

Nonzero-temperature transport near quantum critical points

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We describe the nature of charge transport at nonzero temperatures (T) above the two-dimensional (d) superfluid-insulator quantum-critical point. We argue that the transport is characterized by inelastic collisions among thermally excited carriers at a rate of order $k_B T/\hbar$. This implies that the transport at frequencies $\omega \ll k_B T/\hbar$ is in the hydrodynamic, collision-dominated (or incoherent) regime, while $\omega \gg k_B T/\hbar$ is the collisionless (or phase-coherent) regime. The conductivity is argued to be e^2/h times a nontrivial universal scaling function of $\hbar\omega/k_B T$, and not independent of $\hbar\omega/k_B T$, as has been previously claimed or implicitly assumed. The experimentally measured dc conductivity is the hydrodynamic $\hbar\omega/k_B T \rightarrow 0$ limit of this function, and is a universal number times e^2/h , even though the transport is incoherent. Previous work determined the conductivity by incorrectly assuming it was also equal to the collisionless $\hbar\omega/k_B T \rightarrow \infty$ limit of the scaling function, which actually describes phase-coherent transport with a conductivity given by a *different* universal number times e^2/h . We provide a computation of the universal dc conductivity in a disorder-free boson model, along with explicit crossover functions, using a quantum Boltzmann equation and an expansion in $\epsilon = 3 - d$. The case of spin transport near quantum-critical points in antiferromagnets is also discussed. Similar ideas should apply to the transitions in quantum Hall systems and to metal-insulator transitions. We suggest experimental tests of our picture and speculate on a route to self-duality at two-dimensional quantum-critical points. [S0163-1829(97)07438-9]

I. INTRODUCTION

The charge transport properties of systems near a zero-temperature quantum phase transition have been the subject of a large number of experimental studies in the last few years. Systems in two spatial dimensions have been of special interest, and important examples are the superfluid-insulator transition in disordered thin films¹⁻⁵ and Josephson junction arrays,⁶ the transitions from quantized Hall plateaus,^{7,8} a quantum-critical point in the doped cuprate compounds,⁹ and a recent quantum transition in Si metal-oxide-semiconductor field-effect transistors (MOSFET's).¹⁰ In three dimensions, the metal-insulator transition in doped semiconductors has seen many years of study,¹¹ and evidence for scaling collapse of data near the quantum-critical point has finally begun to emerge.¹² We shall present the results of this paper using the language of the superfluid-insulator transition in thin films in zero external magnetic field, but we believe our ideas are much more general and also have significant implications for the other systems noted above. We shall also discuss the extension of our results to the case of *spin* transport near quantum-critical points in two-dimensional antiferromagnets.¹³

A particular focus of the experiments on the superfluid-insulator transition has been the theoretical prediction, made in the seminal work of Fisher *et al.*¹⁴ and Cha *et al.*,¹⁵ that the conductivity at the critical point in two dimensions is nonzero and equals the quantum unit of conductance (e^2/h) times a universal number. However, there remains an important dichotomy between experimental and theoretical studies of the superfluid-insulator transition in disordered thin films, which is crucial to our discussion. All experiments are performed at a low, but nonzero temperature T and have measured (dc) conductivities at frequencies ω , which easily sat-

isfy $\hbar\omega \ll k_B T$. In contrast, all of the theoretical analytical¹⁴⁻²⁰ and exact diagonalization²¹ work has computed the universal conductivity at $T=0$, which is equivalent to the regime $\hbar\omega \gg k_B T$. Numerical Monte Carlo work has also measured the universal conductivity,^{15,22-25} but it involves an analytical continuation from imaginary Matsubara frequencies $i2\pi p k_B T/\hbar$ ($p \geq 1$, integer) to real frequencies: Such a procedure is insensitive to any structure in the conductivity at $\hbar\omega \ll k_B T$, and so these measurements are effectively also in the regime $\hbar\omega \gg k_B T$. This discrepancy, between theory and experiment, in the orders of limit of $\omega \rightarrow 0$ and $T \rightarrow 0$ was noted by Cha *et al.*¹⁵ and Wallin *et al.*,²⁵ but they asserted that the conductivity approaches a universal value, at low temperatures and frequencies, which is *independent* of the value of the ratio $\hbar\omega/k_B T$.

In this paper, we shall argue that the critical conductivity is in fact *not* independent of $\hbar\omega/k_B T$, and is instead a highly nontrivial, but universal, function of $\hbar\omega/k_B T$; this shall be explicitly demonstrated in a computation of the crossover function in a model of disorder-free bosons. We shall show that the physics of the $\hbar\omega \gg k_B T$ and $\hbar\omega \ll k_B T$ regimes are quite distinct, and shall provide a model computation of the universal properties in the limit $\hbar\omega \ll k_B T$ relevant to dc transport measurements. We hope that our results will stimulate experimental measurements of the ac conductivity at frequencies which can explore the crossover at $\hbar\omega \sim k_B T$.

Our arguments rely on the physical picture of the $T > 0$ dynamics at quantum-critical points developed in the context of quantum antiferromagnets.^{26,13,28,29} It was argued in these works that the order parameter dynamics is *relaxational*, with a relaxation rate $\sim k_B T/\hbar$. Using the orientation of the order parameter to define a phase, we then obtain a phase relaxation rate $1/\tau_\phi \sim k_B T/\hbar$, and this rate is analogous to the phase relaxation rate of disordered metals³⁰ (the latter rate is

much smaller than $k_B T/\hbar$ in the disordered metal,³⁰ but becomes of order $k_B T/\hbar$ at the metal insulator transition—see Sec. IV D). At time scales of order τ_φ , the relaxational dynamics cannot be described by effective classical models of the types discussed in Ref. 31, as a necessary condition for classical behavior is that the relaxation rate be much smaller than $k_B T/\hbar$: The dynamics was therefore dubbed “quantum relaxational.” Related ideas were applied to charge transport in the insightful article by Sondhi *et al.*⁷ Dynamic order parameter fluctuations also carry charge, and therefore inelastic collisions between thermally excited charge-carrying excitations will lead to a transport relaxation time τ_{tr} . As the typical energy exchanged in a collision is $k_B T$, τ_{tr} is also of order τ_φ , and therefore³²

$$\frac{1}{\tau_{tr}} \sim \frac{k_B T}{\hbar}. \quad (1.1)$$

The missing coefficient in Eq. (1.1) is a universal number whose value will depend upon the precise definition of $1/\tau_{tr}$. It is perhaps worth noting explicitly here that Eq. (1.1) holds for all values of the dynamic exponent z . [A scenario under which Eq. (1.1) could be violated is discussed in Sec. IV B]. The collisions leading to Eq. (1.1) conserve total energy and, in the case of translationally invariant systems, total momentum. Note that conservation of total momentum should not be confused with conservation of the total *charge current*—the latter is not conserved in any of the systems of interest here, even in the continuum scaling limit (the total *charge density* is, of course, always conserved).

Now general considerations³³ suggest that there are two qualitatively different regimes of charge transport at nonzero frequencies: (i) $\omega\tau_{tr} \ll 1$: the hydrodynamic, incoherent, collision-dominated regime, where charge transport is controlled by repeated, inelastic scatterings between preexisting thermally, excited carriers; the conductivity should exhibit a “Drude” peak as a function of frequency. (ii) $\omega\tau_{tr} \gg 1$: the high-frequency, phase-coherent, collisionless regime, where the excitations created by the external perturbation are solely responsible for transport, and collisions with thermally excited carriers can be neglected. Essentially all previous theoretical analyses have been restricted to the collisionless regime.

We can sharpen our description of the crossovers in the vicinity of $\omega \sim 1/\tau_{tr}$ by expressing them in universal scaling forms. We will consider systems where the order parameter quanta carry “charge” Q . For the superfluid-insulator transition of bosons, we have $Q = 2e$ and we measure charge transport by the dynamic conductivity $\sigma(\omega)$. For the Néel-paramagnet transition in antiferromagnets, we have $Q = g\mu_B$ (μ_B is the Bohr magneton and g is the gyromagnetic ratio) we and measure spin transport by the “spin conductivity” [also denoted by $\sigma(\omega)$] which determines the spin current in response to a uniform gradient in an external magnetic field. In d spatial dimensions the dynamic conductivity σ obeys at the quantum-critical coupling

$$\sigma(\omega) = \frac{Q_2}{\hbar} \left(\frac{k_B T}{\hbar c} \right)^{(d-2)/z} \Sigma \left(\frac{\hbar \omega}{k_B T} \right), \quad (1.2)$$

where z is the dynamic critical exponent, c is a nonuniversal microscopically determined quantity with the dimensions of

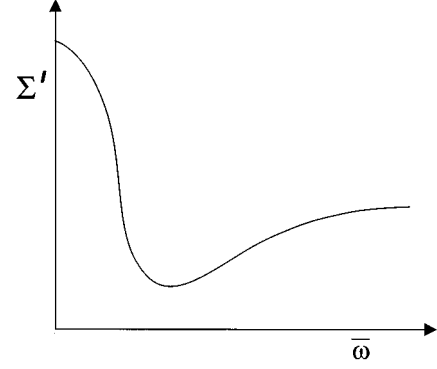


FIG. 1. A sketch of the expected form of the real part Σ' of the universal scaling function Σ appearing in the scaling form (1.2) for the conductivity, as a function of $\bar{\omega} = \hbar\omega/k_B T$. There is a Drude-like peak from the inelastic scatterings between thermally excited carriers at $\bar{\omega}$ of order unity. At larger $\bar{\omega}$, there is a crossover to the collisionless regime where $\Sigma' \sim \bar{\omega}^{-(d-2)/z}$ as $\bar{\omega} \rightarrow \infty$.

(length) ^{z} (time) ^{-1} (for $z = 1$, c is a velocity), and $\Sigma(\bar{\omega})$ is a universal scaling function (note that we are using \hbar rather than h to define the scale of the conductivity). We note again that the dependence on a universal function of $\hbar\omega/k_B T$, with no arbitrary frequency scale factors, holds for all values of the dynamic exponent z .²⁶ Scaling as a function of $\hbar\omega/k_B T$ was also noted recently by Sondhi *et al.*,⁷ but they used it only to establish the frequency dependence of the conductivity away from the critical coupling, with an eye to understanding recent dynamical conductivity measurements in the quantum Hall system³⁴ (we will comment on these measurements in Sec. IV D). Indeed, their scaling forms are consistent with a conductivity which is independent of $\hbar\omega/k_B T$ right at the critical point, and hence with implicit or explicit assumptions in earlier theoretical results.^{14–25} One of our points here is that there is a nontrivial dependence on $\hbar\omega/k_B T$ already at the critical coupling, and that this dependence means that previous analyses of the universal conductivity at the critical coupling either did not compute,^{14–21} or were not particularly sensitive to,^{22–25} the value of the dc conductivity.

Let us now turn to the expected dependence of the conductivity on the $\hbar\omega/k_B T$ as expressed in the universal function $\Sigma(\bar{\omega})$. The discussion here will apply to realistic systems at their critical point for $\hbar\omega$ and $k_B T$ much less than a noncritical microscopic energy scale, e.g., the repulsion energy U between two bosons on the same site of a lattice model. The physical arguments below Eq. (1.1) have been recast in a qualitative sketch of the real part of the function $\Sigma(\bar{\omega})$ in Fig. 1. The dc conductivity is determined by the value of the real universal number $\Sigma(0)$. At small $\bar{\omega}$ there is a Drude-like peak coming from the energy-exchanging collisions among thermally excited carriers. At larger frequencies there is crossover to transport by particle-hole pairs created by the external source. As $\bar{\omega} \rightarrow \infty$ we expect that $\Sigma(\bar{\omega}) \sim (-i\bar{\omega})^{-(d-2)/z}$ so that σ becomes independent of T in the collisionless regime. In $d = 2$, $\Sigma(\infty)$ is a real, finite, universal number determining the high-frequency conductivity; $\Sigma(\infty)$ was the number computed in earlier analyses,^{14–21} and

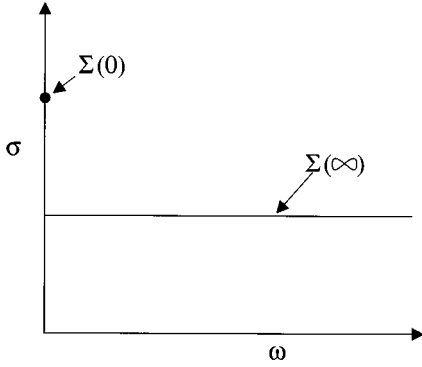


FIG. 2. Universal form of the conductivity $\sigma(\omega, T \rightarrow 0)$ in $d=2$; the vertical scale is measured in units of \hbar/Q^2 . Only the $\omega=0$ value is given by the universal number $\Sigma(0)$. For all $\omega>0$, $(\hbar/Q^2)\sigma = \Sigma(\infty)$. Q is the ‘‘charge’’ of the order parameter: For the superfluid-insulator transition $Q=2e$, while for quantum anti-ferromagnets $Q=g\mu_B$.

not the dc conductivity which is given by $\Sigma(0)$. This is illustrated in Fig. 2 which plots the form of $(\hbar/Q^2)\sigma(\omega, T \rightarrow 0)$ in $d=2$: Its value at $\omega=0$ is given by $\Sigma(0)$, while for all $\omega>0$ it equals $\Sigma(\infty)$. It is likely, although not established, that $\Sigma(0) > \Sigma(\infty)$. Note the difference from Fermi liquid theory, where the Drude peak becomes a δ function with nonzero weight as $T \rightarrow 0$. In the present situation, the weight in the Drude-like peak vanishes like $\sim T^{(d+z-2)/z}$ as $T \rightarrow 0$, and in $d=2$ and the Drude-like peak reduces to the *single* point $\omega=0$ where the conductivity is given by $\Sigma(0)$. The scaling properties of the $d=1$, $z=1$ case are similar to those of a Fermi liquid.

