kv_k, \tag{3.12}$$

where the matrix M_k equals

$$M_k=\begin{pmatrix}\epsilon_k'-\epsilon & i(\epsilon_k''-y)\\ -i(\epsilon_k''-y) & -(\epsilon_k'-\epsilon)\end{pmatrix}, \tag{3.13}$$

and ϵ_k' and ϵ_k'' are the real and imaginary parts of the eigenenergies ϵ_k.

Using the identity

$$\frac{i}{2}\mathrm{Tr}(M_k+i\gamma)^{-1}=\frac{\gamma}{(\epsilon_k'-\epsilon)^2+(\epsilon_k''-y)^2+\gamma^2}, \tag{3.14}$$

one can see that functions $B(\mathbf{r},\mathbf{r}')$, Eq. (3.5), are closely related to the operator $\hat{M}$. The only thing that remains to be done is to express the matrix $(M_k+i\gamma)^{-1}$ and then the operator $(\hat{M}+i\gamma)^{-1}$ in terms of a Gaussian integral over supervectors.

The operator $\hat{M}$ is Hermititian, and its eigenvectors u_k and v_k, Eqs. (3.10), are known, and therefore we can follow the standard procedure of the derivation. Changing from the Hamiltonian H to the operator $\hat{M}$, we have to double the size of the relevant matrices. This means that in order to write proper Gaussian integrals we should use, as usual, eight-component supervectors $\psi(\mathbf{r})$. In fact, one comes to supervectors ψ with exactly the same structure as previously,[7,8]

$$\psi^m=\begin{pmatrix}\vartheta^m\\ r^m\end{pmatrix},\quad \vartheta^m=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi^{m*}\\ \chi^m\end{pmatrix},\quad r^m=\frac{1}{\sqrt{2}}\begin{pmatrix}S^{m*}\\ S^m\end{pmatrix}. \tag{3.15}$$

$m=1,2$; and χ^m and S^m are anticommuting and commuting variables, respectively.

Let us present several important intermediate steps of the reduction of the operator $(\hat{M}+i\gamma)^{-1}$, to the functional integral over $\psi(\mathbf{r})$. First, we have

$$(i\gamma+M_k)^{-1}=-i\int (a_k a_k^*+b_k b_k^*)\exp(-L_k)dR_k$$
$$=-i\int (\sigma_k\sigma_k^*+\rho_k\rho_k^*)\exp(-L_k)dR_k, \tag{3.16}$$

where a_k, b_k and σ_k, ρ_k are commuting and anticommuting variables, respectively, and dR_k stands for the elementary volume in the space of these variables. The function L_k in Eq. (3.16) equals

$$L_k=-i(a_k^* \quad b_k^*)(i\gamma+M_k)\begin{pmatrix} a_k \\ b_k \end{pmatrix}$$
$$-i(\sigma_k^* \quad \rho_k^*)(i\gamma+M_k)\begin{pmatrix} \sigma_k \\ \rho_k \end{pmatrix}. \tag{3.17}$$

The vector fields $\vec{\chi}(\mathbf{r})$ and $\vec{S}(\mathbf{r})$ are introduced as

$$\vec{\chi}(\mathbf{r})=\begin{pmatrix} \chi^1(\mathbf{r}) \\ \chi^2(\mathbf{r}) \end{pmatrix}=\sum_k [a_k u_k(\mathbf{r})+b_k v_k(\mathbf{r})],$$
$$\vec{S}(\mathbf{r})=\begin{pmatrix} S^1(\mathbf{r}) \\ S^2(\mathbf{r}) \end{pmatrix}=\sum_k [\sigma_k u_k(\mathbf{r})+\rho_k v_k(\mathbf{r})], \tag{3.18}$$

where the vectors u_k and v_k are defined in Eqs. (3.10).

With these definitions, one can express the functions $B(\mathbf{r},\mathbf{r}')$, Eq. (3.5), in terms of Gaussian integrals over the vector fields $\vec{\chi}(\mathbf{r})$ and $\vec{S}(\mathbf{r})$. The derivation is based on the identity

$$\int \vec{S}^*(\mathbf{r})\hat{M}\vec{S}(\mathbf{r})d\mathbf{r}=\sum_k (a_k^* \quad b_k^*)\hat{M}_k\begin{pmatrix} a_k \\ b_k \end{pmatrix}, \tag{3.19}$$

that can be proved using Eqs. (2.8) and (3.10) [the same for $\vec{\chi}(\mathbf{r})$]. Less trivial is the expression

$$i\gamma\int S^*(\mathbf{r})\vec{S}(\mathbf{r})d\mathbf{r}. \tag{3.20}$$

Using the expansion, Eq. (3.18), we can see that the integral, Eq. (3.20), contains nondiagonal terms with respect to k,k'. For example, there is the following term:

$$\frac{i\gamma}{2}\sum_{k,k'} a_{k'}^* a_k\int \phi_{k'}^*(\mathbf{r})\phi_k(\mathbf{r})d\mathbf{r}. \tag{3.21}$$

For Hermitian Hamiltonians the integral in Eq. (3.21) would give $\delta_{kk'}$. However, generally it is not zero for arbitrary k and k' because the orthogonality relation, Eq. (2.8), contains ϕ_k but not ϕ_k^*. Fortunately, this does not create the difficulties in the limit of small ''vector potential'' h that are the main focus of interest in the present work, because the difference between ϕ_k and ϕ_k^* is small. (Moreover, we are interested in the limit $\gamma\to 0$). This allows us to write

$$\sum_k L_k=\int [\vec{\chi}^*(\mathbf{r})(i\gamma+\hat{M})\vec{\chi}(\mathbf{r})+\vec{S}^*(\mathbf{r})(i\gamma+\hat{M})\vec{S}(\mathbf{r})]d\mathbf{r}. \tag{3.22}$$

Although one can use Eq. (3.22) as an effective Lagrangian, it is convenient to unify all components of the vectors $\vec{\chi}$, $\vec{\chi}^*$, $\vec{S}$, and $\vec{S}^*$ into the supervector ψ of the form, Eq. (3.15). As a result, one comes to integration with the weight $\exp(-\mathcal{L})$, where the Lagrangian $\mathcal{L}$ takes the form

$$\mathcal{L}=-i\int \overline{\psi}(\mathbf{r})[\mathcal{H}_0+U(\mathbf{r})]\psi(\mathbf{r})d\mathbf{r}, \tag{3.23}$$

and the ''charge-conjugate'' supervector $\overline{\psi}(\mathbf{r})$ is the same as in Refs. 7 and 8. The 8×8 matrix operator $\mathcal{H}_0$ can be written as

$$\mathcal{H}_0=\mathcal{H}_{00}+\mathcal{H}_{01},$$
$$\mathcal{H}_{00}=H_0'-\epsilon+i\gamma\Lambda, \quad \mathcal{H}_{01}=i\Lambda_1(H''+y\tau_3). \tag{3.24}$$

In the continuum model, the ''imaginary'' part H'' of the Hamiltonian H, Eqs. (2.1) and (3.7), has the form

$$H''=-i\frac{\mathbf{h}\nabla}{m}. \tag{3.25}$$

The diagonal matrices Λ and τ_3 are the same as in Refs. 7 and 8. The matrix Λ_1 anticommutes with the matrix Λ and consists of unit 4×4 blocks as well. The explicit forms of these matrices are

$$\Lambda=\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \Lambda_1=\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{3.26}$$

Equation (3.23) is similar to the corresponding equation for localization problems,[7,8] and in the absence of $\mathcal{H}_{01}$ these equations would coincide. All new physics comes from the operator $\mathcal{H}_{01}$. A magnetic field can be included into $\mathcal{H}_{00}$ in a standard way.

All subsequent manipulations are the same as in Refs. 7 and 8. First, one averages over the random potential $U(\mathbf{r})$ using Eq. (2.2) and comes, instead of Eq. (3.23), to a regular Lagrangian $\mathcal{L}$

$$\mathcal{L}=\int\left[-i\overline{\psi}(\mathbf{r})\mathcal{H}_0\psi(\mathbf{r})+\frac{1}{4\pi\nu\tau}[\overline{\psi}(\mathbf{r})\psi(\mathbf{r})]^2\right]d\mathbf{r}. \tag{3.27}$$

Then one decouples the interaction term in Eq. (3.27) by integration over a supermatrix Q, and integrates over the supervector ψ, assuming that the supermatrix Q varies in space slowly. After that one comes to an integral over Q with the weight $\exp(-F[Q])$. The functional integral over Q is calculated using a saddle point approximation. At the saddle point the supermatrix Q does not depend on coordinates, and in the limit of small $\mathcal{H}_{01}$ and γ one obtains the standard equation

$$Q(\mathbf{r})=\frac{1}{\pi\nu}\left(\left[-i\mathcal{H}_{00}+\frac{Q(\mathbf{r})}{2\tau}\right]^{-1}\right)_{\mathbf{rr}}, \tag{3.28}$$

which leads to the constraint $Q^2=1$ and to the form $Q=V\Lambda V$, $VV=1$. Now, one has to expand the free-energy functional $F[Q]$ near the saddle point in H_{01}, γ and ∇Q. As a result, the functional $F[Q]$ acquires the form of a σ model,

$$F[Q]=\frac{\pi\nu}{8}\int S\mathrm{Tr}\{D_0(\nabla Q+\mathbf{h}[Q,\Lambda_1])^2 - 4(\gamma\Lambda+y\Lambda_1\tau_3)Q\}d\mathbf{r}, \quad (3.29)$$

where D_0 is the classical diffusion coefficient, [.,.] is commutator, and STr stands for a supertrace. Equation (3.29) is written in the absence of a magnetic field. The expansion near the saddle point leading to Eq. (3.29) is justified provided $y\ll\tau^{-1}$ and $h\ll l^{-1}$, where l is the mean free path. The supermatrices Q are the same as those for the orthogonal ensemble.[7,8] This case corresponds, in particular, to the problem of vortices in superconductors with line defects.[18] If for some other problems one has to include in the Hamiltonians H and H_L the physical vector-potential $\mathbf{A}$ corresponding to a magnetic field, the standard derivation shows that the proper σ-model is obtained from Eq. (3.29) by the replacement

$$\nabla Q\rightarrow\nabla Q-\frac{ie}{c}\mathbf{A}[Q,\tau_3]. \quad (3.30)$$

In the limit of a strong magnetic field one can neglect fluctuations of a certain symmetry (cooperons). Then Eq. (3.29) is still valid, but the supermatrices Q should have a symmetry corresponding to the unitary ensemble.