There is a certain critical phenomena and scaling perspective in which the result (1.2), and its implications discussed above, may seem quite natural, and even somewhat ‘‘trivial’’ as they follow directly from the fact that ω and T have the same scaling dimension. Nevertheless, its importance has been overlooked in essentially all previous work. This is probably because there are complementary perspectives, more common among investigators in this field, from which Eq. (1.2) implies physics that is surprising and even somewhat radical. In particular, current ideas in quantum transport theory³⁵ and dissipative quantum mechanics³⁶ can lead one to rather different picture, as we now itemize explicitly.

(i) There have been a number of previous situations in which charge transport properties have been found to be universally related to the quantum unit of conductance, e^2/h ; these include the quantized Landauer conductance of ballistic transport in one-dimensional wires, and the universal conductance fluctuations of mesoscopic metals.^{35,37} However, in all previous cases, these universal properties have arisen in a ‘‘phase-coherent’’ regime, i.e., they are associated with physics at scales shorter than the mean distance between inelastic scattering events between the carriers. For the case of a $d=2$ quantum-critical point discussed above, the universal number $\Sigma(\infty)$ is associated with quantum coherent transport, and is therefore the analog of these earlier results. In contrast, the value of $\Sigma(0)$ is controlled by repeated inelastic scattering events, and therefore the dc transport is clearly in what would traditionally be identified as the ‘‘incoherent’’ regime. Nevertheless, we have argued above that $\Sigma(0)$ is a

universal number, and remarkably, the dc conductance remains universally related to e^2/h .

(ii) The community has gained much intuition on the nonzero-temperature transport properties of interacting quantum systems from recent studies of dissipative quantum mechanics and a number of related quantum-impurity problems.^{36,38–40} The scaling properties of such models are given by the theory of *boundary* critical phenomena, in contrast to the *bulk* critical phenomena of interest in this paper. We discuss the transport properties of a class of boundary problems in Appendix A: The leading term in the transport coefficient is found to be independent of ω/T , and dependence on ω/T arises only upon consideration of subleading terms at low T . However, as we argue in Appendix A, this behavior is understood by the fact that the fixed point controlling the low- T behavior is simply a *free-field* theory, and ω/T dependence arises only upon considering the leading irrelevant operator. In contrast, the bulk theories of interest here differ in a crucial respect: They have *interacting* critical theories and therefore, we argue, contain ω/T dependence already in the leading term, before the inclusion of any irrelevant operators.

(iii) Scaling as a function of ω/T does not hold for the Anderson localization transition of noninteracting electrons. The frequency obeys conventional scaling with dimension z , but the behavior of temperature is rather different and non-universal. Analytic theories for such transition are available only for $d>2$, and in these the primary effect of temperature is in the nonuniversal, superlinear in T dependence of the phase-breaking rate which acts like a finite-size-like infrared cutoff to the critical properties. See also Sec. IV B. In contrast, the models of interest here have interactions, and inelastic scattering is central to understanding their universal critical properties; in some respects scaling with respect to temperature is simpler, as its naive scaling dimension of z is now valid.

In this paper we will provide explicit results for the crossover function $\Sigma(\bar{\omega})$ in a simple, disorder-free field-theoretic model for the superfluid insulator transition that was introduced by Cha *et al.*¹⁵ Near the quantum-critical point, this model becomes equivalent to the familiar particle-hole-symmetric ϕ^4 field theory with the effective imaginary time (τ) action

$$\mathcal{S} = \int_0^{\hbar/k_B T} d\tau \int d^d x \left\{ \frac{1}{2} [(\partial_\tau \phi_\alpha)^2 + c^2 (\nabla_x \phi_\alpha)^2 + (m_{0c}^2 + t_0) \phi_\alpha^2] + \frac{u_0}{4!} (\phi_\alpha^2)^2 \right\}. \quad (1.3)$$

Here ϕ_α is a n -component field and the action has $O(n)$ symmetry [the $O(n)$ index α is implicitly summed over]. The spatial and temporal gradient terms are both second order, and so the action has a ‘‘Lorentz’’ invariance with c the velocity of light, and as a result the dynamic critical exponent $z=1$. The bare ‘‘mass’’ term has been written as $m_{0c}^2 + t_0$ so that the $T=0$ quantum-critical point is at $t_0=0$, and u_0 measures the strength of the quartic nonlinearity. The superfluid-insulator transition is described by the case $n=2$ where $\Psi = \phi_1 + i\phi_2$ is the usual complex superfluid order

parameter. We shall also be interested in the case $n=3$ which applies to quantum-critical points in quantum antiferromagnets.²⁹

It should be noted that the continuum model \mathcal{S} for $n=2$, with its double time derivative term, differs from the usual continuum action for nonrelativistic bosons⁴¹ which has only a single time derivative of a complex scalar field. The total momentum and charge current of the latter model are proportional to each other. When these nonrelativistic bosons are placed under the influence of an effective lattice potential, with an average of an integer number of bosons per lattice site, then after integrating out certain local high-energy modes, one obtains the continuum model \mathcal{S} as a low-energy effective theory.⁴² The charge current and momentum operators of \mathcal{S} are now no longer simply related to each other as \mathcal{S} has excitations with both positive and negative charges. As we will see in Sec. III, the total momentum is proportional to the *sum* of the currents of the positive and negative charges, while the total charge current is proportional to their *difference*. This lack of a direct relationship between the charge current and the momentum should not seem surprising as lattice effects were required to obtain \mathcal{S} , and are therefore implicitly accounted for in the continuum theory.

It was asserted by Cha *et al.*¹⁵ that the $T>0$ transport properties of \mathcal{S} were ‘‘pathological’’ in that there was zero resistance to charge transport in the continuum field theory in Eq. (1.3); they suggested that a nonzero resistance appeared only upon considering additional lattice corrections (beyond those required to derive the continuum theory \mathcal{S}), in which momentum could jump in units of reciprocal lattice vectors at scattering events (the so-called ‘‘umklapp’’ scattering events). We shall show here that this is incorrect. The model \mathcal{S} has a finite, and universal, dc resistance at any $T>0$ already in the scaling, continuum limit. Indeed, this is already clear from a recent study⁴³ of \mathcal{S} for the case $d=1$ $n=3$, where a simple and physically transparent argument obtained the exact (and finite) low-temperature value of the spin diffusivity. Rather than being pathological, we claim that the transport properties of the disorder-free boson model \mathcal{S} are generic, and essentially identical in their scaling structure to those of disordered boson systems. The error in Cha *et al.*¹⁵ appears to be due to their ignoring the presence of independent positive and negative charge excitations, and the resulting difference between the total momentum and the total charge current of \mathcal{S} .⁴⁴

As T appears in \mathcal{S} only in the upper limit of the imaginary time integral, it is clear that the scaling form (1.2) is nothing but a standard finite-size scaling result for observables as a function of ‘‘wave vector’’ ω and ‘‘inverse size’’ T . It might then seem that our job is relatively straightforward, and we merely have to obtain standard finite-size scaling results on the familiar model \mathcal{S} . This is far from being the case. The point is that these standard results exist only at imaginary frequencies $i2p\pi k_B T/\hbar$ ($p \geq 1$, integer), and we are especially interested in real frequencies $\ll k_B T/\hbar$. More importantly, it has been shown^{26,13,29} that the operations of analytic continuation and expansion in $1/n$ or $\epsilon=3-d$ (which are the only non-numeric tools for analyzing the critical point of \mathcal{S}) *do not commute*.

The proper tool for analyzing transport at the critical point of \mathcal{S} is a quantum Boltzmann equation (QBE) for the charge

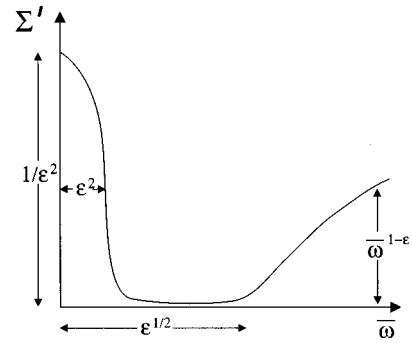


FIG. 3. Structure of the real part Σ' of the universal scaling function Σ in Eq. (1.2) for the conductivity at the quantum-critical coupling of the model \mathcal{S} defined in Eq. (1.3). The spatial dimensionality $d=3-\epsilon$, and ϵ is assumed to be small. As before $\bar{\omega}=\hbar\omega/k_B T$. The Drude peak at small $\bar{\omega}$ has a width of order ϵ^2 and a height of order $1/\epsilon^2$: This feature of the conductivity is denoted later in the paper by σ_1 . The collisionless contribution (denoted σ_{II} later) begins at $\bar{\omega}$ of order $\epsilon^{1/2}$; as $\bar{\omega}\rightarrow\infty$, this contribution is a number of order unity times $\bar{\omega}^{1-\epsilon}$.

carriers. In general, solution of such a QBE is a dauntingly difficult task, but we shall find that it is possible to reduce the solution to a simple linear integral equation in an expansion^{19,29} in $\epsilon=3-d$ (very similar results can be obtained in a related analysis of \mathcal{S} in an expansion in $1/n$; ^{15,26,13} this will be described elsewhere, and we will only discuss the $n=\infty$ results here). For small ϵ , the nonlinearities are weak, and it becomes possible to give a quasiparticlelike interpretation to the excitations of \mathcal{S} at the quantum-critical point. Such an interpretation will be useful in our intuitive understanding, and will help us use standard methods to simplify the QBE in the $\hbar\omega\ll k_B T$ limit. Although the quasiparticle interpretation fails for the physical case $\epsilon=1$, we do not expect any qualitative change in the structure of our results for larger ϵ . The QBE formulation is quite general, and the quasiparticle representation is mainly a useful technical tool towards obtaining its numerical solution. In a more general context, our approach may be considered as an expansion in powers of the anomalous dimension η , which is responsible for replacing the quasiparticle pole by a continuum at the $T=0$ critical point. The continuum associated with a nonzero η is important only for $\hbar\omega\gg k_B T$, while for $\hbar\omega\sim k_B T$ very different thermal damping processes quench the critical fluctuations,²⁸ and are best treated by the QBE. This thermal damping also acts as an effective infrared cutoff which ensures that no qualitatively new physics emerges at higher orders in the expansion in powers of η .

The basic structure of the dynamical conductivity at the critical point of \mathcal{S} for small ϵ is illustrated in Fig. 3. Notice the presence of ‘‘boundary layers’’ which make the analysis of the $\epsilon\rightarrow 0$ limit quite subtle, and is responsible for the noncommutativity of analytic continuation and the naive ϵ expansion noted earlier. The hydrodynamic regime of the conductivity (denoted later in the paper by σ_I) is represented by a Drude peak of width in frequency $\omega\sim\epsilon^2 k_B T/\hbar$ and has a height of order $1/\epsilon^2$. In particular, the dc conductivity is determined by the universal number $\Sigma(0)$, for which we find for $n=2$

$$\Sigma(0) = \frac{0.1650}{\epsilon^2}, \quad (1.4)$$

to leading order in ϵ ; the structure of the higher-order corrections to Eq. (1.4) is quite complex, and was generally discussed in Ref. 29. Determination of the coefficient in Eq. (1.4) required the numerical solution of a QBE, and the uncertainty in the numerics is believed to be restricted to the fourth decimal place. In $d=2$, Eq. (1.4) implies a universal dc conductivity

$$\sigma(0) = 2\pi \Sigma(0) \frac{4e^2}{h} \approx 1.037 \frac{4e^2}{h} \quad (1.5)$$

at the superfluid-insulator transition. This result is remarkably close to the self-dual value $4e^2/h$,^{16,17} and to the results of many experiments;² we will comment further on this in Sec. IV E.

There is a clean separation between the hydrodynamic and collisionless regimes of σ : the latter does not begin until $\omega \sim \epsilon^{1/2} k_B T / \hbar$ (Fig. 3). The $T=0$ collisionless transport is characterized by $\Sigma(\bar{\omega} \rightarrow \infty)$ for which¹⁹ we have at $n=2,3$

$$\Sigma'(\bar{\omega} \rightarrow \infty) = \frac{2^{1-2d} \pi^{1-d/2}}{d\Gamma(d/2)} [1 + \mathcal{O}(\epsilon^2)] \bar{\omega}^{-1-\epsilon}. \quad (1.6)$$

As noted earlier, for $\epsilon=1$ ($d=2$) $\Sigma(\infty)$ is a pure number, but notice that it bears no relationship to $\Sigma(0)$; indeed $\Sigma(\infty)$ is of order unity as $\epsilon \rightarrow 0$, and so these two quantities are of distinct orders in ϵ . For the superfluid-insulator transition in $d=2$, Eq. (1.6) gives a high-frequency conductivity of $0.3927 \times (4e^2/h)$, which is in rough agreement with other analyses in the collisionless regime.¹⁹

The body of the remainder of the paper is devoted to obtaining the above properties of the model \mathcal{S} . Readers not interested in calculations specific to the model \mathcal{S} should now go directly to the concluding Sec. IV where we discuss implications of our results for a number of experimental systems. In Sec. II we will obtain one loop results for the transport properties of \mathcal{S} using the familiar Kubo formalism. Then Sec. III will include two-loop effects using a quantum transport analysis needed to describe the hydrodynamic regime, and obtain Eq. (1.4).

In all of Secs. II and III and the appendixes we will work in units in which $\hbar = k_B = c = 1$; we will reinstate these constants in Sec. IV.

II. ONE-LOOP RESULTS FROM THE KUBO FORMULA

We will begin our analysis of the transport properties of \mathcal{S} by examining the results of a direct evaluation from the Kubo formula,^{15,19} but now working at $T > 0$. The physical interpretation of the results will motivate an analysis using a quantum Boltzmann equation^{33,46,47} which will be carried out in subsequent sections.