The free-energy functional $F[Q]$, Eq. (3.29), has two additional with respect to the functional used for ''conventional'' disorder problems terms. These terms contain the matrix Λ_1 and serve as new effective ''external fields'' in the free energy. We see from Eqs. (3.29) and (3.30) that $\mathbf{h}$ and $\mathbf{A}$ enter $F[Q]$ in a different way. A simple replacement $\mathbf{A}\rightarrow i\mathbf{h}$ in the σ model of Refs. 7 and 8 would give a wrong result. This reflects the fact that a nonzero $\mathbf{A}$ violates the time-reversal symmetry, while $\mathbf{h}$ can break only the symmetry with respect to inversion of coordinates.

In order to express the density function $P(\epsilon,y)$, Eqs. (3.4) and (3.5), in terms of a functional integral over Q, one should know not only the weight $\exp(-F[Q])$ but also a pre-exponential functional $A[Q]$. This can be derived from Eqs. (3.4) and (3.5) in a standard way. One of the functions B can be written using the first line of Eq. (3.16), and the other using the second one. As a result, one obtains in the pre-exponential a product of four different components of the supervector ψ; two of them are at the point $\mathbf{r}$, while the other two at the point $\mathbf{r}'$. After averaging over the random potential $U(\mathbf{r})$ and decoupling of the effective interaction in Eq. (3.27) by integration over the supermatrix Q, one has to compute Gaussian integrals over Ψ. This can be done using the Wick theorem. In the limit $\tau^{-1}\gg(\nu V)^{-1}$ one may take into account only pairing of two ψ at coinciding points. The rest of the calculation is simple, and one obtains

$$P(\epsilon,y)=-\frac{\pi\nu^2}{4V}\lim_{\gamma\rightarrow 0}\int A[Q]\exp(-F[Q])dQ, \quad (3.31)$$

where

$$A[Q]=\int\{[Q_{42}^{11}(\mathbf{r})+Q_{42}^{22}(\mathbf{r})][Q_{24}^{11}(\mathbf{r}')+Q_{24}^{22}(\mathbf{r}')]-[Q_{42}^{21}(\mathbf{r}) + Q_{42}^{12}(\mathbf{r})][Q_{24}^{21}(\mathbf{r}')+Q_{24}^{12}(\mathbf{r}')]\}d\mathbf{r}\,d\mathbf{r}'. \quad (3.32)$$

Numeration of the matrix elements in Eq. (3.32) is standard.[7,8]

Equations (3.29)–(3.32) solve the problem of mapping of the density of complex eigenvalues for disorder models with a direction onto a supermatrix σ model. The density function $P(\epsilon,y)$ depends on the real part ϵ of the eigenenergies through the parameters ν and D_0 that are dependent on ϵ. The dependence on the imaginary part y is more complicated. Remarkably, the σ model derived differs from the σ model for localization problems by additional ''external fields'' only. This simplifies calculations because one can use well-developed computational schemes.

The σ model, Eqs. (3.29)–(3.32) can be used in any dimension. The one-dimensional version describes ''quantum wires'' or, in the language of the superconductor model, to vortices in a slab. According to a discussion of Ref. 18, in one-dimensional models there has to be a localization-delocalization transition. If this is true for thick wires the one-dimensional σ model should undergo a phase transition when changing the value of h. However, study of the one-dimensional model is more difficult than of the zero-dimensional one. Leaving higher-dimensional problems for future investigation let us concentrate in Sec. IV on calculating the density function $P(\epsilon,y)$ for a sample with a finite volume. This situation is described by the zero-dimensional σ model.

IV. DENSITY OF COMPLEX EIGENVALUES IN A LIMITED VOLUME: UNITARY ENSEMBLE

If disorder is not very strong, there is a regime when physical quantities can be obtained from the zero-dimensional (0D) σ model. This is the limiting case when one considers only supermatrices Q that do not vary in space. For the problem of level statistics in Hermitian models the 0D σ model is obtained in the limit $\omega\ll E_c$, where $E_c=\pi^2D_0/L^2$ is the Thouless energy (L is the sample size).[7,8] If the sample is connected with leads and the energy levels are smeared, the 0D case is possible, provided the level width does not exceed E_c. If the disorder is strong or the sample has one- or two-dimensional geometry, such that the localization length L_c is smaller than the sample size, the 0D limit cannot be achieved.

It is clear that the situation with the directed problems involved should be similar, and one can come to the 0D σ model provided h, y, and γ in Eq. (3.29) are not very large and disorder is not very strong. For the model of vortices in a superconductor the 0D limit for the σ model would correspond to a sample with a finite cross section perpendicular to the line defects.

Neglecting all nonzero space harmonics in the free-energy functional $F[Q]$, one can rewrite Eq. (3.29) as follows:

$$F[Q]=S\mathrm{Tr}\left(\frac{a^2}{16}[Q,\Lambda_1]^2-\frac{x}{4}\Lambda_1\tau_3Q-\frac{\widetilde{\gamma}}{4}\Lambda Q\right), \quad (4.1)$$

where

$$a^2=\frac{2\pi D_0h^2}{\Delta},\quad x=\frac{2\pi y}{\Delta},\quad \widetilde{\gamma}=\frac{2\pi\gamma}{\Delta}, \quad (4.2)$$

and $\Delta=(\nu V)^{-1}$ is the mean level spacing.

Although obtaining the 0D limit of the σ model is trivial in the absence of the vector potential, the derivation needs some care for $\mathbf{h}\neq 0$. In case of a real magnetic field one can neglect all nonzero harmonics, provided the London gauge for the vector potential is chosen.[8] The same is correct for $\mathbf{h}$. One can come to the London gauge considering nonzero harmonics perturbatively. The free-energy functional of fields describing deviations from the 0D σ model contains both linear and quadratic terms. In order to obtain a good perturbation theory near the 0D σ model, one should get rid of the linear terms. This can be done for an arbitrary function $\mathbf{h}\ (\mathbf{r})$ if the following equations are satisfied:

$$\mathrm{div}\mathbf{h}\ (\mathbf{r})=0, \quad \mathbf{h}|_S=0, \tag{4.3}$$

where $\mathbf{h}|_S$ is the component perpendicular to the surface. Equation (4.3) is the London gauge for $\mathbf{h}$.

It is clear that the second condition in Eq. (4.3) cannot be fulfilled for a closed sample, and $\mathbf{h}=$const. But this is not an interesting case because, as follows from the general consideration of Sec. II, in such a situation $\mathbf{h}$ can be removed by the transformation, Eq. (2.9) (if the sample is confined by hard walls), and one obtains the standard 0D σ model without the first term in Eq. (4.1).

Equations (4.3) can be satisfied if the sample has a ring geometry, and $\mathbf{h}\ (\mathbf{r})$ is directed along the circumference (this corresponds to the periodic boundary conditions of Ref. 18). Then, $\mathbf{h}$ is approximately constant if the radius of the ring is large. Thus Eqs. (4.1) and (4.2) are applicable to samples with a ring geometry. Of course, one can also use these equations for an arbitrary geometry, but the function $\mathbf{h}\ (\mathbf{r})$ must obey Eqs. (4.3) and the quantity h^2V entering Eq. (4.2) should be replaced by $\int \mathbf{h}^2(\mathbf{r})d\mathbf{r}$.

Removing, with the help of Eqs. (4.3), terms in the free-energy functional linear in the deviations from the 0D σ model, and evaluating the contribution of the quadratic ones, one comes to the conclusion that the nonzero space harmonics can be neglected in the limits

$$y\ll E_c, \quad \tilde{\gamma}\ll E_c, \quad h\ll L^{-1}, \tag{4.4}$$

where L is the sample size.

The density function $P(\epsilon,y)$, Eqs. (3.31) and (3.32), takes the forms

$$P(\epsilon,y)=-\frac{\pi\nu}{4\Delta}\lim_{\tilde{\gamma}\to 0}\int A[Q]\exp(-F[Q])dQ, \tag{4.5}$$

$$A[Q]=(Q_{42}^{11}+Q_{42}^{22})(Q_{24}^{11}+Q_{24}^{22})-(Q_{42}^{21}+Q_{42}^{12})(Q_{24}^{21}+Q_{24}^{12}),$$

with $F[Q]$ determined by Eq. (4.1).

To obtain the function $P(\epsilon,y)$ one should calculate in Eq. (4.5) a definite integral over the supermatrices Q. The structure of supermatrices Q is the same as in Refs. 7 and 8 and, in principle, the way how to compute the integral is clear. As usual, all manipulations are simpler for the unitary ensemble, and therefore let us start with this case.

However, before an explicit calculation of the integral in Eq. (4.5), an interesting observation can be made. We know that the 0D version of the σ model for Hermitian disordered systems can also be derived from random matrix models.[9] In fact, this is the way the equivalence between disordered systems in a limited volume and random matrix theory (RMT) was finally established. Now, a natural question arises: do the random models with a direction considered in the present work correspond to RMT?

Of course, this cannot be a model of Hermitian or real symmetric matrices, because in this case all eigenvalues must be real. Thus, one should think of ensembles of random real asymmetric or complex non-Hermitian matrices. Study of random complex matrices without the requirement of Hermiticity has started quite long ago[28] and since then models of non-Hermitian or real asymmetric random matrices have been considered in a number of publications.[13,14,29,30,32–34] The ensembles of real symmetric random matrices have found applications in, e.g., neural network dynamics,[33,34] while the ensembles of complex random matrices appear in study of dissipative quantum maps.[30,14] One of results obtained is that, for Gaussian ensembles in the limit of a large size N of the matrices, the eigenvalues are uniformly distributed inside an ellipse.[29,32,31]

Recently, an ensemble of ''weakly non-Hermitian'' random matrices X was introduced.[37] These matrices have the form

$$\hat{X}=\hat{A}+i\alpha N^{-1/2}\hat{B}, \tag{4.6}$$

with $N\times N$ statistically independent Hermitian matrices A and B, and a number α of the order of unity. The matrices $\hat{A}$ and $\hat{B}$ obeyed Gaussian distributions with the probability densities

$$\mathcal{P}(\hat{A})\propto\exp\left(-\frac{N}{2J^2}\mathrm{Tr}\hat{A}^2\right), \quad \mathcal{P}(\hat{B})\propto\exp\left(-\frac{N}{2J^2}\mathrm{Tr}\hat{B}^2\right) \tag{4.7}$$

where J has the order of unity.

The parameter $\alpha N^{-1/2}$ is a measure of the non-Hermiticity, and is always small for $N\to\infty$ and α finite. The authors of Ref. 37 calculated a density of complex eigenvalues similar to the function $P(\epsilon,y)$, Eq. (2.13), and demonstrated that this function has a finite limit when $N\to\infty$. At the same time they did not point out any direct physical applications. For computation of the function $P(\epsilon,y)$, they used the supersymmetry technique. Remarkably, the σ model derived in Ref. 37 is exactly the same (although numeration of elements of the matrix Q is somewhat different) as the unitary version of 0D σ model, Eq. (4.1). The preexponential is different, but this is natural because another (less direct) way of calculating the function $P(\epsilon,y)$ was used.

The same form of the σ model obtained for these two different models shows that the directed disordered model with broken time-reversal invariance in a finite volume is equivalent to the model of weakly non-Hermitian matrices. Apparently, the same equivalence holds between the time-reversal-invariant model of disorder and models of weakly nonsymmetric real matrices. However, it is relevant to emphasize that not every non-Hermitian Hamiltonian corresponds to the models of non-Hermitian or nonsymmetric real matrices. For example, models of open chaotic billiards are described by Hamiltonians with additional imaginary terms (see, e.g., Refs. 8 and 9). These Hamiltonians do not seem to be equivalent to the random matrix models of Ref. 37.

Now, let us show how explicit calculations in Eqs. (4.1) and (4.5) can be performed. First of all one should choose a proper parametrization of the supermatrices Q. The authors of Ref. 37 used the parametrization of Ref. 7 (''standard parametrization'' in the terminology of Ref. 8). This parametrization has been used for solving many interesting problems. However, due to the presence of new terms in the free energy $F[Q]$, Eq. (4.1), this parametrization is not as convenient as before,[8] because now $F[Q]$ contains not only the ''eigenvalues'' $\hat{\theta}$ but also many other variables.

As concerns the unitary ensemble, the computation of the function $P(\epsilon,y)$, although very lengthy, is still feasible.[37] At the same time, calculations for the orthogonal case using the standard parametrization do not seem to be possible at all due to unsurmountable technical problems.

Fortunately, one more parametrization is possible that is perfectly suitable for the present problem. To some extent it resembles the parametrization used to study the crossover between the orthogonal and unitary ensembles.[39,8] Of course, it should be written for the orthogonal and unitary ensembles in a different way, but the main structure is the same. Let us show in this section how the function $P(\epsilon,y)$ can be obtained for the unitary ensemble using this new parametrization (it can be called ''non-Hermitian parametrization''). The orthogonal ensemble will be considered in Sec. V.

The supermatrix Q in the non-Hermitian parametrization is written in the form

$$Q=TQ_0\overline{T}, \tag{4.8}$$

where T should be chosen to satisfy the relations $[T,\Lambda_1]=0$ and $T\overline{T}=1$. The bar stands for the ''charge conjugation'' defined in Refs. 7 and 8. It is clear that with such a choice the function $F[Q]$ depends on Q_0 only (for the unitary ensemble one has also $[Q_0,\tau_3]=0$).

The central part Q_0 in Eq. (4.8) is taken in the form

$$Q_0=\begin{pmatrix} \cos\hat{\varphi} & -\tau_3\sin\hat{\varphi} \\ -\tau_3\sin\hat{\varphi} & -\cos\hat{\varphi} \end{pmatrix}, \quad \hat{\varphi}=\begin{pmatrix} \varphi & 0 \\ 0 & i\chi \end{pmatrix}, \tag{4.9}$$

while the supermatrix T can be chosen as

$$T=\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} \cos(\hat{\theta}/2) & -i\sin(\hat{\theta}/2) \\ -i\sin(\hat{\theta}/2) & \cos(\hat{\theta}/2) \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}. \tag{4.10}$$

The supermatrices $\hat{\theta}$, u, and v are equal to

$$\hat{\theta}=\begin{pmatrix} \theta & 0 \\ 0 & i\theta_1 \end{pmatrix}, \tag{4.11}$$

$$u=\begin{pmatrix} 1-2\eta\overline{\eta} & 2\eta \\ -2\overline{\eta} & 1-2\overline{\eta}\eta \end{pmatrix}, \quad v=\begin{pmatrix} 1-2\kappa\overline{\kappa} & 2\kappa \\ -2\overline{\kappa} & 1-2\overline{\kappa}\kappa \end{pmatrix}.$$

The 2×2 matrices φ, χ, θ, and θ_1 are proportional to the unit matrix, the matrices η, κ are

$$\eta=\begin{pmatrix} \eta & 0 \\ 0 & -\eta^* \end{pmatrix}, \quad \kappa=\begin{pmatrix} \kappa & 0 \\ 0 & -\kappa^* \end{pmatrix} \tag{4.12}$$

where η, η^*, κ, and κ^* are anticommuting variables. The conjugate matrices $\overline{\eta}$ and $\overline{\kappa}$ are the same as in Refs. 7 and 8. To understand better the structure of the supermatrix Q given by Eqs. (4.8)–(4.12) it is instructive to write it neglecting all Grassmann variables. Then, one can write separately the compact and noncompact sectors. The compact sector takes the form

$$\begin{pmatrix} \cos\theta\cos\varphi & -\tau_3\sin\varphi+i\sin\theta\cos\varphi \\ -\tau_3\sin\varphi-i\sin\theta\cos\varphi & -\cos\theta\cos\varphi \end{pmatrix}, \tag{4.13}$$

whereas the noncompact sector is written as

$$\begin{pmatrix} \cosh\theta_1\cosh\chi & -i\tau_3\sinh\chi-\sinh\theta_1\cosh\chi \\ -i\tau_3\sinh\chi+\sinh\theta_1\cosh\chi & -\cosh\theta_1\cosh\chi \end{pmatrix}. \tag{4.14}$$

Comparing Eqs. (4.13) and (4.14) with the corresponding expressions for the supermatrix Q in the standard parametrization,[7,8] one can understand that in order to specify the supermatrix Q unambiguously the following inequalities should be imposed:

$$-\infty<\chi<\infty, \quad -\infty<\theta_1<\infty, \quad -\pi<\theta<\pi,$$
$$-\pi/2<\varphi<\pi/2. \tag{4.15}$$

To start the computation with the parametrization, Eqs. (4.8)–(4.12), one should first derive the proper Jacobian. The derivation is presented in the Appendixes. The final result for the elementary volume $[dQ]$ reads

$$[dQ]=J_\varphi J_\theta dR_B dR_F, \quad dR_B=d\theta\, d\theta_1 d\varphi\, d\chi,$$
$$dR_F=d\eta\, d\eta^* d\kappa\, d\kappa^* \tag{4.16}$$

where

$$J_\varphi=\frac{1}{8\pi}\frac{\cos\varphi\cosh\chi}{(\sinh\chi+i\sin\varphi)^2}, \tag{4.17}$$

$$J_\theta=\frac{1}{32\pi}\frac{1}{\sinh^2\frac{1}{2}(\theta_1+i\theta)}. \tag{4.18}$$

Substututing Eqs. (4.8)–(4.12) for Q in Eq. (4.1), one can rewrite the function $F[Q]$ in the limit $\tilde{\gamma}\to0$ as

$$F[Q]=a^2(\sinh^2\chi+\sin^2\varphi)-ix(\sinh\chi+i\sin\varphi) \tag{4.19}$$

(the limit $\tilde{\gamma}\to0$ can be taken in the beginning of the calculations, because in the present parametrization this does not

lead to additional convergence problems). The function $F[Q]$, Eq. (4.19), does not contain the anticommuting variables, and therefore one can easily integrate over the supermatrix u. Writing in Eq. (4.5) the supermatrix Q, Eqs. (4.8)–(4.10), as

$$Q = u\widetilde{Q}\overline{u}, \tag{4.20}$$

with u from Eq. (4.11), and integrating over η and η^*, one obtains

$$P(\epsilon,y) = \frac{\pi\nu}{4\Delta}\int [S\mathrm{Tr}(\tau_3\Lambda_1\widetilde{Q})]^2 \exp(-F[\widetilde{Q}])d\widetilde{Q}$$
$$= \frac{4\pi\nu}{\Delta}\frac{d^2}{dx^2}\int \exp(-F[Q])d\widetilde{Q}, \tag{4.21}$$

where the elementary volume $[d\widetilde{Q}]$ differs from $[dQ]$ by the replacement $dR_F \to d\widetilde{R}_F = d\kappa\, d\kappa^*$, and $F[Q]$ is given by Eq. (4.19). Although Eq. (4.21) is quite simple, one more difficulty should be overcome. The problem is that the integrand in Eq. (4.21) does not contain the variables κ and κ^* and, at first glance, the integral must turn to zero. However, the Jacobian J_θ, Eq. (4.18), is singular for θ, $\theta_1 \to 0$ and this singularity is not compensated for by the integrand. So one obtains an expression of the type $0\times\infty$, which is an usual phenomenon for the supersymmetry technique. Different procedures for how to make the integral well defined have been worked out (for a detailed discussion, see Ref. 8). The simplest way is to rewrite Eq. (4.5) as

$$P(\epsilon,y) = P_m(\epsilon,y) - \frac{\pi\nu}{4\Delta}\int A[Q][\exp(-F[Q])$$
$$-\exp(-F_m[Q])]dQ, \tag{4.22}$$

where

$$P_m(\epsilon,y) = -\frac{\pi\nu}{4\Delta}\int A[Q]\exp(-F_m[Q])dQ, \tag{4.23}$$
$$F_m[Q] - F[Q] = -mS\mathrm{Tr}(T\Lambda\overline{T\Lambda}) \equiv -mS\mathrm{Tr}(\Omega\Lambda).$$

The supermatrix Ω in Eq. (4.23) can be chosen as

$$\Omega = \widetilde{T}\Lambda\overline{\widetilde{T}}, \quad T = u\widetilde{T}. \tag{4.24}$$

The parameter m in Eqs. (4.22) and (4.23) is arbitrary. Using Eq. (4.10), we see that

$$-S\mathrm{Tr}(T\Lambda\overline{T\Lambda}) = 4(\cosh\theta_1 - \cos\theta), \tag{4.25}$$

and, thus, the singularity at $\theta_1 = \theta = 0$ coming from the Jacobian in Eq. (4.22) is compensated for by the integrand. After integration over η and η^*, the integrand does not contain the anticommuting variables κ and κ^*, and the integral vanishes. Therefore, the function $P_m(\epsilon,y)$, Eq. (4.23), does not depend on m, and one can calculate the integral in the limit $m\to\infty$. In this limit only small deviations of the supermatrix Ω from Λ are essential. Using the representation

$$\Omega = \Lambda(1+iW)(1-iW)^{-1}, \quad W = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & \sigma \\ \overline{\sigma} & ib \end{pmatrix} \tag{4.26}$$

expanding Ω in W up to quadratic terms and calculating the Jacobian in this approximation one can see that, in the limit $m\to\infty$,

$$\int \exp[mS\mathrm{Tr}(\Omega\Lambda)]d\Omega = 1\ . \tag{4.27}$$

The supermatrix $\widetilde{T}$ can also be represented through W, and calculating the corresponding Jacobian one may expand up to quadratic in W terms only. As concerns Q in the other terms in the integrand in Eq. (4.23), one should replace in the limit $m\to\infty$ the supermatrices $\widetilde{T}$ by 1. One can also check that now the Jacobian of the transformation from the matrices $\widetilde{T}$ and u to T equals -1 and not J_θ, as it did with the initial parametrization for T, Eq. (4.10). So, calculating the elementary volume $[dQ]$, one should omit in Eq. (4.16) the multiplier $J_\theta d\kappa d\kappa^*$ and change the sign of the rest.