The standard Kubo formula relates the conductivity to a two-point correlator of the conserved $O(n)$ current. We introduce an external vector potential \mathbf{A} associated with the $O(n)$ generator which rotates ϕ_α in the 1,2 plane; the spatial gradient term in \mathcal{S} then undergoes the mapping

$$\sum_{\alpha=1}^n (\nabla_x \phi_\alpha)^2 \rightarrow (\nabla_x \phi_1 - Q\mathbf{A}\phi_2)^2 + (\nabla_x \phi_2 + Q\mathbf{A}\phi_1)^2 + \sum_{\alpha=3}^n (\nabla_x \phi_\alpha)^2. \quad (2.1)$$

The associated $O(n)$ current is then $\delta\mathcal{S}/\delta\mathbf{A}$. We evaluate its two-point correlator using standard diagrammatic perturbation theory to first order in u_0 in the expansion in ϵ , or to leading order in the large n expansion (in which $u_0 \sim 1/n$). The first-order vertex correction vanishes because the interaction is momentum independent and the result in both cases is given simply by^{15,19} (recall we are using units here in which $\hbar = k_B = c = 1$)

$$\sigma(i\omega_n) = -\frac{2Q^2}{\omega_n} T \sum_{\epsilon_n} \int \frac{d^d k}{(2\pi)^d} \times \left[\frac{2k_x^2}{(\epsilon_n^2 + k^2 + m^2)[(\epsilon_n + \omega_n)^2 + k^2 + m^2]} - \frac{1}{\epsilon_n^2 + k^2 + m^2} \right]. \quad (2.2)$$

The first term is the ‘‘paramagnetic’’ contribution, while the second is the ‘‘diamagnetic’’ term proportional to the density.⁴⁸ Here ϵ_n, ω_n are Matsubara frequencies, k_x is the x component of the d dimensional momentum \mathbf{k} , and $k = |\mathbf{k}|$. The ‘‘mass’’ m in the propagators is computed in Appendix D [where it is referred to as $m(T)$] using the $\epsilon=3-d$ expansion developed in Ref. 29, and depends universally upon T and an energy scale measuring the deviation of the ground state from the critical ground state. At the critical coupling, to leading order in ϵ ,

$$m^2 = \epsilon \left(\frac{n+2}{n+8} \right) \frac{2\pi^2 T^2}{3} \quad \text{at } t_0=0. \quad (2.3)$$

The result (2.2) holds for all n , and there is no n dependence at this order in the ϵ expansion, other than that through m . The large n expansion has an identical structure at $n=\infty$, the only difference being in the value of m . Detailed universal expressions for m were given in Ref. 13 in $d=2$; at the critical coupling, the analog of Eq. (2.3) at $n=\infty$ is

$$m = 2 \ln \left(\frac{\sqrt{5}+1}{2} \right) T \quad \text{at } t_0=0, \quad d=2. \quad (2.4)$$

The remaining analysis of this section will apply both to the ϵ and $1/n$ expansions, the only difference being in the values of m given above.

Now insert $1 = \partial k_x / \partial k_x$ in front of the diamagnetic term in Eq. (2.2) and integrate by parts. The surface terms vanish in dimensional or lattice regularization, and the expression for the conductivity becomes

$$\sigma(i\omega_n) = -\frac{2Q^2}{\omega_n} T \sum_{\epsilon_n} \int \frac{d^d k}{(2\pi)^d} \frac{2k_x^2}{\epsilon_n^2 + k^2 + m^2} \times \left[\frac{1}{(\epsilon_n + \omega_n)^2 + k^2 + m^2} - \frac{1}{\epsilon_n^2 + k^2 + m^2} \right]. \quad (2.5)$$

We now evaluate the summation over Matsubara frequencies, analytically continue to real frequencies. The resulting $\sigma(\omega)$ is complex, and we decompose it into its real and imaginary parts $\sigma(\omega) = \sigma'(\omega) + i\sigma''(\omega)$. We will only present results for the real part $\sigma'(\omega)$, and the imaginary part $\sigma''(\omega)$ can be obtained via the standard dispersion relation.

We find that the result for $\sigma'(\omega)$ has two distinct contributions of very different physical origin. We separate these by writing

$$\sigma'(\omega) = \sigma'_I(\omega) + \sigma'_{II}(\omega). \quad (2.6)$$

The first part, $\sigma'_I(\omega)$, is a δ function at zero frequency:

$$\sigma'_I(\omega) = 2\pi Q^2 \delta(\omega) \int \frac{d^d k}{(2\pi)^d} \frac{k_x^2}{\epsilon_k^2} \left(-\frac{\partial n(\epsilon_k)}{\partial \epsilon_k} \right), \quad (2.7)$$

where $n(\epsilon)$ is the Bose function,

$$n(\epsilon) = \frac{1}{e^{\epsilon/T} - 1}, \quad (2.8)$$

and the excitations have the energy-momentum relation

$$\epsilon_k \equiv \sqrt{k^2 + m^2}. \quad (2.9)$$

We will discuss the physical meaning of the δ function in Eq. (2.7) below, and obtain a separate and more physical derivation of its weight in Sec. III A. The second part, $\sigma'_{II}(\omega)$ is a continuum above a threshold frequency of $2m$:

$$\begin{aligned} \sigma'_{II}(\omega) &= \pi Q^2 \int \frac{d^d k}{(2\pi)^d} \frac{k_x^2}{2\epsilon_k^3} [1 + 2n(\epsilon_k)] \delta(|\omega| - 2\epsilon_k) \\ &= \frac{\pi Q^2 S_d}{d} \theta(|\omega| - 2m) \left(\frac{\omega^2 - 4m^2}{4\omega^2} \right)^{d/2} \\ &\quad \times [1 + 2n(\omega/2)] |\omega|^{d-2}, \end{aligned} \quad (2.10)$$

where $S_d = 2[\Gamma(d/2)(4\pi)^{d/2}]$ is a standard phase space factor.

At the critical point, $t_0=0$, it can be verified that the above results for $\sigma(\omega)$ obey the scaling form (1.2) with $z=1$; explicit results for the function $\Sigma(\bar{\omega})$ will appear below and are sketched in Fig. 4.

We now discuss the physical and scaling properties of the two components of the conductivity in turn.

A. σ_I

This is a zero-frequency δ function, and is present only for $T>0$. Clearly, it must be interpreted as the contribution of thermal excitations which propagate ballistically. Indeed, to first order in ϵ ,²⁹ or at $n=\infty$,¹³ the excitations are simply

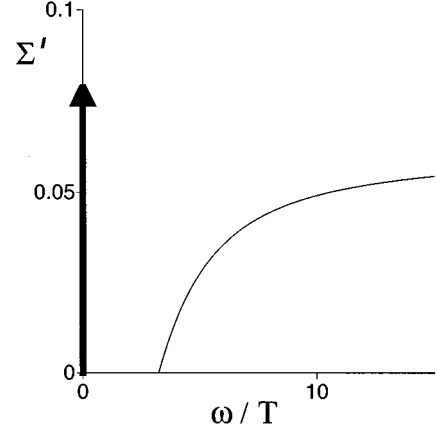


FIG. 4. The real part Σ' of the universal scaling function Σ [see Eq. (1.2)] for the conductivity at the quantum-critical coupling of the model \mathcal{S} , correct to first order in $\epsilon=3-d$. The numerical values are obtained from Eq. (2.3) and (2.10) with $d=2$ ($\epsilon=1$). There is a δ function precisely at $\omega/T=0$ represented by the heavy arrow: The weight of this δ function is given in Eq. (2.7) and (2.11). The δ function contributes to σ_I , and the higher-frequency continuum to σ_{II} .

undamped particles and holes with an infinite lifetime and energy momentum relation ϵ_k . It is necessary to go to second order in ϵ , or to first order in $1/n$, to include collisions which will give the quasiparticles a finite lifetime. We will show in Sec. III that these collisions also broaden the δ function in σ_I . The magnitude of the broadening is expected to be determined by the inverse lifetime of the quasiparticles; at the critical point, this inverse lifetime is of order $\epsilon^2 T$ Ref. 29 in the ϵ expansion, or of order T/n Ref. 13 in the large- n theory. The typical energy of a quasiparticle at the critical point is of order T , and so the quasiparticles are well defined, at least within the ϵ or $1/n$ expansion. Notice, however, that the quasiparticle interpretation breaks down at the physically important value of $\epsilon=1$, $n=2$. The discussion in Sec. III will take place within the context of the ϵ expansion, and will use the quasiparticle interpretation and the arsenal of powerful techniques available^{33,46,47} to describe their quantum transport.

The expression (2.7) is valid everywhere in the normal phase, but here we evaluate it explicitly only at $t_0=0$. Consider first the ϵ expansion. The coefficient of the δ function is a function of the ratio m/T , but notice from Eq. (2.3) that $m \ll T$ at for small ϵ . Evaluating Eq. (2.7) in this limit we find for ϵ small

$$\begin{aligned} \sigma'_I(\omega) &= 2\pi Q^2 T^{d-1} \delta(\omega) \left[\frac{1}{18} - \frac{m}{8\pi T} + \dots \right] \\ &= 2\pi Q^2 T^{d-1} \delta(\omega) \left[\frac{1}{18} - \frac{\sqrt{\epsilon}}{8} \left(\frac{2(n+2)}{3(n+8)} \right)^{1/2} + \dots \right]. \end{aligned} \quad (2.11)$$

Actually the expression (2.7) is good to order ϵ but we have refrained from displaying the next term as it is rather lengthy. The first term in Eq. (2.11) is obtained by evaluating Eq. (2.7) at $m=0$, $d=3$; the second term is from an integral dominated by small $k \sim m \ll T$ and hence the Bose function

can be replaced by its classical limit. An important point to note is that the current carried by the thermally excited carriers is dominated in the leading term of Eq. (2.11) by momenta $k \sim T \gg m$. This will be useful to us in the analysis of collisions in Sec. III where we will simply be able to set $m=0$ to obtain the leading term.

Turning to the large n theory, we evaluate Eq. (2.7) in $d=2$ using the value of m in Eq. (2.4), and obtain after an integration by parts and rescaling of variables

$$\begin{aligned} \sigma'_I(\omega) &= \frac{Q^2 T}{2} \delta(\omega) \left[\int_{\Theta}^{\infty} d\varepsilon \left(1 + \frac{\Theta^2}{\varepsilon^2} \right) \frac{1}{e^\varepsilon - 1} \right] \\ &= \frac{Q^2 T}{2} \delta(\omega) \times 0.689\,402\,548\,116\,632 \dots, \end{aligned} \quad (2.12)$$

where $\Theta = 2 \ln[(\sqrt{5} + 1)/2]$ is a number which plays a central role in the large n theory.^{13,49} Notice that as $m \sim T$, we have now been unable to approximate $\varepsilon_k \approx k$ to get the leading result, as was done in the ϵ expansion.

An interesting numerical property of the above results in $d=2$ is worth noting explicitly. The spectral weight of the δ function to leading order in the ϵ expansion is, from Eq. (2.11), $Q^2 T$ times $\pi/9 = 0.3491 \dots$ (recall that this number was obtained by evaluating a momentum space integral in $d=3$). The same quantity at $n=\infty$, from Eq. (2.12), is $Q^2 T$ times $0.3447 \dots$ (obtained now by evaluating a different momentum space integral in $d=2$), which is remarkably close. Later in Sec. III, we will consider broadening of the δ function in the ϵ expansion, and we will work in the approximation in which the spectral weight is $Q^2 T$ times $\pi/9$ [see Eq. (3.34) later]; the present numerical ‘‘coincidence’’ suggests that the numerical values of the leading order ϵ result are quite accurate.

B. σ_{II}

This is the continuum contribution to σ which vanishes for $\omega < 2m$. At this order in ϵ ($1/n$) there is a sharp threshold at $\omega = 2m$ but we expect that this singularity will be rounded out when collisions are included at order ϵ^2 ($1/n$): We will not describe this rounding out in this paper, however. Although they have a strong effect at the threshold, collisions are not expected to significantly modify the form of $\sigma'_{II}(\omega)$ at higher frequencies where the transport is predominantly collisionless. In particular, the $\omega \rightarrow \infty$ limit is precisely the $T=0$ result obtained earlier:¹⁹

$$\sigma'_{II}(\omega \rightarrow \infty) = \frac{\pi Q^2 S_d}{2^d d} |\omega|^{d-2}. \quad (2.13)$$

III. QUANTUM TRANSPORT EQUATIONS

The general analysis of higher-order corrections to σ is quite complex, and so we will confine ourselves in this section to the answer to a single question: How does the $\delta(\omega)$ term in $\sigma'_I(\omega)$ broaden?

We will address this question exclusively in the context of the ϵ expansion, and generalization to the $1/n$ expansion will

be discussed elsewhere. In principle, the answer can be obtained by including the $\mathcal{O}(\epsilon^2)$ correction to the self-energy and accounting for the associated infinite-ladder vertex corrections.⁴⁸ However, this method is quite inconvenient as it does not allow for easy separation of the distinct phenomena in different frequency regimes. Instead, we shall use the (in principle) equivalent⁴⁸ quantum transport formalism of Kadanoff and Baym.^{33,46,47} The physical content of this formalism is transparent at all stages, and the approximations necessary to focus on the low-frequency conductivity are readily apparent. In particular, we can drop the terms leading to $\sigma_{II}(\omega)$ at an early stage.