As a result of all these manipulations one comes to the following expression for the function $P(\epsilon,y)$:

$$P(\epsilon,y) = -\frac{\pi\nu}{4\Delta}\int [S\mathrm{Tr}(\tau_3\Lambda_1 Q_0)]^2$$
$$\times\exp(-F[Q_0])J_\varphi d\varphi\, d\chi, \tag{4.28}$$

with Q_0 from Eq. (4.9) and J_φ from Eq. (4.17). The function $F[Q_0]$ is given by the right-hand side of Eq. (4.19). The limits of integration over φ and χ are determined in Eqs. (4.15).

The further calculation in Eq. (4.28) is very simple because the function in the preexponential is proportional to J_φ^{-1}. Changing the variables of integration $z = \sinh\chi$ and $t = \sin\varphi$, one calculates a Gaussian integral over z, and the final expression takes the form

$$P(\epsilon,y) = \frac{\nu\sqrt{\pi}}{a\Delta}\exp\left(-\frac{x^2}{4a^2}\right)\int_0^1 \cosh xt\, \exp(-a^2t^2)dt. \tag{4.29}$$

The function $P(\epsilon,y)$ is properly normalized, and one obtains, using Eq. (4.2),

$$\int P(\epsilon,y)dy = 1. \tag{4.30}$$

The density of complex eigenvalues $P(\epsilon,y)$, Eq. (4.29), agrees precisely with the corresponding function for weakly non-Hermitian random matrices obtained in Ref. 37 in the limit $N\to\infty$. The parameters a and Δ are related in this case to the parameters in Eq. (4.6) and (4.7) as

$$a = \sqrt{2}\pi J\nu(\epsilon)\alpha, \quad \Delta = [\nu(\epsilon)N]^{-1},$$
$$\nu(\epsilon) = (2\pi J)^{-1}\sqrt{4-(\epsilon/J)^2}, \tag{4.31}$$

and $x = 2\pi\nu(\epsilon)yN$.

The agreement can serve as a proof of the equivalence between the directed disorder models in a finite volume (with broken time-reversal invariance) and the models of non-Hermitian matrices defined by Eqs. (4.6) and (4.7). The function $P(\epsilon,y)$ is represented in Fig. 1, and let us discuss its basic properties following Ref. 37.

The density of complex eigenvalues is a smooth function at any finite a, which means that any finite non-Hermiticity

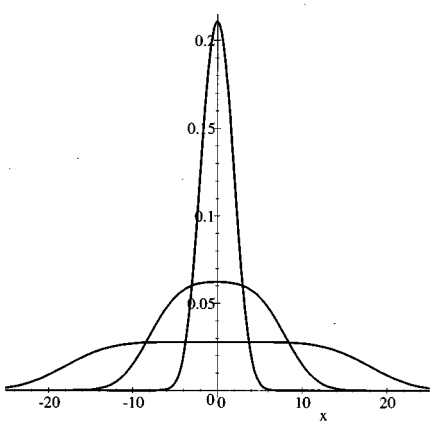

FIG. 1. The density of complex eigenenergies $P(\epsilon,y)$ for the unitary ensemble as a function of the imaginary part $x=2\pi y/\Delta$ for $a=1,2,3$.

smears all eigenenergies, making them complex. The probability of strictly real eigenvalues is negligible. For $a\gg 1$ the integral in Eq. (4.29) can be calculated analytically using the saddle-point method. In the interval $|x|<2a^2$ the integrand as a function of t has a sharp maximum in the domain of the integration, and the integral can be extended to infinity. For $|x|>2a^2$ the function P decays quickly. As a result, one obtains

$$P(\epsilon,y)\simeq\frac{\pi\nu(\epsilon)}{2a^2\Delta}\begin{cases}1, & |x|<2a^2\\ 0, & |x|>2a^2\end{cases}. \tag{4.32}$$

Equation (4.32) shows that for $a\gg 1$ the density of imaginary parts y of eigenvalues at a fixed real part is homogeneous in the interval $x\in(-2a^2,2a^2)$. Using Eq. (4.31) for $\nu(\epsilon)$ and a, we can rewrite the result expressed by Eq. (4.32) in terms of a distribution of eigenvalues in the complex plane. In such a formulation, Eq. (4.32) means that the complex eigenvalues are distributed homogeneously within the ellipse

$$\left(\frac{\epsilon}{2J}\right)^2+\left(\frac{y}{2Jv}\right)^2=1,\quad v=\alpha N^{-1/2}. \tag{4.33}$$

This is the ''elliptic law'' found in Refs. 29 and 31; the agreement is natural because the limit $a\gg 1$ should correspond to a ''strong'' non-Hermiticity. At the same time, the elliptic law is model dependent. For the models of disorder considered in the present paper the density of complex states essentially depends on y only.

In the opposite limit $a\ll 1$, the density of complex states $P(\epsilon,y)$ takes the form

$$P(\epsilon,y)\simeq\frac{\nu\sqrt{\pi}}{a\Delta}\exp\left(-\frac{x^2}{4a^2}\right). \tag{4.34}$$

The Gaussian form of the function P can be easily understood starting from the random matrix model, Eqs. (4.6) and (4.7). The function $P(\epsilon,y)$ can be written as

$$P(\epsilon,y)=N^{-1}\sum_{n=1}^{N}\langle\delta(\epsilon-\epsilon_n')\,\delta(y-\epsilon_n'')\rangle$$
$$=\frac{1}{2\pi N}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dk\;e^{iky}\langle\delta(\epsilon-\epsilon_n')\exp(-ik\epsilon_n'')\rangle, \tag{4.35}$$

where the angular brackets $\langle\cdots\rangle$ stand for the averaging over the matrices $\hat{A}$ and $\hat{B}$, Eq. (4.7). In the limit of small α the imaginary part ϵ_n'' can be obtained using the standard perturbation theory. In first order, one has

$$\epsilon_m''=\phi_m^*\hat{B}\phi_m, \tag{4.36}$$

where ϕ_m is the eigenvector of the matrix $\hat{A}$ corresponding to the eigenvalue ϵ_m'. Substituting Eq. (4.36) into Eq. (4.35), one can immediately average over the matrix $\hat{B}$. Using the orthogonality of the eigenvectors ϕ_m, one can write the result of the averaging as

$$P(\epsilon,y)=\frac{1}{2\pi N}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dk\;e^{iky}\Big\langle\delta(\epsilon-\epsilon_n')$$
$$\times\exp\left[-\frac{1}{2}\left(\frac{\alpha kJ}{N}\right)^2\right]\Big\rangle_A \tag{4.37}$$

where $\langle\cdots\rangle_A$ stands for averaging over $\hat{A}$. Integrating over k and using Eq. (4.31), one comes to Eq. (4.34). As concerns the models of disorder, Eqs. (2.1)–(2.3), even the asymptotics, Eqs. (4.32) and (4.37), have not been known before, and it not clear how to reproduce them using simple arguments.

Are the results obtained in this section general, can one expect anything new for the orthogonal ensemble? Of course, there is no reason to hope that Eq. (4.29) also describes the orthogonal ensemble, but are the asymptotics in the limits $a\gg 1$ and $a\ll 1$, Eqs. (4.32)–(4.34), still correct?

The orthogonal ensemble of random matrices can again be introduced by Eqs. (4.6) and (4.7) but now the matrices $\hat{A}$ and $\hat{B}$ should be real symmetric and antisymmetric, respectively. One should also make the replacement $\alpha\to-i\alpha$ in Eq. (4.6). As concerns the asymptotics in the limit $a\gg 1$, the same elliptic law as in Eq. (4.33) has been recovered.[32] At the same time, one can expect a completely different behavior for $a\ll 1$. This can be seen easily from the fact that the first order of the perturbation theory corresponding to Eq. (4.36) gives zero, and one cannot derive Eq. (4.34) as before. In fact, the density of complex eigenvalues $P(\epsilon,y)$ is singular at $y=0$. A study of the orthogonal ensemble is presented in Sec. V.

V. DENSITY OF COMPLEX EIGENVALUES IN A LIMITED VOLUME: ORTHOGONAL ENSEMBLE

To compute the density of complex eigenvalues $P(\epsilon,y)$ for the orthogonal ensemble, one can start, as previously, from Eqs. (4.1)–(4.4), but now one should use supermatrices Q with the structure corresponding to this case. As mentioned, the presence in Eq. (4.1) of the new term with the matrix Λ_1 makes the calculation very difficult even for the unitary ensemble, and hardly feasible at all for the orthogo-

nal one. So, as in Sec. IV, a new parametrization for Q should be designed.

Let us write the supermatrix Q in the forms

$$Q=ZQ_0\overline{Z}, \quad Z=TY, \tag{5.1}$$

with the supermatrices Q_0 and T specified by Eqs. (4.9)–(4.12), and choose the supermatrix Y as follows:

$$Y=Y_0RS, \quad Y_0=Y_3Y_2Y_1. \tag{5.2}$$

The supermatrix Y_1 entering Eq. (5.2) is

$$Y_1=\begin{pmatrix}\hat{w} & 0\\ 0 & \hat{w}\end{pmatrix}, \quad \hat{w}=\begin{pmatrix}w & 0\\ 0 & 1\end{pmatrix},$$
$$w=\begin{pmatrix}\cos(\mu/2) & -\sin(\mu/2)\\ \sin(\mu/2) & \cos(\mu/2)\end{pmatrix}. \tag{5.3}$$

The supermatrix Y_2 is equal to

$$Y_2=\begin{pmatrix}\cos(\hat{\theta}_2/2) & -i\,\sin(\hat{\theta}_2/2)\\ -i\,\sin(\hat{\theta}_2/2) & \cos(\hat{\theta}_2/2)\end{pmatrix},$$
$$\hat{\theta}_2=\begin{pmatrix}0 & 0\\ 0 & i\theta_2\tau_1\end{pmatrix}, \quad \tau_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{5.4}$$

The supermatrix Y_3 is

$$Y_3=\begin{pmatrix}\exp(i\hat{\beta}/2) & 0\\ 0 & \exp(i\hat{\beta}/2)\end{pmatrix}, \quad \hat{\beta}=\begin{pmatrix}\beta\tau_3 & 0\\ 0 & \beta_1\tau_3\end{pmatrix},$$
$$\tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{5.5}$$

The supermatrices R and S contain remaining Grassmann variables, and are written as

$$R=\begin{pmatrix}\hat{R} & 0\\ 0 & \hat{R}\end{pmatrix}, \quad \hat{R}=\begin{pmatrix}1-2\rho\overline{\rho} & 2\rho\\ -2\overline{\rho} & 1+2\rho\overline{\rho}\end{pmatrix},$$
$$\rho=\begin{pmatrix}\rho & 0\\ 0 & -\rho^*\end{pmatrix} \tag{5.6}$$

and

$$S=\begin{pmatrix}1-2\hat{\sigma}^2 & 2i\hat{\sigma}\\ 2i\hat{\sigma} & 1-2\hat{\sigma}^2\end{pmatrix}, \quad \hat{\sigma}=\begin{pmatrix}0 & \sigma\\ \overline{\sigma} & 0\end{pmatrix},$$
$$\sigma=\begin{pmatrix}\sigma & 0\\ 0 & -\sigma^*\end{pmatrix} \tag{5.7}$$

where $\overline{\rho}$ and $\overline{\sigma}$ are conjugate to ρ and σ.