The transport equation is best studied in the Hamiltonian formalism by casting it in terms of the weakly interacting ‘‘normal modes.’’ So we begin by writing down the Hamiltonian associated with \mathcal{S} (remember that we are using units in which $\hbar = k_B = c = 1$):

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{ext}}. \quad (3.1)$$

The first term, \mathcal{H}_0 , is the free-particle part of \mathcal{S} ,

$$\mathcal{H}_0 = \frac{1}{2} \int [\pi_\alpha^2 + (\nabla_x \phi_\alpha)^2 + m^2 \phi_\alpha^2], \quad (3.2)$$

where $\pi_\alpha(\mathbf{x}, t)$ (t is real time) is the canonically conjugate momentum to the quantum field $\phi_\alpha(\mathbf{x}, t)$ and satisfies the equal-time commutation relations

$$[\phi_\alpha(\mathbf{x}, t), \pi_\beta(\mathbf{x}', t)] = i \delta_{\alpha\beta} \delta^d(\mathbf{x} - \mathbf{x}'). \quad (3.3)$$

For future convenience, we have already included the Hartree-Fock correction from the interactions into the ‘‘mass’’ m^2 (these correspond to self-consistently summing all the one-loop tadpole diagrams, as discussed in Appendix D). The second term, \mathcal{H}_{int} , is the quartic interaction

$$\mathcal{H}_{\text{int}} = \frac{u_0}{4!} \int d^d x (\phi_\alpha^2)^2, \quad (3.4)$$

and it is understood that the Hartree-Fock term arising from \mathcal{H}_{int} will be omitted. Finally, \mathcal{H}_{ext} contains the coupling to the external space and time-dependent potentials $U^a(\mathbf{x}, t)$ [$a = 1 \dots n(n-1)/2$]:

$$\mathcal{H}_{\text{ext}} = Q \int d^d x U^a(\mathbf{x}, t) L^a_{\alpha\beta} \pi_\alpha(\mathbf{x}, t) \phi_\beta(\mathbf{x}, t). \quad (3.5)$$

Here the L^a are $n \times n$ real, antisymmetric matrices that are i times the generators of the Lie algebra of $O(n)$. The U^a are coupled to the conserved $O(n)$ charge densities⁵⁰ of $\mathcal{H} + \mathcal{H}_{\text{int}}$. We shall be interested only in the linear response of the current to the ‘‘electric field’’ $\mathbf{E}^a = -\vec{\nabla}_x U^a(\mathbf{x}, t)$, and it will be assumed below that \mathbf{E}^a is independent of \mathbf{x} . Notice that we are making a gauge choice different from that in Eq. (2.1), and coupling now to the scalar and not the vector potential; this is for convenience, and should not change the final gauge-invariant results. The ‘‘charge’’ current \mathbf{J}^a is defined by the expectation value⁵⁰

$$\mathbf{J}^a = Q L^a_{\alpha\beta} \langle \phi_\alpha \vec{\nabla}_x \phi_\beta \rangle, \quad (3.6)$$

and it will also be independent of \mathbf{x} . Making the Fourier expansion

$$\mathbf{E}^a(t) = \int \frac{d\omega}{2\pi} \mathbf{E}^a(\omega) e^{-i\omega t}, \quad (3.7)$$

and similarly for \mathbf{J}^a , we can define the dynamical conductivity $\sigma(\omega)$ by the expected linear response relation

$$\mathbf{J}^a(\omega) = \sigma(\omega) \mathbf{E}^a(\omega). \quad (3.8)$$

For completeness, let us also note here the expression for the total momentum⁵⁰ \mathbf{P} :

$$\mathbf{P} = \langle \pi_\alpha \vec{\nabla}_x \phi_\alpha \rangle. \quad (3.9)$$

Notice that it is quite distinct from \mathbf{J}^a . In particular, in the absence of an external potential, \mathbf{P} is conserved (i.e., it obeys an equation of the form $\partial_t \mathbf{P} + \vec{\nabla} \cdot \vec{\mathbf{T}} = 0$ for some $\vec{\mathbf{T}}$), while \mathbf{J}^a is not.

We now make the mode expansion

$$\begin{aligned} \phi_\alpha(\mathbf{x}, t) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{2\varepsilon_k}} \\ &\times [a_\alpha(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}} + a_\alpha^\dagger(\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{x}}], \\ \pi_\alpha(\mathbf{x}, t) &= -i \int \frac{d^d k}{(2\pi)^d} \sqrt{\frac{\varepsilon_k}{2}} \\ &\times [a_\alpha(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}} - a_\alpha^\dagger(\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{x}}], \end{aligned} \quad (3.10)$$

where the $a(\mathbf{k}, t)$ operators satisfy the equal-time commutation relations

$$\begin{aligned} [a_\alpha(\mathbf{k}, t), a_\beta^\dagger(\mathbf{k}', t)] &= \delta_{\alpha\beta} (2\pi)^d \delta^d(\mathbf{k} - \mathbf{k}'), \\ [a_\alpha(\mathbf{k}, t), a_\beta(\mathbf{k}', t)] &= 0. \end{aligned} \quad (3.11)$$

It can now be verified that Eq. (3.3) is satisfied, and \mathcal{H}_0 is given by

$$\mathcal{H}_0 = \int \frac{d^d k}{(2\pi)^d} \varepsilon_k [a_\alpha^\dagger(\mathbf{k}, t) a_\alpha(\mathbf{k}, t) + 1/2]. \quad (3.12)$$

We will also need the expression for the current \mathbf{J}^a in terms of the a and a^\dagger . We will only be interested in the case where the system carries a position-independent current: For this case, inserting Eq. (3.10) into Eq. (3.6), we find

$$\begin{aligned} \mathbf{J}^a(t) &= \mathbf{J}_I^a(t) + \mathbf{J}_{II}^a(t), \\ \mathbf{J}_I^a(t) &= iQL_{\alpha\beta}^a \int \frac{d^d k}{(2\pi)^d} \frac{\mathbf{k}}{\varepsilon_k} \langle a_\alpha^\dagger(\mathbf{k}, t) a_\beta(\mathbf{k}, t) \rangle, \\ \mathbf{J}_{II}^a(t) &= -iQL_{\alpha\beta}^a \int \frac{d^d k}{(2\pi)^d} \frac{\mathbf{k}}{2\varepsilon_k} \langle a_\alpha^\dagger(-\mathbf{k}, t) a_\beta^\dagger(\mathbf{k}, t) \rangle + \text{H.c.} \end{aligned} \quad (3.13)$$

It should be evident that processes contributing to \mathbf{J}_{II}^a require a minimum frequency of $2m$, and so \mathbf{J}_{II}^a only contributes to $\sigma_{II}(\omega)$. We will therefore drop the \mathbf{J}_{II}^a contribution below and approximate $\mathbf{J}^a \approx \mathbf{J}_I^a$. The ease with which the high-

frequency components of $\sigma(\omega)$ can be separated out is an important advantage of the present formulation of the quantum transport equations.

A central object in transport theory is the Green's function^{33,47}

$$\begin{aligned} g_{\alpha\beta}^<(\mathbf{k}, \Omega, \mathbf{R}, t) &= \int \frac{d^d K}{(2\pi)^d} \int dt_1 e^{i\mathbf{K} \cdot \mathbf{R} + i\Omega t_1} \\ &\times \langle a_\beta^\dagger(\mathbf{k} - \mathbf{K}/2, t - t_1/2) a_\alpha(\mathbf{k} + \mathbf{K}/2, t + t_1/2) \rangle. \end{aligned} \quad (3.14)$$

For the case of a system carrying a spatially independent current, $g^<$ will be independent of \mathbf{R} and this will implicitly be assumed below by dropping the \mathbf{R} argument. We also define the particle distribution function

$$f_{\alpha\beta}(\mathbf{k}, t) = \int \frac{d\Omega}{2\pi} g_{\alpha\beta}^<(\mathbf{k}, \Omega, t), \quad (3.15)$$

in terms of which the current is

$$\mathbf{J}^a(t) = iQL_{\alpha\beta}^a \int \frac{d^d k}{(2\pi)^d} \frac{\mathbf{k}}{\varepsilon_k} f_{\alpha\beta}(\mathbf{k}, t). \quad (3.16)$$

The corresponding expression for the momentum density is

$$\mathbf{P}(t) = \int \frac{d^d k}{(2\pi)^d} \mathbf{k} f_{\alpha\alpha}(\mathbf{k}, t). \quad (3.17)$$

Notice the difference in the structure of the $O(n)$ indices between Eqs. (3.16) and (3.17).

We note here that our formulation of the transport theory in terms of Green's functions of the a_α , a_α^\dagger rather than those of the ϕ_α , π_α was motivated in part by recent exact results in $d=1$ for time-dependent correlations of models equivalent to \mathcal{S} for $n=1,3$.^{51,43} In the latter cases it was evident that the physics is most simply described by following the propagators of particles created or annihilated by $a_\alpha^\dagger/a_\alpha$ through their collisions.

A. Collisionless transport

In this section we will examine the transport equations for the collisionless case where $\mathcal{H}_{\text{int}}=0$; we remind the reader that interactions have already been included at the Hartree-Fock level in \mathcal{H}_0 (see Appendix D). Strictly speaking, we also have to remember that the mass m can in general depend upon \mathbf{E}^a ; however, for the case of a momentum-independent local interaction u_0 , such a ‘‘vertex’’ correction vanishes to lowest order in \mathbf{E}^a , and will therefore be omitted from our discussion.

While it is possible to discuss the general $O(n)$ case, in the interest of simplicity and to keep the physical content transparent, we will restrict our attention here to the special case $n=2$. The generalization to $n>2$ is discussed in Appendix B.

For $n=2$ ($\alpha=1,2$), there is only one real antisymmetric matrix, and therefore the index a can be dropped. We choose $L_{1,2} = -L_{2,1} = 1$ and $L_{1,1} = L_{2,2} = 0$. This matrix is off diago-

nal and it is helpful to transform to a basis where the external field is diagonal. We therefore define

$$a_{\pm}(\mathbf{k}, t) \equiv \frac{a_1(\mathbf{k}, t) \pm ia_2(\mathbf{k}, t)}{\sqrt{2}}. \quad (3.18)$$

The current now becomes

$$\begin{aligned} \mathbf{J} &= Q \int \frac{d^d k}{(2\pi)^d} \sum_{\lambda} \lambda \frac{\mathbf{k}}{\varepsilon_k} \langle a_{\lambda}^{\dagger}(\mathbf{k}, t) a_{\lambda}(\mathbf{k}, t) \rangle \\ &= Q \int \frac{d^d k}{(2\pi)^d} \sum_{\lambda} \lambda \frac{\mathbf{k}}{\varepsilon_k} f_{\lambda}(\mathbf{k}, t), \end{aligned} \quad (3.19)$$

where the index λ is assumed here and below to extend over the values ± 1 , and f_{λ} are the particle distribution functions which are now diagonal in λ space. Notice that there are two species of charged particles with charges $\pm Q$: These are the particlelike and holelike excitations of the bosonic insulator. Let us also note the expression for the momentum density:

$$\mathbf{P} = \int \frac{d^d k}{(2\pi)^d} \sum_{\lambda} \mathbf{k} f_{\lambda}(\mathbf{k}, t). \quad (3.20)$$

An important difference between Eqs. (3.19) and (3.20) is the λ inside the summation in Eq. (3.19) which is absent from Eq. (3.20). Thus the ‘‘charge’’ current is proportional to the difference of the particle and hole number currents, while the momentum density is proportional to their sum.

It is now easy to use standard methods^{33,47} to derive the following transport equation in the collisionless limit described earlier:

$$\left(\frac{\partial}{\partial t} + \lambda Q \mathbf{E}(t) \cdot \frac{\partial}{\partial \mathbf{k}} \right) f_{\lambda}(\mathbf{k}, t) = 0. \quad (3.21)$$

In deriving this equation we have made approximations to the charge density appearing in \mathcal{H}_{ext} similar to those made for \mathbf{J} : Upon expressing \mathcal{H}_{ext} in terms of the a, a^{\dagger} we have dropped all terms involving the product of two a 's or a^{\dagger} 's as these will only contribute to the high-frequency σ_{II} . Equations (3.19) and (3.21) are therefore accurate to first order in u_0 provided we are limiting ourselves to frequencies $\omega \ll 2m$.

Let us now solve Eqs. (3.19) and (3.21) in linear response. In the absence of \mathbf{E} , the distribution function has the equi-

librium value given by Bose function $f_{\lambda}(\mathbf{k}, t) = n(\varepsilon_k)$. We Fourier transform from time t to frequency ω , and parametrize to linear order in \mathbf{E} :

$$f_{\lambda}(\mathbf{k}, \omega) = 2\pi \delta(\omega) n(\varepsilon_k) + \lambda Q \mathbf{k} \cdot \mathbf{E}(\omega) \psi(k, \omega), \quad (3.22)$$

where we have used the fact that only \mathbf{E} breaks spatial rotational invariance and $O(2)$ symmetry to conclude that ψ is independent of \mathbf{k}/k and λ . Now inserting in Eq. (3.21), and using $\partial \varepsilon_k / \partial \mathbf{k} = \mathbf{k} / \varepsilon_k$ it is simple to solve for ψ to leading order in \mathbf{E} :

$$\psi(k, \omega) = \frac{1}{-i\omega} \frac{1}{\varepsilon_k} \left(-\frac{\partial n(\varepsilon_k)}{\partial \varepsilon_k} \right). \quad (3.23)$$

Finally we insert in Eq. (3.19) and deduce the conductivity

$$\sigma(\omega) = \frac{2Q^2}{-i\omega} \int \frac{d^d k}{(2\pi)^d} \frac{k_x^2}{\varepsilon_k^2} \left(-\frac{\partial n(\varepsilon_k)}{\partial \varepsilon_k} \right). \quad (3.24)$$

The real part of this agrees with Eq. (2.7). Notice that the leading factor of 2 comes from the sum over λ . The current is therefore carried equally by the thermally excited particles and holes: They move in opposite directions to create a state with vanishing momentum but non-zero charge current.

We will see in the next section that this charge current can be relaxed by collisions among the particles and holes. There is no need to invoke umklapp scattering as a momentum sink and source (as was done by Cha *et al.*¹⁵) as the current-carrying state does not have a net momentum to begin with.

B. Collision-dominated transport

We have seen in Sec. III A that to order $u_0(\epsilon)$ the transport is described by the ballistic motion of undamped particles of two charges. We now consider the collisions of these particles which appear at order ϵ^2 . As we noted in the discussion below Eq. (2.11) the typical energy of a particle contributing to the transport was of order T . Their collisions will lead to a broadening of the single-quasiparticle pole of order $\epsilon^2 T$;²⁹ for ϵ small, this broadening is weak, and it is then permissible to argue in terms of a quasiparticle interpretation.

Applying the standard methods of transport theory discussed in Refs. 33 and 47, we generalize Eq. (3.21) to include the collision term which appears at order u_0^2 , and obtain to linear order in \mathbf{E} :

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \lambda Q \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{k}} \right) f_{\lambda}(\mathbf{k}, t) &= -\frac{2u_0^2}{9} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{1}{16\varepsilon_k \varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{k_3}} (2\pi)^d \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) 2\pi \delta(\varepsilon_k + \varepsilon_{k_1} - \varepsilon_{k_2} - \varepsilon_{k_3}) \\ &\times \{ 2f_{\lambda}(\mathbf{k}, t) f_{-\lambda}(\mathbf{k}_1, t) [1 + f_{\lambda}(\mathbf{k}_2, t)] [1 + f_{-\lambda}(\mathbf{k}_3, t)] + f_{\lambda}(\mathbf{k}, t) f_{\lambda}(\mathbf{k}_1, t) [1 + f_{\lambda}(\mathbf{k}_2, t)] \\ &\times [1 + f_{\lambda}(\mathbf{k}_3, t)] - 2[1 + f_{\lambda}(\mathbf{k}, t)] [1 + f_{-\lambda}(\mathbf{k}_1, t)] f_{\lambda}(\mathbf{k}_2, t) f_{-\lambda}(\mathbf{k}_3, t) - [1 + f_{\lambda}(\mathbf{k}, t)] \\ &\times [1 + f_{\lambda}(\mathbf{k}_1, t)] f_{\lambda}(\mathbf{k}_2, t) f_{\lambda}(\mathbf{k}_3, t) \}. \end{aligned} \quad (3.25)$$

The collision term on the right-hand side of Eq. (3.25) can also be obtained by a simple argument based on Fermi's golden rule of the type described in Ref. 33. A number of simplifications have been made in deriving Eq. (3.25), and we now describe and justify them.

(i) The collision term is initially expressed in terms of full two-point Green's functions like $g_{\alpha\beta}^<$. However, as is conventional,^{33,47} we assume we can neglect damping in these Green's functions and express them in terms of the particle distribution functions $f_{\alpha\beta}$. This is permissible for a system with well-defined quasiparticles, as is the case here for small ϵ . This approximation will not affect the conductivity at order ϵ^2 .

(ii) In addition to the collision terms, there are also self-energy terms affecting the quasiparticle energies at order u_0^2 . These include terms that couple the usual Green's functions to the anomalous ones (involving expectation values of pairs of creation or annihilation operators). Such terms will, in general, modify the conductivity at order ϵ^2 . However, they do not affect the nature of broadening of the δ function in $\sigma_1'(\omega)$. The total spectral weight in $\sigma_1'(\omega)$ will change from that in Eq. (2.11) at order ϵ^2 due to these terms, but the functional form of $\sigma_1'(\omega)$ at frequencies of order $\epsilon^2 T$ will not be affected.

(iii) We have neglected collisions which involve creation or annihilation of particle-hole pairs as they have negligible phase space. Thus a collision in which, e.g., a positively charged particle of momentum \mathbf{k} turns into two positively charged particles and a negatively charged hole with momenta \mathbf{k}_1 , \mathbf{k}_2 , and \mathbf{k}_3 , respectively, is permitted by the symmetries of the problem. However, it remains to evaluate the phase space over which such collisions conserve total energy and momentum. Notice that the "mass" m of the particles and holes is of order $\sqrt{\epsilon} T$ [Eq. (2.3)] while their momentum is of order T . So to leading order in ϵ we may just replace the energy-momentum relation (2.9) by [see also the discussion below Eq. (2.11)]

$$\varepsilon_k = k. \quad (3.26)$$

We will use this simplified energy-momentum relation throughout this subsection. The particle-hole pair-creation collision now requires that $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3$ and $k = k_1 + k_2 + k_3$. This is only possible if all three momenta are collinear, and this process therefore has vanishing phase space. More generally, for a nonzero m , the phase space vanishes as $\epsilon \rightarrow 0$.

We now insert the parametrization (3.22) in Eq. (3.25), linearize the resulting equation in the external electric field \mathbf{E} , and obtain a linear integral equation for the unknown function $\psi(k, \omega)$. Further, to leading order in ϵ , we may set $d=3$ in the collision term in Eq. (3.25), and replace the coupling u_0 by its fixed point value for $n=2$ (see Appendix D):

$$u_0 = \frac{24\pi^2 \epsilon}{5}. \quad (3.27)$$

Further details of this linearization procedure are given in Appendix C. The final integral equation for $\psi(k, \omega)$ can be written as

$$-i\omega\psi(k, \omega) + \frac{1}{k} \frac{\partial n(k)}{\partial k} = -\epsilon^2 \int_0^\infty dk_1 [F_1(k, k_1)\psi(k, \omega) + F_2(k, k_1)\psi(k_1, \omega)]. \quad (3.28)$$

The expressions for the functions $F_{1,2}$ are quite lengthy and are discussed in Appendix C.

It is now useful to scale out the dependence of all functions on ϵ and T so that the final integral equation is written in dimensionless form, and has all couplings of order unity. From the expressions in Appendix C we know that $F_{1,2}$ are homogeneous functions of momenta and T which satisfy

$$F_i(k, k_1) = \Phi_i(k/T, k_1/T) \quad (3.29)$$

for some Φ_i , with $i=1,2$. (Here, and everywhere in this section, the dimensional analysis is performed for at $d=3$: It is not difficult to extend it to general d , but we will not in the interests of simplicity, and because it is unnecessary to obtain results to leading order in ϵ .) Now by examining Eq. (3.28) we see that it is useful to introduce the function Ψ defined by

$$\psi(k, \omega) = \frac{1}{\epsilon^2 T^3} \Psi\left(\frac{k}{T}, \frac{\omega}{\epsilon^2 T}\right). \quad (3.30)$$

In terms of Φ_i , Ψ the integral equation (3.28) takes the dimensionless form

$$\begin{aligned} -i\tilde{\omega}\Psi(\bar{k}, \tilde{\omega}) - \frac{1}{4\bar{k}\sinh^2(\bar{k}/2)} \\ = - \int_0^\infty d\bar{k}_1 [\Phi_1(\bar{k}, \bar{k}_1)\Psi(\bar{k}, \tilde{\omega}) \\ + \Phi_2(\bar{k}, \bar{k}_1)\Psi(\bar{k}_1, \omega)], \end{aligned} \quad (3.31)$$

where $\bar{k} = k/T$ and $\tilde{\omega} = \omega/\epsilon^2 = \omega/\epsilon^2 T$. Now, from Eq. (3.19) we see that the conductivity $\sigma_1(\omega)$ obeys

$$\sigma_1(\omega) = \frac{Q^2 T^{d-2}}{\epsilon^2} \Sigma_1\left(\frac{\omega}{\epsilon^2 T}\right), \quad (3.32)$$

where the scaling function Σ_1 is given by

$$\Sigma_1(\tilde{\omega}) = \frac{1}{3\pi^2} \int_0^\infty \bar{k}^3 d\bar{k} \Psi(\bar{k}, \tilde{\omega}). \quad (3.33)$$

We can now already see from the structure of Eq. (3.32) that $\sigma_1'(\omega=0)$ has a value of order T^{d-2}/ϵ^2 , and that the width of the "Drude" peak will be of order $\omega \sim \epsilon^2 T$. Further, the collision term will not modify the total spectral weight in σ_1 , which will therefore be identical to that in Eq. (2.11); this implies that the function Σ_1 , as defined by Eqs. (3.31) and (3.33) should satisfy

$$\int_0^\infty d\tilde{\omega} \tilde{\omega} \Sigma_1'(\tilde{\omega}) = \frac{\pi}{18}. \quad (3.34)$$

It should be noted that this sum rule is special to the leading order in ϵ being considered here. For ϵ of order unity, there is no sharp distinction between σ_I and σ_{II} and there is no

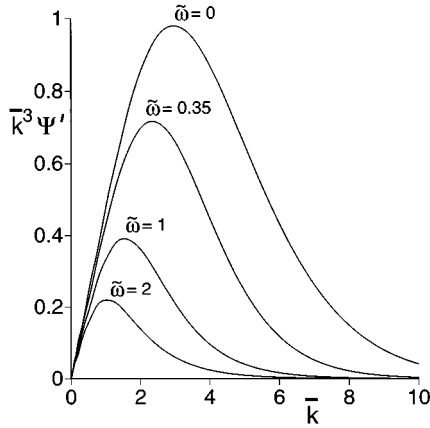


FIG. 5. Real part of the universal function $\bar{k}^3\Psi(\bar{k},\tilde{\omega})$ as a function of \bar{k} for a few values of $\tilde{\omega}$. The function Ψ is defined in Eqs. (3.30) and (3.22), and was obtained by numerical solution of the linearized quantum Boltzmann equation (3.31). At $\tilde{\omega}=0$, Ψ is real, but is complex for general $\tilde{\omega}$. Here $\bar{k}=k/T$, and $\tilde{\omega}=\bar{\omega}/\epsilon^2=\omega/\epsilon^2T$ (in physical units $\bar{k}=\hbar ck/k_B T$, $\tilde{\omega}=\hbar\omega/\epsilon^2 k_B T$).

sum rule: Indeed the integral in Eq. (3.34) when carried out over the total σ will be divergent. For any realistic lattice model there is a large microscopic energy scale ($\sim U$ the repulsion between bosons on the same site) beyond which the universal scaling results do not apply, and the entire spectral weight (including frequencies beyond U) is not divergent; this latter spectral weight satisfies a sum rule related to nonuniversal microscopic quantities, and is unrelated to the universal result (3.34).

It now remains to numerically solve Eq. (3.31) to determine $\Psi(\bar{k},\tilde{\omega})$, and then to obtain Σ_1 from Eq. (3.33). The integral equation (3.31) was solved by a straightforward numerical iteration, and we found very rapid convergence to its unique solution. We show a plot of its solution at a few values of $\tilde{\omega}$ in Fig. 5. The final result for the universal function $\Sigma_1'(\tilde{\omega})$ is shown in Fig. 6.

IV. CONCLUSIONS

The central message of this paper is simply stated: Understanding the universal dc conductivity of a two-dimensional system at its quantum-critical point requires a nonzero temperature analysis of the hydrodynamic, collision-dominated regime where $\hbar\omega\ll k_B T$. *The transport in this low-frequency regime is incoherent, but nevertheless the remarkable fact is that the conductivity is still a universal number times e^2/h .* Computations carried out exactly at $T=0$, with $\omega\rightarrow 0$, do not yield the dc conductivity, and are controlled by very different physical processes involving phase-coherent, collisionless transport. A related comment is that a theoretical analysis for the dc conductivity must necessarily be formulated in real time, as the imaginary-time Matsubara frequencies are of order $2\pi k_B T/\hbar$ or larger, and cannot easily capture the singular structure in Fig. 2. These criticisms apply not only to computations of the universal conductivity at the superfluid-insulator transition^{14–25} but also to the transitions between the Hall plateaus,^{52,53} all of whom computed the analog of $\Sigma(\infty)$. There are also mappings between these

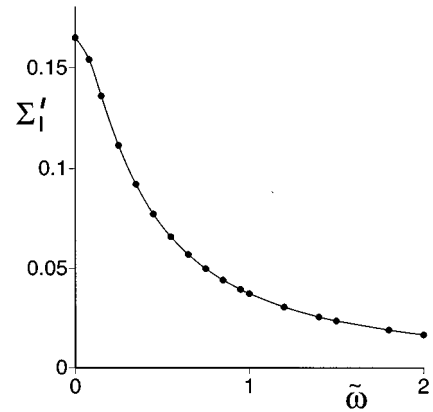


FIG. 6. The real part of the universal function $\Sigma_1(\tilde{\omega})$, which is related to the low-frequency part of the conductivity [$\sigma_1(\omega)$] by Eq. (3.32). The results are obtained by the numerical solution of Eq. (3.31), followed by the integration in Eq. (3.33). This function describes the inelastic collision-induced broadening of the $\omega=0$ δ function in Fig. 4 at a frequency scale of order $\epsilon^2 T$. The conductivity has an additional continuum contribution [$\sigma_{II}(\omega)$] at frequencies larger than $\omega\sim\epsilon^{1/2}T$ which is not shown above (see Fig. 3).

models,⁵⁴ but they presumably hold for all values of the ratio $\hbar\omega/k_B T$. There are also special self-dual models^{55,56} in which the conductivity is claimed⁵⁷ to be independent of $\hbar\omega/k_B T$, but this is not expected to be the generic physical situation. The computation of the finite T conductivity at the superfluid-insulator transition in one dimension by Giamarchi and Schulz⁵⁸ (and close to one dimension by Herbut⁵⁹) seems closer in spirit to our approach, but it would be useful to have an explicit computation as a function of $\hbar\omega/k_B T$ to verify this.

Many experiments on two-dimensional films appear not to observe a universal conductivity at the superfluid-insulator transition. Our results imply that this is very likely due to crossovers caused by *inelastic* scattering mechanisms other than those contained in the critical theory. Measurements of the conductivity as a function of frequency (Sec. IV D) should help disentangle these effects.

In the remainder of this section we comment on some general experimental and theoretical issues related to the results of the paper.

A. Imaginary time Monte Carlo simulation

Most existing numerical studies^{15,22–25} of σ at a two-dimensional quantum-critical point used a Monte Carlo simulation in imaginary time. The simulation measures the values of σ at the nonzero Matsubara frequencies, i.e., at $\sigma(2\pi nTi)$ where n is a nonzero, positive integer. The limit $\Sigma(z,|z|\rightarrow\infty)$ is expected to be the same number $\Sigma(\infty)$ for all values of the phase of z , and therefore its value can be deduced from a relatively straightforward analysis of the numerical data. The value of $\Sigma(0)$ is more problematical. All of the interesting structure in $\Sigma(z)$ discussed in this paper occurs for z of order, or less than unity, and it is difficult to extract this information from its values at the nonzero quantized Matsubara frequencies. Cha *et al.*¹⁵ numerically examined the model S and found little dependence on ω_n/T . Note

that this is the model for which we have computed the strongly ω/T -dependent conductivity here, but, as expected, this is apparently not evident at the imaginary Matsubara frequencies. Similarly, Wallin *et al.*²⁵ also found little evidence for a significant dependence of the critical conductivity of disordered models on ω_n/T .

We are of the opinion that it will be difficult to determine $\Sigma(0)$ from this method, unless highly accurate numerical results are obtained at the imaginary frequencies. The difficulty is also apparent by a glance at Fig. 2: At $T=0$, the value of $\Sigma(0)$ appears only at a single point which carries zero weight under any integral over frequencies. The accurate numerical data should then be analyzed in the following manner. First, from observations at a number of different values of ω_n and T , the universal scaling part of σ , dependent only on the ratio ω_n/T , should be obtained. Note that this universal scaling part reaches a constant value $[(4e^2/\hbar)\Sigma(\infty)]$ as $\omega_n \rightarrow \infty$, while the full σ vanishes as $1/\omega_n^2$ for frequencies larger than a microscopic lattice scale. Then a Padé analysis, general enough to allow the structure in Fig. 1, should be used to analytically continue *only* the universal scaling part to real frequencies.