The parametrization for Y, Eqs. (5.2)–(5.7), is chosen in such a way that $[Y,\Lambda_1]=0$. To specify the supermatrix Q unambiguously, one should restrict variations of the variables by certain intervals. This can be done as in Sec. IV by comparing the bosonic ''skeleton'' of Q written in the parametrization, Eqs. (5.1)–(5.7) (let us call it ''nonsymmetric parametrization'') with the standard parametrization of Refs. 7 and 8. As a result, one can write the following inequalities:

$$0<\chi<\infty, \quad -\pi/2<\varphi<\pi/2, \quad -\infty<\theta_1<\infty,$$
$$-\pi<\theta<\pi,$$
$$0<\theta_2<\infty, \quad 0<\mu<\pi, \quad 0<\beta<\pi, \quad 0<\beta_1<2\pi. \tag{5.8}$$

The next step is to calculate the Jacobian. The derivation is presented in Appendixes, and the final result for the elementary volume $[dQ]$ is

$$[dQ]=J_\varphi J_\theta J_\mu J_c dR_B dR_F dR_{1B} dR_{1F}. \tag{5.9}$$

In Eq. (5.9), J_φ, J_θ, dR_B, and dR_F are given by Eqs. (4.16)–(4.18). The additional quantities entering Eq. (5.9) are equal to

$$J_\mu=\frac{1}{2^8\pi^2}\,\frac{\sinh\theta_2\sin\mu}{(\cosh\theta_2-\cos\mu)^2}, \tag{5.10}$$

$$J_c=\frac{4\sin^2\varphi}{(\sinh\chi-i\,\sin\varphi)^2}, \tag{5.11}$$

and

$$dR_{1B}=d\mu\; d\theta_2 d\beta\; d\beta_1, \quad dR_{1F}=d\sigma\; d\sigma^* d\rho\; d\rho^*\,. \tag{5.12}$$

The free energy $F[Q]$, Eq. (4.1), takes, in the limit $\gamma\to 0$, the following form:

$$F[Q]=a^2(\sin^2\varphi+\sinh^2\chi)+x[(\cos\mu\;\sin\varphi$$
$$-i\;\cosh\theta_2\;\sinh\chi)+4(\sigma\sigma^*+\rho\rho^*)(\cosh\theta_2-\cos\mu)$$
$$\times(\sin\varphi-i\;\sinh\chi)]. \tag{5.13}$$

The nonsymmetric parametrization given by Eqs. (5.1)–(5.12) looks rather complicated. The calculation of the Jacobian is most lengthy, but this has to be done only once. At the same time, the Jacobian does not contain Grassmann variables, and the free energy $F[Q]$, Eq. (5.13), is simple enough. Moreover, the supermatrix Q can be written as in Sec. IV in the form of Eq. (4.20) (although the supermatrix $\widetilde{Q}$ is now different from that for the unitary ensemble). This allows one to integrate first over the matrix u and obtain Eq. (4.21).

Further simplifications come from the fact that, as previously, one obtains an uncertainty of the type $0\times\infty$ because the integrand in Eq. (4.21) does not contain the variables κ and κ^*, whereas the Jacobians J_θ, Eq. (4.18), and J_μ, Eq. (5.10), are singular at θ, θ_1, θ_2, $\mu\to 0$. We saw in Sec. IV that the uncertainties can be rather easily avoided and, as a result, one obtains a more simple integral. The ''regularization'' procedure, Eqs. (4.22)–(4.27), led to the integral, Eq. (4.28), that contained the variables φ and χ only.

Similar transformations can be performed for the orthogonal ensemble. Proceeding as for the unitary ensemble, let us introduce the function $F_{mn}[Q]$,

$$F_{mn}=F[Q]-mS\mathrm{Tr}(\Lambda T\Lambda\overline{T})-nS\mathrm{Tr}(\tau_3 Y\tau_3\overline{Y}). \tag{5.14}$$

The second term in Eq. (5.14) can also be written in the form of Eq. (4.25). Using Eqs. (5.2)–(5.7) we can write the third term as

$$-nS\mathrm{Tr}(\tau_3 Y \tau_3 \overline{Y}) = 4n(\cosh\theta_2 - \cos\mu). \quad (5.15)$$

In analogy with the transformation of the integrand in Eqs. (4.22) and (4.23), we represent $\exp(-F[Q])$ as follows:

$$e^{-F} = e^{-F_{mn}} + e^{-F_{m0}}(1-e^{-F^{(n)}}) + e^{-F_{0n}}(1-e^{-F^{(m)}}) + e^{-F}(1-e^{-F^{(m)}})(1-e^{-F^{(n)}}), \quad (5.16)$$

where

$$F^{(m)} = F_{mn} - F_{0n}, \quad F^{(n)} = F_{mn} - F_{m0}.$$

The parameters m and n in Eqs. (5.15) and (5.16) are arbitrary. Therefore, substituting Eq. (5.16) into Eq. (4.5), we can take the limit $m, n \to \infty$. The contribution coming from the last term in Eq. (5.16) vanishes because all singularities are compensated for any m and n, but the integrand does not contain the anticommuting variables κ and κ^*. The limit $m \to \infty$ allows one to expand the supermatrix $\widetilde{T}$, Eq. (4.24), near 1 (and the supermatrix Ω near Λ). As explained in Sec. IV, in the limit $m \to \infty$ one can replace $\widetilde{T} \to 1$ everywhere in the integrand omitting simultaneously $J_\theta d\kappa\, \kappa^*$ in the elementary volume $[dQ]$. The same is correct now and one should remove $J_\theta d\kappa\, d\kappa^*$ from $[dQ]$, Eq. (5.9) (changing the sign).

The other singularity at $\theta_2, \mu \to 0$ in the first and third terms in Eq. (5.16) can be avoided in a similar way. In the limit $n \to \infty$ the supermatrix Y, Eqs. (5.2)–(5.7), is also close to 1. To make an expansion in small deviations Y from 1 one can use the following parametrization:

$$Y = (1-iX)(1+iX)^{-1}, \quad X = \begin{pmatrix} i\hat{A} & \hat{L} \\ \hat{L} & i\hat{A} \end{pmatrix}. \quad (5.17)$$

The blocks $\hat{A}$ and $\hat{L}$ satisfy the constraints $\overline{A} = -A$, $\overline{L} = L$, $\{A, \tau_3\} = 0$, and $\{L, \tau_3\} = 0$, where $\{\cdots\}$ is an anticommutator. These blocks can be written in an explicit form as

$$\hat{A} = \begin{pmatrix} f & \xi \\ -\overline{\xi} & 0 \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & \zeta \\ \overline{\zeta} & il \end{pmatrix}, \quad (5.18)$$

where the 2×2 matrices f and l contain conventional complex numbers f and l, whereas ξ and ζ consist of anticommuting variables ξ and ζ. The explicit forms of these matrices are

$$f = \begin{pmatrix} 0 & -f \\ f^* & 0 \end{pmatrix}, \quad l = \begin{pmatrix} 0 & l \\ l^* & 0 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 0 & \zeta \\ -\zeta^* & 0 \end{pmatrix}. \quad (5.19)$$

In Eq. (5.19), l is an arbitrary complex number, while for f one should integrate over the domain $\mathrm{Im}\, f > 0$. The structure of ξ is the same as of ζ.

Substituting Eqs. (5.17)–(5.19) into Eq. (5.14), one should expand the term $S\mathrm{Tr}(\tau_3 Y \tau_3 \overline{Y})$ up to quadratic terms in X and replace Y by 1 everywhere else in the integrand. Calculating the Jacobian we can see that the factor $J_\mu dR_{1B} dR_{1F}$ should be replaced by 1. Of course, this concerns only the first and the third terms in Eq. (5.16) because the second term does not lead to any singularity in the integrand at $\theta_2 = \mu = 0$. In fact, the contribution from the third term in Eq. (5.16) is zero because it is not singular at $\theta = \theta_1 = 0$ and does not contain the variables κ and κ^*. At the same time, the singularity at $\theta_2 = \mu = 0$ is avoided by using the parametrization, Eqs. (5.17)–(5.19).

As a result of these manipulations one should replace finally Eq. (4.5) by

$$P(\epsilon, y) = P^{(1)}(\epsilon, y) + P^{(2)}(\epsilon, y), \quad (5.20)$$

$$P^{(1)}(\epsilon, y) = -\frac{\pi\nu}{4\Delta} \lim_{m,n\to\infty} \int A[Q] \exp(-F_{mn}[Q])\, dQ, \quad (5.21)$$

$$P^{(2)}(\epsilon, y) = -\frac{\pi\nu}{4\Delta} \lim_{m,n\to\infty} \int A[Q][\exp(-F_{m0}) - \exp(-F_{mn})]\, dQ. \quad (5.22)$$

The integrand in Eq. (5.21) has both singularities. Therefore, one has to replace $\widetilde{T}$ and Y by 1 simultaneously everywhere in the integrand, replacing $J_\theta J_\mu d\kappa d\kappa^* dR_{1B} dR_{1F}$ in the elementary volume $[dQ]$, Eq. (5.9), by -1. As concerns Eq. (5.22), the integrand has only the singularity at $\theta = \theta_1 = 0$, and one should replace by 1 the supermatrix $\widetilde{T}$ only. In the elementary volume $J_\theta d\kappa d\kappa^*$ should be replaced by -1.

The subsequent manipulations are rather straightforward. Integrating over the supermatrix u, one obtains for $P^{(1)}(\epsilon, y)$ and $P^{(2)}(\epsilon, y)$ analogs of Eq. (4.21). Then the function $P^{(1)}(\epsilon, y)$ is expressed in terms of the integral over the variables $t = \sin\varphi$ and $z = \sinh\chi$,

$$P^{(1)}(\epsilon, y) = \frac{\nu}{4\Delta} \frac{d^2}{dx^2} \int e^{-a^2(t^2+z^2) - x(t-iz)} \frac{4t^2 dt\, dz}{(t^2+z^2)^2}. \quad (5.23)$$

In the integral in Eq. (5.22), one has to integrate first over the variables ρ, ρ^*, σ, and σ^*, and then, the function $P^{(2)}(\epsilon, y)$ reduces to

$$P^{(2)}(\epsilon, y) = \frac{\nu}{4\Delta} \frac{d^2}{dx^2} \int e^{-a^2(t^2+z^2) - x(t\omega - i\lambda z)} \times \frac{(t-iz)^2 x^2 t^2}{(t^2+z^2)^2} dt\, dz\, d\omega\, d\lambda \quad (5.24)$$

where $\omega = \cos\mu$, and $\lambda = \cosh\theta_2$. The integration in Eqs. (5.23) and (5.24) is performed over t and z in the intervals $-1 < t < 1$ and $-\infty < z < \infty$, and over ω and λ in the intervals $-1 < \omega < 1$ and $1 < \lambda < \infty$.

The integration over ω and λ in Eq. (5.24) can be carried out immediately. However, to provide the convergence of the integral over λ one should shift the contour of integration over z into the complex plane $z \to z + i\delta\, \mathrm{sgn}(x)$, where δ is an infinitesimal positive number and

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

Integrating over ω and λ and adding Eqs. (5.23) and (5.24), we obtain, for $P(\epsilon, y)$, Eq. (5.20),

$$P(\epsilon,y)=\frac{\nu}{4\Delta}\frac{d^2I(x)}{dx^2}, \tag{5.25}$$

$$I(x)=\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-a^2(t^2+z_-^2)}[e^{-x(t-iz_-)}(t+iz_-)^2 - e^{x(t+iz_-)}(t-iz_-)^2]\frac{t}{iz_-}\frac{dt\ dz}{(t^2+z_-^2)^2}, \tag{5.26}$$

where $z_-=z+i\delta\ \mathrm{sgn}(x)$.

As is clear from the form of the function $I(x)$, it is convenient to differentiate first over x and then calculate the integral. However, one should be careful performing this, at first glance trivial, manipulation. The problem is that z_- contains x, which can result in an additional contribution.

To avoid lengthy calculations let us consider first the case when x is finite nonzero number. Then, the derivatives dz_-/dx and d^2z_-/dx^2 vanish, and one has to differentiate the exponentials only. Shifting the contour of integration $z\to z+(ix/2a^2)$, which can be done without crossing singularities in the complex plane and changing the new variable z as $z\to z/a$, one obtains

$$P_c(\epsilon,y)=\frac{\nu}{a\Delta}\exp\left(-\frac{x^2}{4a^2}\right)\int_0^1 xt\ \sinh xt \times\exp(-a^2t^2)dt\int_0^\infty\frac{\exp(-z^2)dz}{z^2+\frac{x^2}{4a^2}} \tag{5.27}$$

[the variables x and y are related to each other through Eq. (4.2)].

Equation (5.27) holds for any finite x, but is it the final result? It would be the final result if the density function were continuous at $x=0$. As concerns the unitary ensemble, we already know that the function $P(\epsilon,y)$ is continuous [see Eq. (4.29)], but does the continuity follow from a physical principle? In fact, it does not and the function $P(\epsilon,y)$ for the orthogonal ensemble contains a δ function at $x=0$.

To extract the δ function let us expand the exponentials in the integrand in Eq. (5.26). In the first two orders one obtains

$$P(\epsilon,y)\simeq\frac{\nu}{2\Delta}\frac{d^2}{dx^2}\left[\int_{-1}^{1}\int_{-\infty}^{\infty}t^2e^{-a^2(t^2+z_-^2)}\left[\frac{2}{(t^2+z_-^2)^2} - \frac{x}{iz_-}\frac{1}{t^2+z_-^2}\right]dt\ dz\right]. \tag{5.28}$$

The first term in the integrand in Eq. (5.28) has no singularities, and one can shift the contour of the integration over z such that the variables z_- are replaced by z. Then this part of the integrand does not contain x, and the differentiation gives zero. The contribution involved comes from the second term in the integrand. Writing z_-^{-1} as

$$\frac{1}{z_-}=\frac{z-i\delta\ \mathrm{sgn}(x)}{z^2+\delta^2},$$

one can represent the function $P(\epsilon,y)$ for $x\to0$ as

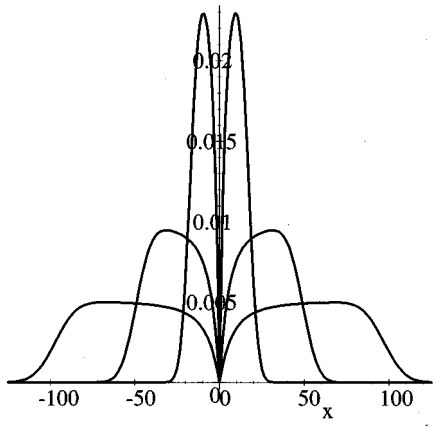

FIG. 2. The density of complex eigenenergies (with nonzero imaginary part) $P_c(\epsilon,y)$ for the orthogonal ensemble as a function of the imaginary part $x=2\pi y/\Delta$ for $a=3,5,7$.

$$P(\epsilon,y)_{x\to0}=\frac{\nu}{2\Delta}\frac{d^2}{dx^2}\lim_{\delta\to0}\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-a^2(t^2+z^2)} \times\frac{t^2|x|}{t^2+z^2}\frac{\delta}{z^2+\delta^2}dt\ dz.$$

The integration over z in the limit $\delta\to0$ is elementary, and for the anomalous contribution $P_r(\epsilon,y)$ one obtains the following expression:

$$P_r(\epsilon,y)=\frac{2\pi\nu}{\Delta}\delta(x)\int_0^1\exp(-a^2t^2)dt. \tag{5.29}$$

By making simple transformations in Eq. (5.27), the final result for the density of complex eigenvalues $P(\epsilon,y)$ can be written as

$$P(\epsilon,y)=P_r(\epsilon,y)+P_c(\epsilon,y), \tag{5.30}$$

where $P_r(\epsilon,y)$ is given by Eq. (5.29), and $P_c(\epsilon,y)$ equals

$$P_c(\epsilon,y)=\frac{2\pi\nu}{\Delta}\frac{1}{2}\Phi\left(\frac{|x|}{2a}\right)\int_0^1 t\ \sinh(|x|t)\exp(-a^2t^2)dt \tag{5.31}$$

where $\Phi(v)=(2/\sqrt{\pi})\int_v^\infty\exp(-u^2)du$. It is not difficult to check that the function $P(\epsilon,y)$, Eqs. (5.29)–(5.31), satisfies the normalization condition, Eq. (4.30), and the singular part $P_r(\epsilon,y)$ gives an essential contribution that becomes small only in the limit $a\to\infty$. The function $P_c(\epsilon,y)$ is represented in Fig. 2.

The existence of the anomalous part $P_r(\epsilon,y)$, Eq. (4.33), means that a finite fraction of all eigenvalues remains real for any imaginary vector potential h in the models of disorder, Eqs. (2.1) and (2.3), or degree of asymmetry α for the real random matrix models. At the same time, the function $P_c(\epsilon,y)$ decays when $y\to0$, which corresponds to a vanishing probability of eigenstates with small but nonzero imaginary parts.

In contrast to the unitary ensemble, the function $P(\epsilon,y)$ for $a\ll1$ can hardly be obtained from a perturbation theory. Most of the eigenvalues are in this case real. In the opposite

limit $a \gg 1$ one should distinguish between several regions. In the limit $|x| \ll a$ the asymptotics is determined by the expression

$$P_c(\epsilon,y) \simeq \frac{\pi\nu}{2a^2\Delta} \frac{\sqrt{\pi}|x|}{2a}, \tag{5.32}$$

showing a linear decay of the density as $|x| \to \infty$.

In the region $|x| \gg 2a$ the density of complex eigenvalues is constant for $|x| < 2a^2$, and falls off outside this interval. Its value in this region is the same as in the unitary case, Eq. (4.32). This corresponds to the elliptic law, Eq. (4.33). For an ensemble of strongly asymmetric real random matrices with a Gaussian distribution, this law was proven in Refs. 31 and 32. The authors of this paper have also found numerically that the portion of real eigenvalues for their ensemble decays as $N^{-1/2}$, where N is the size of the matrices. Apparently, this behavior corresponds to the δ-functional part $P_r(\epsilon,y)$, Eq. (5.29), in the eigenvalue density for the case of weak asymmetry [the orthogonal analog of Eqs. (4.7)].

VI. DISCUSSION

The results presented in the previous sections demonstrate that disorder models with a direction are interesting, and can be efficiently studied using the supersymmetry technique. The σ model derived, Eq. (3.29), can be used in any dimension. It is relevant to emphasize that, as usual,[7,8] the dimensionality is determined by the geometry of the sample. So the one-dimensional version of the σ model corresponds to a thick wire with a directed hopping. In the language of vortices in a superconductor,[18] the 1D model can describe the vortices in a slab with line defects and the magnetic field parallel to the surface. Such a model is somewhat more realistic than the purely 1D model of Ref. 18. The 2D σ model is supposed to describe the vortices in a bulk superconductor with line defects. In addition, one can imagine a situation when the sample is long but has a small cross section. If the line defects are aligned in the longitudinal direction, one comes to the 0D σ model considered in the present paper.

Of course, the directed non-Hermitian Hamiltonians can arise not only from the vortex model but also correspond to nonequilibrium processes. A very interesting possibility is the directed hopping model, Eq. (2.3), that can be considered as a quantum counterpart of the directed percolation model.[21] Applications to other physical systems that can be reduced to models of a disorder with a direction also deserve an attention. The problem of turbulence is one of most famous. The main features of the turbulence are believed to be described by the Burgers equation.[22,23,40] Reduction of the Burgers equation to a linear equation allows one to use well-developed methods of disorder physics. A similarity of the linear equation to equations used in a study of problems of directed polymers have already inspired application of the replica method to study the problem of turbulence.[25] Use of supersymmetry for the problems of the turbulence might be one more interesting direction of research.

Leaving these interesting problems for future study, let us summarize the results obtained in the present work. The σ model, Eq. (3.29), differs from the σ models used in the localization and mesoscopic problems[7,8] by the term with the matrix Λ_1. Although the Hamiltonians with the direction of Eqs. (2.1) and (2.3) can be obtained from conventional Hermitian Hamiltonians in a magnetic field by the formal replacement $\mathbf{A} \to i\mathbf{h}$, the same replacement in the conventional σ models would not lead to Eq. (3.29). This reflects an essential symmetry difference between systems in a magnetic field where the time-reversal invariance is broken, and the models with direction that are time reversal invariant.