It would also be interesting to explore newer methods: perhaps examining an open system in which it is possible to have a net current in thermodynamic equilibrium, or doing a computation in real time. The exact diagonalization approach of Runge²¹ can perhaps be extended to $T>0$ without a great deal of difficulty.

B. Dangerously irrelevant interactions

It is possible to violate the basic scaling result (1.1) if quantum-mechanical interactions between the elementary critical excitations happen to be dangerously irrelevant;⁶⁰ in that case we expect

$$\frac{1}{\tau_{\text{tr}}} \sim u T^{1+\theta_u/z}, \quad (4.1)$$

where u is proportional to some power of the irrelevant interaction, $\theta_u > 0$ is the associated crossover exponent, and z is the dynamic critical exponent. Anderson localization transitions, with interactions leading only to an infrared cutoff as a phase-breaking rate, are a realization of such a scenario. (Quantum-impurity critical points^{36,38-40} have some similarities to bulk systems with dangerously irrelevant interactions, and are discussed in more detail in Appendix A.) However, this scenario is much less likely to be realized at *bulk* two-dimensional quantum-critical points, and we consider it unlikely that interactions can be neglected for the superfluid-insulator transition. The quantum Hall transition has primarily been studied using noninteracting electrons,⁶¹ but there is evidence that Coulomb interactions are relevant at quantum Hall transitions.^{62,63} Experimental measurements of z in the quantum Hall system⁶⁴ indicate the value $z=1$ for the dynamic critical exponent, and this is incompatible with free-electron models. The ac conductivity measurements of Engel *et al.*³⁴ in a quantum Hall system (discussed further in Sec. IV D below) show a characteristic frequency scale $\omega \sim k_B T/\hbar$ which is inconsistent with the result (4.1) for the case of irrelevant interactions, but is consistent with the interacting theory result (1.1).

C. Luttinger liquids

A well-studied critical system in $d=1$ is the gapless Luttinger-liquid ground state of interacting fermions or bosons away from commensurate filling fractions. Here we discuss how this familiar system fits into the general framework of this paper. The nonzero- T conductivity of the critical theory of the Luttinger liquid is given by

$$\sigma'(\omega) = K \delta(\hbar\omega), \quad (4.2)$$

where K is some T -independent constant. Notice that there is no broadening of the δ function, even for $T>0$, in the scaling limit. (Related considerations also apply to Fermi liquids in $d>1$,^{65,66} but for many purposes these are better thought of as the analog of Goldstone phases rather than critical phases.) Let us now rewrite Eq. (4.2) as

$$\sigma'(\omega) = K(k_B T)^{-1} \delta\left(\frac{\hbar\omega}{k_B T}\right). \quad (4.3)$$

Now notice that Eq. (4.3) is consistent with the general scaling form (1.2) for $d=1$, and the known dynamic exponent $z=1$. The scaling function Σ takes the simple and singular form $\Sigma'(\bar{\omega}) = \delta(\bar{\omega})$. For this special form of Σ , the limits $\omega \rightarrow 0$ and $T \rightarrow 0$ do happen to commute.

These singular properties of the critical theory of the Luttinger liquid are clearly a consequence of the absence of scattering between carriers in the critical theory: Like many other critical theories in $d=1$, it is conformally invariant, and correlation functions (including those for $T>0$) factorize into independent components given by the left and right movers. In contrast, for critical theories in $d>1$, like the one studied in this paper, no such analogous factorization exists. Further, for $d>1$ models below their upper critical dimension, there is scattering between the carriers already in the scaling limit. To introduce scattering in the Luttinger liquid, it is necessary to go beyond the scaling limit, and consider corrections to scaling.⁴⁵ These corrections are therefore dangerously irrelevant, and will destroy simple δ function form of the conductivity in Eq. (4.2), and the limits $\omega \rightarrow 0$ and $T \rightarrow 0$ will no longer commute.⁴⁵

D. Measurements of the ac conductivity

A finite- ω measurement of the conductivity in a situation related to that discussed in this paper has been carried out by Engel *et al.*³⁴ They examined the transition between integer quantum Hall plateaus by studying the dependence of the conductivity on ω , and the deviation of the field from its critical value $\delta \equiv (B - B_c)/B_c$. This should obey⁷ the scaling form

$$\sigma = \frac{e^2}{\hbar} \Sigma \left(\frac{\hbar\omega}{k_B T}, \frac{\delta}{T^{1/\nu z}} \right), \quad (4.4)$$

which generalizes Eq. (1.2) to $\delta \neq 0$. In their analysis, Engel *et al.*³⁴ focused mainly on the δ dependence at $\hbar\omega \gg k_B T$: They measured the width of the transition region in $\delta = \Delta B$, and found $\Delta B \sim \omega^{1/\nu z}$, in agreement with Eq. (4.4). Further, the ω dependence of ΔB saturated for $\omega < k_B T/\hbar$, in agreement with the ideas we have discussed here.^{29,7} However, Engel *et al.*³⁴ did not analyze the ω and T dependence of σ

precisely at the critical field $\delta=0$. It appears to us that it should be relatively straightforward to extend their measurements at $\delta=0$ to test the validity of the scaling form (1.2). Further, it should also be possible to theoretically determine the expected form of the scaling function Σ in the models for the quantum Hall transitions considered in Refs. 52 and 53.

A limited test of Eq. (1.2) should also be possible in measurements of the ac conductivity at relatively low frequencies at which $\hbar\omega \ll k_B T$. At such frequencies, we can expand the scaling function $\Sigma(\bar{\omega})$ about $\bar{\omega}=0$, and analyticity of $T>0$ properties then implies the form

$$\sigma'(\omega) = \frac{Q^2}{\hbar} \left(\frac{k_B T}{\hbar c} \right)^{(d-2)/z} \left[\Sigma(0) - \frac{1}{2} \Sigma^{(2)}(0) \left(\frac{\hbar\omega}{k_B T} \right)^2 + \dots \right], \quad (4.5)$$

where $\Sigma(0)$ and $\Sigma^{(2)}(0)$ are expected to be positive universal numbers of order unity. Experimentally, one can test if the frequency-dependent correction to σ has the $1/T^2$ dependence predicted by Eq. (4.5).

There do not appear to be any existing measurements of the ac conductivity near quantum-critical points in other systems. The results (1.2) and (4.5) should apply also to the superfluid-insulator transition in thin films,^{1,2} to the quantum transition in the doped cuprates⁹ and two-dimensional MOSFET's,¹⁰ and to the metal-insulator transition in three-dimensional Si:P;^{11,12} we hope that experiments will be undertaken, as the results will be central to our theoretical understanding of these systems. In particular, if scaling as a function of $\hbar\omega/k_B T$ is observed, it would establish quite conclusively that interactions are an essential ingredient in the critical theory.

Also, we note that our picture suggests a rather interesting nonmonotonic ω dependence of σ in at the metal-insulator transition in $d=3$: For small $\hbar\omega/k_B T$ σ should decrease with increasing ω as predicted by Eq. (4.5), but for larger $\hbar\omega/k_B T$ it should increase as $\sim \omega^{1/z}$. It would be worthwhile to undertake analytic calculations for disordered, interacting, electronic systems to search for this nonmonotonic ω dependence: There are cases where the critical theory is accessible at low orders in the $2+\epsilon$ expansion.¹¹ There are existing calculations for the ω and T dependence of the conductivity in the weakly disordered metal³⁰ in which $1/\tau_\varphi \ll k_B T/\hbar$; these need to be extended to the critical point where we expect that $1/\tau_\varphi \sim k_B T/\hbar$.

E. Self-duality

Of great current interest is the issue of ‘‘boson-vortex duality’’ at two-dimensional quantum-critical points. At the superfluid-insulator transition, many of the experimentally measured values of $\Sigma(0)$ appear to be tantalizingly close² to a value predicted by self-duality arguments.^{16,17} For the quantum Hall transitions, experimental evidence for self-duality has been presented recently.⁶⁷ This self-duality appears rather surprising as there is no fundamental reason for an equivalence between the underlying boson and vortex Hamiltonians.¹⁶ We suggest here that these inequivalent Hamiltonians will be apparent in the value of the $T=0$ conductivity, given by $\Sigma(\infty)$, which, incidentally, is the quantity that has been explicitly or implicitly studied in earlier

work.^{14–25,52–56} In contrast, the dc conductivity, given by $\Sigma(0)$, is controlled by the crossover to the hydrodynamic, collision-dominated regime, and we propose that this could be less sensitive to the details of the boson-vortex Hamiltonian. This proposal is easily subject to experimental tests: Measure $\Sigma(\infty)$ by determining the conductivity in the regime $\hbar\omega \gg k_B T$, and see if the self-duality predictions continue to hold—we predict they will *not*.

Our suggestion that $\Sigma(0)$ could be insensitive to differences in the interaction Hamiltonian of bosons and vortices is motivated by the expectation that an important role of interactions is in the collisions that establish local thermodynamic equilibrium.^{33,68} As a result, the equations governing the net hydrodynamic flow of bosons and vortices in the collision-dominated regime could be more symmetrical than the underlying Hamiltonian. In other words, we are proposing here that a true understanding of the experimentally observed duality near the quantum-critical point will emerge from a study of the crossover from the microscopic quantum-critical physics of the elementary excitations to the low-frequency collision-dominated regime best described by a quantum Boltzmann equation.

We suggest that even the simple model \mathcal{S} is self-dual for $d=2$, $n=2$ in its dc transport, i.e., $2\pi\Sigma(0)=1$ exactly for the two-dimensional quantum XY rotor model. This model has boson particle-hole excitations with short-range interactions, while the vortices in the dual representation have long-range logarithmic (Coulomb in $d=2$) interactions. However, at nonzero T , this logarithmic interaction will be screened in manner analogous to the classical Debye screening above the Kosterlitz-Thouless temperature T_{KT} ; indeed the region $g=g_c$, $T>0$ of interest here is continuously connected to the $g<g_c$, $T>T_{KT}$ region. So the effective interactions between the bosons and vortices are both short range for $T>0$, leaving open the possibility of self-dual behavior at low frequencies. There is now general agreement that $2\pi\Sigma(\infty) \approx 0.3$ for this model.¹⁹ This paper contains the first computation of $\Sigma(0)$, and as we noted earlier near Eqs. (1.4) and (1.5), it is quite remarkable, though possibly fortuitous, that our leading-order result for $\Sigma(0)$ in the ϵ expansion differs from the self-dual value by less than 4%. Definitively establishing this self-duality would, however, require techniques other than expansion in $\epsilon=3-d$, or $1/n$, as it is only possible precisely at $d=2$, $n=2$. It would be of great interest to undertake higher-precision quantum Monte Carlo, exact diagonalization, or high-temperature series studies while carefully examining the reliability of the analytic continuation to real frequencies. [The reader may be interested in a recent article⁶⁹ by one of us discussing this work in the context of recent results on nonzero-temperature dynamical properties of \mathcal{S} for other values of d and n , including exact solutions in $d=1$.]

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APPENDIX A: TRANSPORT IN DISSIPATIVE QUANTUM MECHANICS

We will consider the transport properties of a well-studied and representative model from the subject of dissipative quantum mechanics and related quantum-impurity problems.³⁶ Our purpose here is to contrast the ω and T dependence of the transport coefficient in such a situation with that of bulk critical points studied in the body of the paper. We believe that such an exercise will help clarify the significance of our results for the reader. We thank E. Fradkin for posing the questions which led to the analysis below. We will use units with $\hbar = k_B = 1$ in this appendix.

Consider the motion of a quantum ‘‘particle’’ in a periodic potential in the presence of a linear coupling to a Ohmic heat bath.^{36,70–77,38–40} We represent the time-dependent coordinate of the particle by $X(t)$, and its analytic continuation to imaginary-time by $X(\tau)$, and the Fourier transform to imaginary Matsubara frequencies by $X(\omega_n)$. The imaginary-time effective action obtained after integrating out the degrees of freedom of the heat bath is

$$S_1 = \frac{T\alpha}{4\pi} \sum_{\omega_n} |\omega_n| |X(\omega_n)|^2 - y \int d\tau \cos[X(\tau)]. \quad (\text{A1})$$

Here α is a dimensionless coupling constant characterizing the strength of the Ohmic dissipation, and y measures the strength of the periodic potential. Models like Eq. (A1) describe tunneling in a superconducting quantum interference device (SQUID), where X is interpreted as the flux in the SQUID, or tunneling between Luttinger liquids and quantum Hall edge states,^{38,40,76,77} where X now becomes a bosonic phase field.

The action (A1) can be written in a form local in time if we extend the quantum degree of freedom $X(t)$ to an infinite number of degrees of freedom $X(x, t)$ lying along the line $-\infty < x < \infty$, with $X(t) \equiv X(x=0, t)$; then S_1 is equivalent to^{71,73,74,76,38–40}

$$S_2 = \frac{\alpha c}{8\pi} \int dx d\tau \left([\partial_x X(x, \tau)]^2 + \frac{1}{c^2} [\partial_\tau X(x, \tau)]^2 \right) - y \int d\tau \cos[X(\tau)], \quad (\text{A2})$$

where c is an arbitrary velocity. Notice that the cosine interaction acts only along the single line $x=0$, identifying this as a boundary critical phenomena problem.