In contrast to average density of states for Hermitian disorder problems which is always smooth, the joint probability density of complex eigenenergies considered in the previous sections is a nontrivial quantity. The σ model was derived to describe this quantity, and it is expected to be sensitive to localization-delocalization transitions in one- and higher-dimensional systems.[18]

The form of the 0D version of the σ model obtained above demonstrates the equivalence between the directed disorder models in a limited volume and ensembles of random weakly non-Hermitian or weakly asymmetric real matrices that have been mapped onto the 0D σ model previously.[37] Complex random non-Hermitian matrices appear in a study of dissipative quantum maps,[30,14] whereas random real asymmetric matrices have applications in neural network dynamics.[33,34] Thus the σ model can describe completely different phenomena in an unified manner.

The supermatrix σ model can serve as a useful calculational tool for all these non-Hermiatian problems. Although the new term with the matrix Λ_1 in the σ model, Eq. (3.29), makes the use of previous parametrizations[8] difficult, the new parametrization suggested in the present paper allows one to circumvent the difficulties and obtain explicit results for the 0D case in a straightforward manner. Weakly non-Hermitian random matrices can also be studied using more traditional methods of orthogonal polynomials.[41] However, a study of weakly non-symmetric real matrices with this method seems be more difficult and the density of complex eigenvalues, Eqs. (5.29)–(5.31), has been calculated for the first time. In addition, the σ model approach is not dependent on details of the model considered, and can be applied not only to Gaussian models. It can also be used to study the directed models in one and higher dimensions, where one can expect localization-delocalization transitions.

Equations (5.29)–(5.31) demonstrate that at any finite disorder and ''imaginary vector potential'' a finite portion of eigenvalues remain real, whereas this does not occur if the time-reversal invariance is broken [Eq. (4.29)]. This phenomenon has manifested itself in numerical study of different models. In Refs. 31 and 32 ensembles of random strongly asymmetric matrices (symmetric and antisymmetric parts had the same order of magnitude) were considered. It was found that the fraction of real eigenvalues decayed as $N^{-1/2}$ for large matrix sizes N. Apparently, this corresponds to the finite fraction of the real eigenvalues $P_r(\epsilon,y)$, Eq. (5.29), because in the ensemble of weakly nonsymmetric matrices involved the magnitude of the antisymmetric part of the random matrices is $N^{1/2}$ times smaller than that of the symmetric one.

A finite fraction of real eigenenergies was found in a numerical study of the 2D model, Eq. (2.3), (without magnetic interactions) near the center of the band.[18] Although the 2D case was not considered in the present paper, and nothing

can be said about a possibility of a mixture of eigenstates with real and complex eigenvalues, one can argue that perhaps the parameters of the model of Ref. 18 corresponded to the 0D case. This might easily happen because the localization length in weakly disordered 2D systems is exponentially large and can exceed the sample size, which would correspond to the 0D regime. If this is really so, the results of the present study are in an agreement with the numerical investigation.

The phenomenon that some finite portion of eigenvalues lies on a certain line in the complex plane occurs also in other models with a randomness. Recently, it was found that a finite fraction of all roots of random self-inversive polynomials lies on the unit circle.[42] At the same time, if the polynomials are not self-inversive the density of complex roots is smooth everywhere in the complex plane. It is clear from the preceding discussion that the directed disorder models deserve further investigation.

APPENDIX A: NON-HERMITIAN PARAMETRIZATION (UNITARY ENSEMBLE)

Let us calculate for the unitary ensemble the Jacobian for the parametrization given by Eqs. (4.8)–(4.12) (it was suggested to call it ''non-Hermitian parametrization''). As usual,[7,8] it is convenient to consider the length $S\mathrm{Tr}(dQ)^2$. With Eq. (4.8), it can be written as

$$S\mathrm{Tr}(dQ)^2 = S\mathrm{Tr}\{(dQ_0)^2 + [\delta T, Q_0]^2 + 4\,\delta T\,\delta Q_0\}, \tag{A1}$$

where $\delta T = \overline{T}\,dT$, $\delta Q_0 = Q_0 dQ_0$ and $[\cdots]$ is the commutator.

It is easy to see from Eq. (4.9) that

$$\delta Q_0 = \begin{pmatrix} 0 & -\tau_3 d\hat{\varphi} \\ \tau_3 d\hat{\varphi} & 0 \end{pmatrix}, \tag{A2}$$

and hence

$$\{\delta Q_0, \Lambda_1\} = 0 \tag{A3}$$

where $\{\cdots\}$ is the anticommutator.

Then, using the relation $[\delta T, \Lambda_1] = 0$ and Eq. (A3) we obtain

$$S\mathrm{Tr}(\delta T\,\delta Q_0) = S\mathrm{Tr}(\Lambda_1 \delta T\,\delta Q_0 \Lambda_1) = -S\mathrm{Tr}(\delta T\,\delta Q_0) = 0 \tag{A4}$$

which shows that Jacobians are the product of Jacobians corresponding to dQ_0 and δT. As concerns dQ_0, we have

$$S\mathrm{Tr}(dQ_0)^2 = 4[(d\varphi)^2 + (d\chi)^2]. \tag{A5}$$

Writing Eq. (4.10) as

$$T = uT_0 v, \tag{A6}$$

one obtains

$$\delta T = \overline{v\,\overline{T}_0}\,\delta u T_0 v + \overline{v}\,\overline{\delta T}_0 v + \delta v, \tag{A7}$$

where, with Eq. (4.11) and (4.12)

$$\delta T_0 = -\frac{i}{2}\begin{pmatrix} 0 & d\hat{\theta} \\ d\hat{\theta} & 0 \end{pmatrix}, \tag{A8}$$

$$\delta u = \delta u_{\parallel} + \delta u_{\perp},$$

$$\delta u_{\parallel} = 2\tau_3(\eta\, d\eta^* - d\eta\, \eta^*), \quad \delta u_{\perp} = 2\begin{pmatrix} 0 & d\eta \\ -d\overline{\eta} & 0 \end{pmatrix},$$

and similar equations can be written for δv.

Substituting Eqs. (4.11), (4.12), and (A8) into Eq. (A7) one can represent the supermatrix δT as

$$\delta T = \delta T^{\parallel} + \delta T^{\perp}, \tag{A9}$$

$$\begin{aligned}\delta T^{\parallel} = {} & 2\cos\frac{\theta - i\theta_1}{2}\begin{pmatrix} 0 & d\eta \\ -d\overline{\eta} & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & d\kappa \\ -d\overline{\kappa} & 0 \end{pmatrix} \\ & + 2\tau_3(\eta\, d\eta^* - d\eta\, \eta^* + \kappa\, d\kappa^* - d\kappa\, \kappa^*) \\ & + 4\tau_3 \cos\frac{\theta - i\theta_1}{2}(\kappa^* d\eta - d\eta^* \kappa),\end{aligned} \tag{A10}$$

$$\begin{aligned}\delta T^{\perp} = {} & i\Lambda_1\Bigg[2\sin\frac{\theta - i\theta_1}{2}\begin{pmatrix} 0 & d\eta \\ d\overline{\eta} & 0 \end{pmatrix} - (d\theta - i d\theta_1)\begin{pmatrix} 0 & \kappa \\ \overline{\kappa} & 0 \end{pmatrix} \\ & - \frac{1}{2}\begin{pmatrix} d\theta(1 - 4\kappa\kappa^*) & 0 \\ 0 & i d\theta_1(1 + 4\kappa\kappa^*) \end{pmatrix} \\ & + 4\sin\frac{\theta - i\theta_1}{2}(\kappa^* d\eta + d\eta^* \kappa)\Bigg].\end{aligned} \tag{A11}$$

In Eqs. (A9)–(A11), $\delta T^{\parallel}$ commutes with Λ, and $\delta T^{\perp}$ anticommutes with Λ. The second line in Eq. (A10) does not contribute to $[\delta T, Q_0]$ in Eq. (A1). In Eqs. (A10) and (A11), one can change the variables

$$d\theta(1 - 4\kappa\kappa^*) \to d\theta, \quad d\theta_1(1 + 4\kappa\kappa^*) \to d\theta_1, \tag{A12}$$

and make the shifts

$$\begin{aligned}\frac{1}{2}\begin{pmatrix} d\theta & 0 \\ 0 & i d\theta_1 \end{pmatrix} \to {} & \frac{1}{2}\begin{pmatrix} d\theta & 0 \\ 0 & i d\theta_1 \end{pmatrix} + 4(\kappa^* d\eta + d\eta^* \kappa) \\ & \times \sin\frac{\theta - i\theta_1}{2},\end{aligned} \tag{A13}$$

$$2\sin\frac{\theta - i\theta_1}{2} d\eta \to 2\sin\frac{\theta - i\theta_1}{2} d\eta + (d\theta - i d\theta_1)\kappa, \tag{A14}$$

$$d\kappa \to d\kappa - \cos\frac{\theta - i\theta_1}{2} d\eta. \tag{A15}$$

The transformations, Eqs. (A12)–(A15), do not change the Jacobian, and $\delta T^{\parallel}$ and $\delta T^{\perp}$ take more simple forms,

$$\delta T^{\parallel} = \mathbf{1}\left[i\tau_3\begin{pmatrix} dc & 0 \\ 0 & dc \end{pmatrix} + 2\begin{pmatrix} 0 & d\kappa \\ -d\overline{\kappa} & 0 \end{pmatrix}\right], \tag{A16}$$

$$\delta T^{\perp}=i\Lambda_1\left[-\frac{1}{2}\begin{pmatrix} d\theta & 0 \\ 0 & id\theta_1\end{pmatrix}+2\sin\frac{\theta-i\theta_1}{2}\begin{pmatrix} 0 & d\eta \\ d\overline{\eta} & 0\end{pmatrix}\right], \tag{A17}$$

where $i\tau_3 dc$ is the second line of Eq. (A10), and **1** is the unit 8×8 matrix.

Further computation is already simple. Changing once more

$$\sin\frac{\theta-\theta_1}{2}d\eta\to d\eta, \tag{A18}$$

one obtains a contribution to the Jacobian proportional to J_θ, Eq. (4.18). Writing the second term in Eq. (A1) through the new variables we have

$$\begin{aligned} S\mathrm{Tr}[\delta T,Q_0]^2=4[(d\theta)^2\cos^2\varphi+(d\theta_1)^2\cosh^2\chi] \\ +128\left(\cos^2\frac{\varphi+i\chi}{2}d\eta\, d\eta^* \right. \\ \left. +\sin^2\frac{\varphi-i\chi}{2}d\kappa\, d\kappa^*\right). \end{aligned} \tag{A19}$$

Equations (A5) and (A19) lead to the elementary volume $[dQ]$, Eq. (4.16).