We shall be interested here in the linear response of the system to a time-dependent force, $F(t)$, acting on the particle. In imaginary time, in the presence of such a force

$$S_1 \rightarrow S_1 - \int d\tau F(\tau) X(\tau). \quad (\text{A3})$$

In the presence of a time-independent force, we expect the particle to acquire a finite velocity $V \equiv dX/dt$ in steady state: This allows us to define a *mobility* G by $V = GF$. More generally, we expect a frequency-dependent response $G(\omega)$, defined by

$$V(\omega) = G(\omega) F(\omega). \quad (\text{A4})$$

It is our purpose here to describe the behavior of the dynamic mobility $G(\omega)$ at low ω and T , and to compare it with the results for $\sigma(\omega)$ obtained in the main part of the paper. We also note that $G(\omega)$ is the *conductance* associated with tunneling between quantum Hall edge states⁴⁰ at a value $g = 1/\alpha$ in the notation of Moon *et al.*⁷⁷

Consider first the properties of S_1 under a $T=0$ renormalization group transformation under which $\tau \rightarrow \tau e^{-\ell}$. It is found^{70–72} that α remains unrenormalized, while the potential strength y obeys the simple flow equation

$$\frac{dy}{d\ell} = - \left(\frac{1}{\alpha} - 1 \right) y. \quad (\text{A5})$$

For $\alpha \neq 1$, this flow has fixed points only at $y=0$ and $y=\infty$ (we will not consider the case $\alpha=1$ here). For $\alpha < 1$, the $y=0$ fixed point is stable and the $y=\infty$ fixed point is unstable; for $\alpha > 1$ the stability of the fixed points is interchanged. Notice that *both* fixed points are free-field theories, supplemented by free or fixed boundary conditions at $x=0$. This is a crucial difference from the bulk theory S in which the critical theory is interacting for $d < 3$, and is primarily responsible for the differences in the structure of $G(\omega)$ and $\sigma(\omega)$ we shall find below.

Let us now write down the general scaling predictions for $G(\omega)$ which follow from the renormalization group arguments. The system will be completely characterized by a single nonuniversal energy scale T_K , which measures its deviation from the *unstable* fixed point. From the flow equations (A5) we can deduce $T_K \sim y^{\alpha/(\alpha-1)}$ for $\alpha > 1$; the perturbation theory to be discussed below shows $T_K \sim y^{-\alpha/(1-\alpha)}$ also for $\alpha < 1$. The mobility G has a zero scaling dimension, and therefore obeys the scaling form

$$G(\omega) = \mathcal{G}_\alpha \left(\frac{\omega}{T}, \frac{T}{T_K} \right), \quad (\text{A6})$$

where \mathcal{G}_α is a universal function. We now describe the low- T, ω form of \mathcal{G}_α for the cases $\alpha < 1$ and $\alpha > 1$ separately.

Consider first $\alpha < 1$. In this case y flows to zero, and therefore the periodic potential has vanishing strength at long times and the particle is delocalized. Naive perturbation theory in y is expected to be reliable. The mobility can computed in this perturbation theory using a Kubo formula: Such a computation was carried out in Appendix A of Ref. 76, and we find from their results

$$G(\omega) = \frac{2\pi}{\alpha} - \left(\frac{T}{T_K} \right)^{2(1/\alpha-1)} \mathcal{P}_\alpha \left(\frac{\omega}{T} \right) + \dots, \quad (\text{A7})$$

where the ellipsis represents terms which are higher order in T/T_K —this result holds for $\omega \ll T_K$, $T \ll T_K$, but ω/T can be arbitrary. Notice that Eq. (A7) is consistent with Eq. (A6). The explicit form of \mathcal{P}_α can be deduced after some analysis of Eq. (A7) of Ref. 76:

$$\begin{aligned}
\mathcal{P}_\alpha(\bar{\omega}) &= \left| \Gamma\left(\frac{1}{\alpha} + \frac{i\bar{\omega}}{2\pi}\right) \right|^2 \frac{\sinh(\bar{\omega}/2)}{\bar{\omega}} - i \frac{\tan(\pi/\alpha)}{\bar{\omega}} \\
&\times \left\{ \left| \Gamma\left(\frac{1}{\alpha} + \frac{i\bar{\omega}}{2\pi}\right) \right|^2 \cosh(\bar{\omega}/2) - \Gamma^2\left(\frac{1}{\alpha}\right) \right\}, \\
\mathcal{P}_\alpha(\bar{\omega} \rightarrow 0) &= \frac{1}{2} \Gamma^2\left(\frac{1}{\alpha}\right), \\
\mathcal{P}_\alpha(\bar{\omega} \rightarrow \infty) &= \left(\frac{\bar{\omega}}{\pi}\right)^{2(1/\alpha-1)} 2^{1-2/\alpha} [1 - i \tan(\pi/\alpha)], \\
\mathcal{P}_{1/2}(\bar{\omega}) &= \frac{1}{2} + \frac{\bar{\omega}^2}{8\pi^2},
\end{aligned}
\tag{A8}$$

a simple special case (Refs. 78 and 76). (A8)

The overall normalization of \mathcal{P}_α is arbitrary, as it can be absorbed into a redefinition of T_K . The most important property of Eqs. (A7) and (A8) is that the leading term in the mobility, $2\pi/\alpha$, is independent of ω/T . Alternatively stated, we have

$$G(\omega \rightarrow 0, T=0) = G(\omega=0, T \rightarrow 0) = \frac{2\pi}{\alpha}. \tag{A9}$$

However, it is clear that this leading term is a property of the $y=0$ fixed point, and the independence on ω/T is a consequence of its free-field nature. The next, subdominant, term arises from the leading irrelevant operator in the theory, and does indeed lead to a nontrivial dependence on ω/T in the universal function \mathcal{P}_α . In contrast, the bulk model \mathcal{S} , not being a free field at the critical point, had such a ω/T dependence already in the leading term, without the inclusion of any irrelevant operators. This distinction is the central point of this appendix.

Now turn to $\alpha > 1$. In this case y flows to ∞ , implying that the periodic potential localizes the particle at $T=0$. The low- T mobility clearly cannot be computed by a naive perturbation theory in y . Instead one can use a self-duality property of S_1 (Refs. 70, 72, and 76) under which $\alpha \leftrightarrow 1/\alpha$, and then use perturbation theory. The $y=\infty$ fixed point now implies that the leading scaling result for the mobility is simply $G=0$, and so the $\omega \rightarrow 0$ and $T \rightarrow 0$ limits commute again for a trivial reason. The leading ω and T dependence is given by an irrelevant operator, which now yields

$$G(\omega) = \left(\frac{T}{T_K}\right)^{2(\alpha-1)} \mathcal{P}_{1/\alpha}\left(\frac{\omega}{T}\right) + \dots, \tag{A10}$$

where the scaling function $\mathcal{P}_{1/\alpha}$ was defined in Eq. (A8), the corrections are higher order in T/T_K , and the result holds for $\omega \ll T_K$, $T \ll T_K$, but ω/T arbitrary. Notice also the result

$$G(\omega \rightarrow 0, T=0) = G(\omega=0, T \rightarrow 0) = 0, \tag{A11}$$

which arises from the $y=\infty$ free-field fixed point.

We also note for completeness the high- T behavior, with $T \gg T_K$, to allow contact with the quantum Hall edge state tunneling results of Ref. 77. The key is to note that the $\alpha \leftrightarrow 1/\alpha$ duality interchanges small and large y . So the result (A10) also describes the $T \gg T_K$ limit of the $\alpha < 1$ case, while Eq. (A7) describes $T \gg T_K$ for $\alpha > 1$. The main change is that the values of T_K (denoted \bar{T}_K) inserted in these expressions will not be the same as those in the low T limit. Perturbation theory cannot determine the universal relationship between T_K and \bar{T}_K : This requires use of the exact integrability of S_1 , as discussed in Refs. 38 and 73–75.

Finally, we suggest that the above commutativity of the limits $\omega \rightarrow 0$ and $T \rightarrow 0$ (and also $\omega \rightarrow \infty$, $T \rightarrow \infty$, where ∞ refers to a scale much larger than T_K), and the free-field nature of the fixed points may be the reason for the success of the imaginary-time Monte Carlo simulation of Moon *et al.*⁷⁷

APPENDIX B: GENERAL TRANSPORT EQUATION

Here, we will generalize the transport equation (3.25) of the model \mathcal{S} to arbitrary $n \geq 2$.

First consider $O(n=3)$. This case applies to quantum antiferromagnets¹³ and the external ‘‘potentials’’ U^a ($a=1,2,3$) correspond to the three components of a magnetic field. Let us take the field pointing in the three-direction: $U^a = (0,0,H)$. For the antisymmetric matrices we choose $L_{\alpha\beta}^a = \epsilon_{a\alpha\beta}$, the third-rank antisymmetric tensor. Note that such a field is the same as that in Eq. (2.1). The distribution functions become diagonal by changing basis from $a_{1,2,3}$ to

$$a_\pm(\mathbf{k}, t) \equiv \frac{a_1(\mathbf{k}, t) \pm ia_2(\mathbf{k}, t)}{\sqrt{2}}, \quad a_3(\mathbf{k}, t). \tag{B1}$$

The particle distribution function now has the three diagonal components f_\pm, f_3 . The ‘‘magnetization’’ current is nonzero only in the $a=3$ direction and equals

$$\mathbf{J}^3 = Q \int \frac{d^d k}{(2\pi)^d} \frac{\mathbf{k}}{\epsilon_k} [f_+(\mathbf{k}, t) - f_-(\mathbf{k}, t)]. \tag{B2}$$

A key simplifying feature is that the distribution function f_3 must be even in the external field, and will therefore only be modified at quadratic order in H . We will be satisfied by working in linear response, in which case $f_3(\mathbf{k}, t) = n(\epsilon_k)$.

The $n > 3$ case is very similar: The only difference is that there are now $n-2$ values of α for which $f_\alpha(\mathbf{k}, t) = n(\epsilon_k)$ in linear response.

The generalization of the transport equation (3.25) to $O(n)$ is now easily obtained by an application of Fermi’s golden rule:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \lambda Q \vec{\nabla} H \cdot \frac{\partial}{\partial \mathbf{k}}\right) f_\lambda(\mathbf{k}, t) = & -\frac{u_0^2}{9} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{1}{16 \varepsilon_k \varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{k_3}} (2\pi)^d \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) 2\pi \delta(\varepsilon_k + \varepsilon_{k_1} - \varepsilon_{k_2} - \varepsilon_{k_3}) \\
& \times \left\{ 4f_\lambda(\mathbf{k}, t) f_{-\lambda}(\mathbf{k}_1, t) [1 + f_\lambda(\mathbf{k}_2, t)] [1 + f_{-\lambda}(\mathbf{k}_3, t)] + 2f_\lambda(\mathbf{k}, t) f_\lambda(\mathbf{k}_1, t) [1 + f_\lambda(\mathbf{k}_2, t)] \right. \\
& \times [1 + f_\lambda(\mathbf{k}_3, t)] + (n-2) f_\lambda(\mathbf{k}, t) n(\varepsilon_{k_1}) [1 + f_\lambda(\mathbf{k}_2, t)] [1 + n(\varepsilon_{k_3})] \\
& + \frac{(n-2)}{2} f_\lambda(\mathbf{k}, t) f_{-\lambda}(\mathbf{k}_1, t) [1 + n(\varepsilon_{k_2})] [1 + n(\varepsilon_{k_3})] - 4[1 + f_\lambda(\mathbf{k}, t)] [1 + f_{-\lambda}(\mathbf{k}_1, t)] \\
& \times f_\lambda(\mathbf{k}_2, t) f_{-\lambda}(\mathbf{k}_3, t) - 2[1 + f_\lambda(\mathbf{k}, t)] [1 + f_\lambda(\mathbf{k}_1, t)] f_\lambda(\mathbf{k}_2, t) f_\lambda(\mathbf{k}_3, t) - (n-2)[1 + f_\lambda(\mathbf{k}, t)] \\
& \left. \times [1 + n(\varepsilon_{k_1})] f_\lambda(\mathbf{k}_2, t) n(\varepsilon_{k_3}) - \frac{(n-2)}{2} [1 + f_\lambda(\mathbf{k}, t)] [1 + f_{-\lambda}(\mathbf{k}_1, t)] n(\varepsilon_{k_2}) n(\varepsilon_{k_3}) \right\} \quad (\text{B3})
\end{aligned}$$

with $\lambda = \pm 1$. The analysis of the linearized form of this equations is similar to that for (3.25), but will not be presented here. Such an analysis will lead to a determination of the spin conductivity, σ , which is related to the spin diffusion constant D_s by the Einstein relation

$$\sigma = D_s \chi, \quad (\text{B4})$$

where χ is the uniform spin susceptibility. Results for the χ at nonzero T above the critical point have been given earlier in the $1/n$ (Refs. 27 and 13) and ϵ (Ref. 29) expansions.

APPENDIX C: COMPUTATIONS WITH THE COLLISION TERM

We describe here some of the steps in between the original transport equation (3.25) and the linearized form (3.28).

First we insert the parametrization (3.22) into Eq. (3.25), and linearize in the the external electric field. Then, notice that in the collision term the unknown function ψ appears only in the integrals over the radial components of the momenta. The angular integrals involve only known functions and can be performed analytically. As already noted in Sec. III B, the integrals in the collision term can be done directly in $d=3$ to obtain the leading result, and so all the computations here are for $d=3$. One needed angular integral is

$$\int dk_3 g(k_3) \int d\Omega_{\mathbf{k}_1} d\Omega_{\mathbf{k}_2} d\Omega_{\mathbf{k}_3} \delta^3(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) \delta(k + k_3 - k_1 - k_2) = \frac{8\pi^2}{kk_1k_2(k_1+k_2-k)} g(k_1+k_2-k) I_1(k, k_1, k_2), \quad (\text{C1})$$

where $g(k_3)$ is some function of k_3 . The result involves the function I_1 which is given by

$$I_1(k, k_1, k_2) = \begin{cases} 0 & k_1 + k_2 \leq k, \\ k_1 + k_2 - k & k_1 + k_2 \geq k \text{ and } k_1 \leq k \text{ and } k_2 \leq k, \\ k_1 & \text{for } k_1 \leq k \text{ and } k_2 \geq k, \\ k & k_1 \geq k \text{ and } k_2 \geq k, \\ k_2 & k_1 \geq k \text{ and } k_2 \leq k. \end{cases} \quad (\text{C2})$$

A second angular integral has a single vector momentum in the integrand

$$\int dk_3 g(k_3) \int d\Omega_{\mathbf{k}_1} d\Omega_{\mathbf{k}_2} d\Omega_{\mathbf{k}_3} \mathbf{k}_1 \delta^3(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(k + k_1 - k_2 - k_3) = -\frac{8\pi^2 \mathbf{k}}{3k^3 k_1 k_2 (k + k_1 - k_2)} g(k + k_1 - k_2) I_2(k, k_1, k_2), \quad (\text{C3})$$

where now

$$I_2(k, k_1, k_2) = \begin{cases} 0 & k_2 \geq k + k_1, \\ k^3 + k_1^3 + 2k_2^3 - 3k_2^2(k + k_1) + 3kk_1k_2 & k_2 \leq k + k_1 \text{ and } k_2 \geq k \text{ and } k_2 \geq k_1, \\ k^3 & \text{for } k_2 \geq k \text{ and } k_2 \leq k_1, \\ -2k_2^3 + 3k_2^2(k + k_1) - 3kk_1k_2 & k_2 \leq k \text{ and } k_2 \leq k_1, \\ k_1^3 & k_2 \geq k_1 \text{ and } k_2 \leq k. \end{cases} \quad (\text{C4})$$

Finally, we also need a variation of Eq. (C3):

$$\int dk_3 g(k_3) \int d\Omega_{\mathbf{k}_1} d\Omega_{\mathbf{k}_2} d\Omega_{\mathbf{k}_3} \mathbf{k}_1 \delta^3(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_1 - \mathbf{k}_3) \delta(k + k_2 - k_1 - k_3) = \frac{8\pi^2 \mathbf{k}}{3k^3 k_1 k_2 (k + k_2 - k_1)} g(k + k_2 - k_1) I_3(k, k_1, k_2), \quad (\text{C5})$$

where

$$I_3(k, k_1, k_2) = \begin{cases} 0 & k_1 \geq k + k_2, \\ k^3 - k_1^3 - 2k_2^3 - 3k_2^2(k - k_1) + 3kk_1k_2 & k_1 \leq k + k_2 \text{ and } k_1 \geq k \text{ and } k_2 \leq k_1, \\ k^3 & \text{for } k_1 \geq k \text{ and } k_2 \geq k_1, \\ -2k_2^3 - 3k_2^2(k - k_1) + 3kk_1k_2 & k_1 \leq k \text{ and } k_2 \leq k_1, \\ k_1^3 & k_2 \geq k_1 \text{ and } k_1 \leq k. \end{cases} \quad (\text{C6})$$

Using Eqs. (C1)–(C5) in the linearized form of Eq. (3.25), the integral equation satisfied by ψ becomes

$$\begin{aligned} -i\omega\psi(k, \omega) + \frac{1}{k} \frac{\partial n(k)}{\partial k} &= -\frac{\pi\epsilon^2}{75k^4} \left\{ \frac{18\psi(k, \omega)k^2}{n(k)} \int_0^\infty dk_1 dk_2 I_1(k, k_1, k_2) n(k_2) n(k_1) [1 + n(k_1 + k_2 - k)] + 2[1 + n(k)] \right. \\ &\quad \times \int_0^\infty dk_1 dk_2 \frac{\psi(k_1, \omega)}{n(k_1)} I_2(k, k_1, k_2) n(k_2) n(k + k_1 - k_2) \\ &\quad \left. - 4n(k) \int_0^\infty dk_1 dk_2 \frac{\psi(k_1, \omega)}{n(k_1)} I_3(k, k_1, k_2) n(k_2) [1 + n(k + k_2 - k_1)] \right\}. \quad (\text{C7}) \end{aligned}$$

This can be turned into the form (3.28) by evaluating the integrals over k_2 , which can also be done analytically. To do this, it is first necessary to separate the products of two Bose functions each with k_2 in their arguments, into terms in which only one Bose function involves k_2 : This is done by repeated use of the identity

$$n(k_2 + a)n(k_2 + b) = n(b - a)n(k_2 + a) + n(a - b)n(k_2 + b), \quad (\text{C8})$$

which is valid for any a, b . From the forms of the functions $I_{1,2,3}$ above, it is now clear that we only need integrals of k_2 over Bose functions of k_2 , times powers of k_2 . The general integral needed is of the form

$$\begin{aligned} \int_a^b dk_2 \frac{k_2^n}{e^{(k_2+c)/T} - 1} &= \int_a^\infty dk_2 \frac{k_2^n}{e^{(k_2+c)/T} - 1} \\ &\quad - \int_b^\infty dk_2 \frac{k_2^n}{e^{(k_2+c)/T} - 1}, \quad (\text{C9}) \end{aligned}$$

where n is an integer, and c is a real number. Changing variables on the right-hand side, we can rewrite the first term on the right-hand side as

$$\int_a^\infty dk_2 \frac{k_2^n}{e^{(k_2+c)/T} - 1} = \int_0^\infty dk_2 \frac{(k_2 + a)^n}{e^{(k_2+c+a)/T} - 1}, \quad (\text{C10})$$

and similarly for the second term. Expanding the polynomial in the numerator, we finally conclude that all of the integrals over k_2 can be reduced to the following basic integral:

$$\int_0^\infty dk_2 \frac{k_2^n}{e^{(k_2+a)/T} - 1} = T^{n+1} \Gamma(n+1) \text{Li}_{n+1}(e^{-a/T}). \quad (\text{C11})$$

Here $\text{Li}_p(z)$ is the polylogarithm function of order p , defined by the series

$$\text{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}. \quad (\text{C12})$$

Notice $\text{Li}_p(1) = \zeta(p)$.

After performing the integrals over k_2 as described above, Eq. (C7) takes the form (3.28). We will not display explicit expressions for $F_{1,2}(k, k_1)$ here as they are quite lengthy and not particularly informative. We used a CERNLIB routine for numerical evaluation of $\text{Li}_p(z)$ for $0 < z < 1$: This allowed very rapid determination of the kernel of the integral equation (3.28).

APPENDIX D: ORDER PARAMETER SUSCEPTIBILITY

This appendix will review the results of Ref. 29 on the finite-temperature crossovers in the order parameter susceptibility $\chi = \langle \phi_\alpha \phi_\alpha \rangle$ (no summation over α). Reference 29 was concerned with making statements correct to all orders in ϵ and to understanding the crossover to classical critical fluctuations in the vicinity of the finite-temperature phase transition line, and this required a rather involved analysis. Here, we are mainly interested in low-order results at finite temperatures above the quantum-critical point and the quantum disordered phase: In this case the necessary results can

be obtained directly in a self-consistent Hartree-Fock-like analysis, as we now show.

The self-consistent Hartree-Fock susceptibility of \mathcal{S} is obtained by summing all the one-loop tadpole diagrams; this leads to the expression

$$\chi(k, \omega_n) = \frac{1}{\omega_n^2 + k^2 + m^2(T)}, \quad (\text{D1})$$

where $m^2(T)$ is given by the solution of the equation

$$\begin{aligned} m^2(T) &= t_0 + m_{0c}^2 \\ &+ u_0 \left(\frac{n+2}{6} \right) T \sum_{\epsilon_n} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\epsilon_n^2 + q^2 + m^2(T)} \\ &= t_0 + u_0 \left(\frac{n+2}{6} \right) \left[T \sum_{\epsilon_n} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\epsilon_n^2 + q^2 + m^2(T)} \right. \\ &\quad \left. - \int \frac{d^{d+1} p}{(2\pi)^{d+1}} \frac{1}{p^2} \right]. \end{aligned} \quad (\text{D2})$$

In the second equation we have inserted the leading result for the value of m_{0c}^2 . We are following the convention here of denoting $(d+1)$ -dimensional spacetime momenta by p and d -dimensional spatial momenta by k, q . In the critical region, the coupling u_0 is of order ϵ , and so one might think that it is permissible to set $m^2(T) = t_0$ on the right-hand side of Eq. (D2) and obtain a result correct to order ϵ^2 . However, this is not adequate for our purposes for two reasons.²⁹

(i) The resulting expression for $m^2(T)$ is not analytic as a function of t_0 at $t_0 = 0$ for $T > 0$. This analyticity is required on the physical ground that there can be no thermodynamic singularity at $T > 0$ at the quantum-critical coupling $t_0 = 0$.

(ii) At $T > 0$ above the quantum-critical coupling t_0 there is a contribution to $m^2(T)$ of order $\epsilon^{3/2}$ which is missed.

To proceed, we can either use the analysis of Ref. 29, or directly analyze the singularity structure of Eq. (D2) as $u_0 \rightarrow 0$. By either method, it can be shown that it is permissible to set $m^2(T) = t_0$ in only the $\epsilon_n \neq 0$ terms on the right-hand side of Eq. (D2). To describe this, we write

$$m^2(T) = R(T) + \delta m^2(T), \quad (\text{D3})$$

where

$$\begin{aligned} R(T) &= t_0 + u_0 \left(\frac{n+2}{6} \right) \left[\int \frac{d^d q}{(2\pi)^d} \left(T \sum_{\epsilon_n \neq 0} \frac{1}{\epsilon_n^2 + q^2 + t_0} + \frac{T}{q^2} \right) \right. \\ &\quad \left. - \int \frac{d^{d+1} p}{(2\pi)^{d+1}} \frac{1}{p^2} \right] \end{aligned} \quad (\text{D4})$$

was a quantity introduced in Ref. 29, and

$$\delta m^2(T) = u_0 \left(\frac{n+2}{6} \right) T \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2 + m^2(T)} - \frac{1}{q^2} \right). \quad (\text{D5})$$

Finally, on the right hand side of Eq. (D5) we replace $m^2(T)$ by $R(T)$. The resulting expression for $\delta m^2(T)$ when inserted along with Eq. (D4) into Eq. (D3) gives us our final result for $m^2(T)$.

We will now manipulate the above result for $m^2(T)$ into a form which explicitly displays its universal scaling nature. First replace the bare couplings by renormalized couplings; to the order we are computing things, this is equivalent to performing the substitutions^{79,80} $u_0 = \mu^\epsilon g / S_{d+1}$ and $t_0 = t [1 + (n+2)g / (6\epsilon)]$ where μ is a renormalization momentum scale, g is a dimensionless coupling constant, and t is a renormalized coupling measuring the deviation from the critical point. Expanding the result to order g , we find that the poles in ϵ cancel. Finally, we set g at its fixed point value^{79,80} $g = g^* = 6\epsilon / (n+8)$. Our final result for $m^2(T)$ then becomes

$$m^2(T) = R(T) - \epsilon \left(\frac{n+2}{n+8} \right) 2\pi T \sqrt{R(T)}, \quad (\text{D6})$$

where the second term on the right-hand side is the contribution of $\delta m^2(T)$. A somewhat subtle analysis, discussed at some length in Ref. 29, is required to obtain the following final form for $R(T)$:

$$R(T) = t \left(1 + \epsilon \frac{n+2}{n+8} \ln \frac{T}{\mu} \right) + \epsilon T^2 \frac{n+2}{n+8} G \left(\frac{t}{T^2} \right), \quad (\text{D7})$$

with the crossover function $G(y)$ given by

$$\begin{aligned} G(y) &= -2 \int_0^\infty dq \left[\ln \left(2q^2 \frac{\cosh(\sqrt{q^2+y}) - 1}{q^2+y} \right) - q \right. \\ &\quad \left. - \frac{y}{2\sqrt{q^2+1/e}} \right]. \end{aligned} \quad (\text{D8})$$

This form of $G(y)$ is valid for both negative and positive y (when the argument of the square root is negative we use the identity $\cosh ix = \cos x$) and is easily shown to be analytic at $y = 0$ where

$$G(0) = \frac{2\pi^2}{3}, \quad dG/dy(0) = 2.453\,808\,582\dots \quad (\text{D9})$$

The result (D7) can therefore be used on both sides of the critical coupling $t = 0$, and the required analyticity of $T > 0$ properties at the critical coupling has been achieved. Also notice that at $t = 0$, $R(T)$ is of order ϵ , and so the result (D6) contains a term of order $\epsilon^{3/2}$.

Finally, let us express $m^2(T)$ in terms of experimentally measurable energy scales. The energy scales have to be defined differently for $t > 0$ and $t < 0$.

For $t > 0$ we choose the $T = 0$ energy gap, Δ , to measure the deviation from the critical point. Using the relation²⁹ $\Delta^2 = \mu^2 (t/\mu^2)^{2\nu}$ in Eq. (D7) ($\nu = 1/2 + \epsilon(n+2)/[4(n+8)]$ is the correlation length exponent) we get our final universal expression for $R(T)$, valid everywhere above the quantum-disordered (insulating) phase ($t > 0$):

$$R(T) = \Delta^2 \left(1 + \epsilon \frac{n+2}{n+8} \ln \frac{T}{\Delta} \right) + \epsilon T^2 \frac{n+2}{n+8} G \left(\frac{\Delta^2}{T^2} \right). \quad (\text{D10})$$

Notice that the arbitrary momentum scale μ has disappeared, and combined with Eq. (D6) we now have a result for $m^2(T)$ expressed solely in terms of the measurable energy scales Δ and T .

For $t < 0$, our results are confined to the normal phase $T > T_c(t)$. We measure the deviation from the critical point by the value of the superfluid stiffness, ρ_s at $T=0$ (for $n=3$, ρ_s is the spin stiffness of the ordered antiferromagnetic phase). To obtain a quantity with the dimensions of energy we define

$$\tilde{\rho}_s \equiv [(3-d)\rho_s]^{1/(d-1)}. \quad (\text{D11})$$

The numerical factors are for future convenience; also notice that in $d=2$, $\tilde{\rho}_s \equiv \rho_s$. Then we use the expression for ρ_s in

terms of t and μ in Ref. 29 to express t in terms of $\tilde{\rho}_s$ and μ in Eq. (D7). Then all the μ dependences cancel as before, and we get our final result

$$R(T) = -\frac{\tilde{\rho}_s^2}{n+8} \left(1 - \frac{\epsilon}{2(n+8)} + \frac{3\epsilon}{n+8} \ln \frac{2}{n+8} \right) + \epsilon T^2 \frac{n+2}{n+8} G \left(-\frac{\tilde{\rho}_s^2}{(n+8)T^2} \right). \quad (\text{D12})$$

Again combined with Eq. (D6) we now have an expression for $m^2(T)$ in terms of the superfluid stiffness of the ordered phase at $T=0$ for $t < 0$ and $T > T_c(t)$.

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