APPENDIX B: NONSYMMETRIC PARAMETRIZATION (ORTHOGONAL ENSEMBLE)

To calculate the Jacobian of the parametrization, Eqs. (5.1)–(5.7), for the orthogonal ensemble we can use the results obtained for the unitary ensemble, because Eq. (5.1) contains the same supermatrices Q_0 and T, as previously. However, the presence of the supermatrix Y makes the computation quite lengthy. The length $S\mathrm{Tr}(dQ)^2$ is written as in Appendix A,

$$S\mathrm{Tr}(dQ)^2=S\mathrm{Tr}[(dQ_0)^2+[\delta Z,Q_0]^2+4\delta Z\,\delta Q_0], \tag{B1}$$

where $\delta Z=\overline{Z}\,dZ$ can be written as

$$\delta Z=\overline{S\,R(Y_0}\delta TY_0+\delta Y_0+dR\,\overline{R}+R\,dS\,\overline{S\,R)}RS. \tag{B2}$$

The last term in Eq. (B1) is equal to zero [see Eq. (7.4)]. As concerns the supermatrix δT, it can be written after the replacements, Eqs. (A12)–(A15), in the form of Eqs. (A16) and (A17). So, one has to calculate the other differentials entering Eq. (B2). Using Eq. (5.7) one can rewrite $dS\,\overline{S}$ in the form

$$\begin{aligned} dS\,\overline{S}=(dS\,\overline{S})_{\parallel}+(dS\,\overline{S})_{\perp}, \\ (dS\,\overline{S})_{\parallel}=2\tau_3\mathbf{1}(d\sigma\,\sigma^*-\sigma\,d\sigma^*),\quad (dS\,\overline{S})_{\perp}=2i\Lambda_1 d\hat{\sigma}. \end{aligned} \tag{B3}$$

Taking the supermatrix R from Eq. (5.6), one can derive

$$R\,dS\,\overline{SR}=R(dS\,\overline{S})_{\perp}\overline{R}+(dS\,\overline{S})_{\parallel}, \tag{B4}$$

$$R(dS\,\overline{S})_{\perp}\overline{R}=2i\Lambda_1 d\hat{\sigma}+4i\Lambda_1(d\sigma\,\rho^*+\rho\,d\sigma^*), \tag{B5}$$

and

$$dR\,\overline{R}=\mathbf{1}\left[2\begin{pmatrix} 0 & d\rho \\ -d\overline{\rho} & 0\end{pmatrix}+2\tau_3(d\rho\,\rho^*-\rho\,d\rho^*)\right]. \tag{B6}$$

Now we have to calculate δY_0. Using Eqs. (5.3)–(5.5) one can represent this differential in the form

$$\delta Y_0=\delta Y_1+\delta Y_2+\overline{Y_1Y_2}\delta Y_3Y_2Y_1. \tag{B7}$$

Calculating the matrices δY_1, δY_2, and δY_3 we rewrite δY_0 as follows

$$\begin{aligned} \delta Y_0=\mathbf{1}\frac{i}{2}\left[\begin{pmatrix} d\beta\,\overline{w}\tau_3 w & 0 \\ 0 & d\beta_1\tau_3\cosh\theta_2\end{pmatrix}-d\mu\begin{pmatrix}\tau_2 & 0\\ 0 & 0\end{pmatrix}\right] \\ +\Lambda_1\frac{1}{2}\left[-\begin{pmatrix} 0 & 0 \\ 0 & \tau_2 d\beta_1\sinh\theta_2\end{pmatrix}+d\theta_2\begin{pmatrix}0 & 0\\ 0 & \tau_1\end{pmatrix}\right], \end{aligned} \tag{B8}$$

where

$$\tau_2=\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},\quad \overline{w}\tau_3 w=\begin{pmatrix}\cos\mu & -\sin\mu \\ -\sin\mu & -\cos\mu\end{pmatrix}.$$

Making the replacement

$$d\kappa\to d\kappa\,\exp\frac{i(\beta-\beta_1)}{2},\quad d\kappa^*\to d\kappa^*\exp\frac{i(\beta-\beta_1)}{2},$$

and the same for $d\eta$ and $d\eta^*$, one can derive

$$\begin{aligned} \overline{Y_0}\delta TY_0=\mathbf{1}\times 2\left[\cosh\frac{\theta_2}{2}\begin{pmatrix} 0 & d\kappa' \\ -d\kappa' & 0\end{pmatrix}\right. \\ \left. +i\,\sinh\frac{\theta_2}{2}\begin{pmatrix} 0 & d\eta'\tau_1 \\ -\tau_1 d\overline{\eta'} & 0\end{pmatrix}\right] \\ +2i\Lambda_1\left(\cos\frac{\theta_2}{2}\begin{pmatrix} 0 & d\eta' \\ d\overline{\eta'} & 0\end{pmatrix}\right. \\ \left. -i\,\sinh\frac{\theta_2}{2}\begin{pmatrix} 0 & d\kappa'\tau_1 \\ \tau_1 d\overline{\kappa'} & 0\end{pmatrix}-\frac{i}{2}d\hat{\theta}\right), \end{aligned} \tag{B9}$$

where $d\eta'=\overline{w}\,d\eta$ and $d\overline{\eta'}=d\overline{\eta}\,w$, and the same for $d\kappa'$ and $d\overline{\kappa}$. The contribution from $i\tau_2 dc$, Eq. (A16), is not written because it can be removed by a proper shift of $d\hat{\beta}$ and $d\mu$.

Substituting Eqs. (B3)–(B6) into Eqs. (B2) and (B1), we see that the second terms of Eq. (B4) and (B6) do not contribute. After making the replacement in δT, Eq. (A18), and shifting

$$d\sigma=d\sigma_1-\left(\cosh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\eta+i\,\sinh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\kappa^*\right), \tag{B10}$$

$$d\rho=d\rho_1-\left(\cosh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\kappa-i\,\sinh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\eta^*\right),$$

it is convenient to introduce the matrix differentials

$$d\sigma=\begin{pmatrix} d\sigma_1 & d\sigma_2 \\ -d\sigma_2^* & -d\sigma_1^* \end{pmatrix}, \quad d\rho=\begin{pmatrix} d\rho_1 & d\rho_2 \\ -d\rho_2^* & -d\rho_1^* \end{pmatrix}, \tag{B11}$$

where

$$\begin{aligned} d\sigma_2 &= -\cosh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\eta^* - i\,\sinh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\kappa, \\ d\sigma_2^* &= \cosh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\eta - i\,\sinh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\kappa^*, \\ d\rho_2 &= -\cosh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\kappa^* + i\,\sinh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\eta, \\ d\rho_2^* &= \cosh\frac{\theta_2}{2}\sin\frac{\mu}{2}d\kappa + i\,\sinh\frac{\theta_2}{2}\cos\frac{\mu}{2}d\eta^*. \end{aligned} \tag{B12}$$

The Jacobian $\widetilde{J}_\mu$ of the transformation, Eqs. (B12), equals

$$\widetilde{J}_\mu=\frac{4}{(\cosh\theta_2-\cos\mu)^2}. \tag{B13}$$

Then Eq. (B2) can be written as

$$\delta Z=\overline{SR}\,\delta U R S, \tag{B14}$$

$$\delta U=\delta Y_0+i\Lambda_1\left(2\,d\hat{\sigma}-\frac{i}{2}d\hat{\theta}\right)+2k\;d\hat{\rho}\;\mathbf{1}, \tag{B15}$$

where the matrices $d\sigma$ and $d\rho$ entering $d\hat{\sigma}$ and $d\hat{\rho}$, Eq. (5.7), have the structure of Eq. (B11), and

$$k=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can obtain δZ, Eq. (B14) calculating first $\overline{R\,\delta U R}$ and then δZ. The corresponding manipulations are still quite lengthy. One should again make different replacements that do not change the Jacobian. Alternatively, one might write the final result using general symmetry properties of δZ. Finally, one obtains

$$\delta Z=\delta Y_0'+i\Lambda_1(2d\hat{\sigma}-\tfrac{1}{2}\,d\hat{\theta})+2k\;d\hat{\rho}\;\mathbf{1}. \tag{B16}$$

The supermatrix $\delta Y_0'$ entering Eq. (B16) equals

$$\begin{aligned} \delta Y_0' = &-\mathbf{1}\,\frac{i}{2}\left[d\beta\,\sin\mu\begin{pmatrix}\tau_1 & 0\\ 0 & 0\end{pmatrix}\right]+d\mu\begin{pmatrix}\tau_2 & 0\\ 0 & 0\end{pmatrix} \\ &+\Lambda_1\frac{1}{2}\left[-\sinh\theta_2 d\beta_1\begin{pmatrix}0 & 0\\ 0 & \tau_2\end{pmatrix}+d\theta_2\begin{pmatrix}0 & 0\\ 0 & \tau_1\end{pmatrix}\right]. \end{aligned} \tag{B17}$$

Using Eq. (B16) we can calculate $S\mathrm{Tr}[\,\delta Z,Q_0]^2$. The anticommuting part $\delta Z_\perp$ decouples from the commuting one $\delta Z_\parallel$, and one obtains

$$\begin{aligned} S\mathrm{Tr}[\,\delta Z_\perp,Q_0]^2 = 64\{&d\sigma_1 d\sigma_1^*[1+\cos(\varphi+i\chi)]+d\sigma_2 d\sigma_2^*[1 \\ &+\cos(\varphi-i\chi)] \\ &+d\rho_1 d\rho_1^*[1-\cos(\varphi-i\chi)] \\ &+d\rho_2 d\rho_2^*[1-\cos(\varphi+i\chi)]\}. \end{aligned} \tag{B18}$$

The Jacobian $J_{\varphi\chi}$ corresponding to the length, Eq. (B18) equals

$$J_{\varphi\chi}=\frac{1}{2^{24}}\,\frac{1}{(\sin^2\varphi+\sinh^2\chi)^2}. \tag{B19}$$

The commuting part $\delta Z_\parallel$ contributes to the elementary length as

$$\begin{aligned} S\mathrm{Tr}[\,\delta Z_\parallel,Q_0]^2 = 4\{&[(d\mu)^2+(d\beta)^2\sin^2\mu]\sin^2\varphi \\ &+(d\theta)^2\cos^2\varphi+(d\theta_1)^2\cosh^2\chi \\ &+(d\theta_2)^2+(d\beta_1)^2\sinh^2\theta_2\}. \end{aligned} \tag{B20}$$

Combining the contribution to the Jacobian from Eqs. (A5) and (B20) with those written in Eqs. (B13) and (B19), and recalling that the replacement, Eq. (A18), results in an additional multiplier proportional to J_θ one obtains finally the elementary volume $[dQ]$, Eqs. (5.9)–(5.12) and (4.16)–(4.18).

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