Light amplification and absorption in a random medium

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Using an invariant imbedding approach, we derive exact expressions for the distributions of reflection and transmission amplitude coefficients and phases of light backscattered coherently from an amplifying or absorbing random medium in one dimension. The length L of the system is assumed to be small compared to the wavelength, which is itself smaller than the localization length L_c of the light. These distributions and their statistical moments are analyzed in various length ranges defined by the characteristic absorption and amplification lengths. The results for the statistics of the reflection and transmission coefficients are contrasted with analogous results obtained previously for the strong localization domain, $L \gg L_c$, by assuming the random phases to be uniformly distributed. The case where L is larger than the wavelength but small compared to L_c, where the disorder is weak, is also briefly analyzed. [S0163-1829(97)07738-2]

I. INTRODUCTION

Our understanding of the Anderson localization of electrons in disordered systems has considerably broadened with the development of a scaling theory of localization and of electron transport. In the simple one-dimensional case this theory focuses on the scaling of the complex amplitudes of reflection and transmission of an electron (related to resistance and conductance by the Landauer formula) as a function of the length L of the scattering system.[1] Here the characteristic scaling length is the localization length L_c of the electronic states, which, in one dimension, is given by the backscattering mean free path. A detailed description of scaling in regimes of strong localization ($L \gg L_c$) and of weak localization (also called the diffusive or quasimetallic regime) ($L \ll L_c$) has been obtained[2] using the invariant imbedding evolution equations for reflection and transmission amplitudes.[3,4]

It was observed some time ago[5–7] that, in analogy with the quantum Anderson localization of electrons, light waves in a medium with a random dielectric constant may also be localized, i.e., nonpropagating, if the disorder is sufficiently strong (and for any disorder in one dimension). The localization of a classical wave results from the interference of phase-coherent multiple partial reflections and transmissions experienced by the wave in a random medium. If, however, the medium is also optically active, then, in addition to being multiply scattered and, possibly, localized, the wave may also be partially attenuated (absorbed) or amplified as a result of the bosonic nature of photons. Here we only consider *coherent* amplification and attenuation by coherent absorption, which may be modeled by (constant) positive and negative imaginary parts in the dielectric constant, respectively.[8] As is well known, laser action is based on phase-coherent amplification by stimulated emission in an optically active medium. It is also important to recall that absorption (which corresponds to the removal of photons) does not affect the interference of elastically scattered waves leading to the localization, in contrast to inelastic scattering processes, which cause loss of phase memory.

In parallel with the electronic case, scaling studies of the reflection and transmission of optical waves by a medium with a complex random dielectric constant have been developed for both one-dimensional systems[9,10] and for quasi-one-dimensional systems with many transmission channels.[11,12] Pradhan and Kumar[9] have used the invariant imbedding equations[3] and Beenakker and collaborators[11,12] and others[13] used a Fokker-Planck equation for the many-channel case analogous to the Dorokhov-Mello-Pereyra-Kumar equation. In the above treatments the emphasis is on the interplay of amplification (or absorption) and disorder, that is, the local enhancement of the amplification of reflection or transmission by the disorder. This enhancement of amplification is caused by confinement due to spatial localization of the light, as a result of multiple scattering by the disorder. In particular, such an enhancement should be present in a random laser, namely, when amplification is increased up to the lasing threshold.[14] This leads to the interesting possibility of obtaining laser action without mirrors,[9,11] that is, a laser where the feedback of radiation is provided by random scattering by disorder rather than by mirrors, as in conventional lasers.

The previous studies of reflection and transmission probabilities of light in amplifying or absorbing media[9–12] are restricted to long systems ($L \gg L_c$), i.e., to the so-called strong localization regime, apart from a brief discussion of short systems in Ref. 12. The object of this paper to present an extensive analytical treatment for short samples (such that $L \ll L_c$), where the transport is diffusive. Our treatment is based on the method of invariant imbedding,[3,4] of which a simple derivation is given in an appendix and, for most of it (Secs. II and III), is restricted to sample lengths which are short compared to the wavelength $\lambda = 2\pi/k_0$ (with k_0 the wave number). More precisely, we assume that the reduced length $2k_0L \ll 1$. Thus our analysis is valid for disorder strengths ranging from weak disorder ($k_0L_c \gg 1$) up to strong disorders where $k_0L_c \sim 1$, which corresponds to the lower limit of the localization length, usually referred to as the Ioffe-Regel limit.[15] We obtain exact expressions not only for the probabilities of the reflection and transmission coefficients, but also for the probabilities of the phases of the complex reflection and transmission amplitudes. We recall that, besides affecting the probabilities of the reflection and transmission coefficients as a result of strong correlations, these phases also directly relate to physical quantities of interest.[16] In particular, the absolute phase of the transmission amplitude is related to the integrated density of states inside the scattering domain.

The detailed results of Secs. II and III, as well as the (less extensive) results of Sec. IV for the domain $\lambda \lesssim L \ll L_c$, in addition to their intrinsic interest, are expected to be useful for comparisons with the earlier results for the domain $L \gg L_c$ (Refs. 9, 11, and 12) and for testing their validity. Inherent to the treatments for $L \gg L_c$ (Refs. 9, 11, and 12) is the so-called uniform random phase approximation (URPA) for the scattering in the random amplifying or absorbing medium. The URPA assumes the reflection and transmission phases to be independent random variables which are uniformly distributed in the interval $0,2\pi$. In the invariant imbedding treatment of Pradhan and Kumar,[9] the URPA allows one to reduce the Fokker-Planck equation for the joint distribution of the reflection coefficient and phase to a simple diffusion equation for the distribution of the reflection coefficient itself. We recall that in the case of a passive random medium as for the electron transport problem, the URPA has been shown to be valid for the marginal reflection phase distribution, for weak disorder ($k_0 L_c \gg 1$) only, but for both $L \gg L_c$ (Ref. 17) and $L \ll L_c$ (when, in addition, $2k_0 L \gg 1$).[18] The nonuniformity of the transmission phase distribution was recently studied by Freilikher and Pustilnik[19] using, however, a URPA for the reflection phase distribution.

Anderson localization may be more difficult to observe for light than for electrons. This is due to the fact that the localization of light arises from a term in Maxwell's equation given by the square of the wave number $k_0^2 = \omega^2/c^2$ times the random dielectric constant, which is very small for small ω and is responsible for the diverging localization length for $\omega \to 0$.[7] This further emphasizes the importance of the weak localization diffusive domain $L \ll L_c$ studied in the following sections. Our analysis ends with some final remarks in Sec. V.

II. PROBABILITY DISTRIBUTIONS OF REFLECTION AND TRANSMISSION COEFFICIENTS AND PHASES AT SHORT LENGTH SCALES

Consider an optically active one-dimensional disordered dielectric occupying the domain $0 \leqslant x \leqslant L$ of the x axis, which is surrounded by a passive background system of dielectric constant ε_0. The complex amplitude of a wave which is incident from the right obeys the Maxwell equation

$$\frac{\partial^2 E(x)}{\partial x^2} + k_0^2[1+\eta(x)]E(x) = 0, \quad 0 \leqslant x \leqslant L, \tag{1}$$

where $E(x) \equiv E(x,L)$, $k_0^2 = \omega^2 \varepsilon_0 / c^2$ (with k_0, ω, and c, respectively, the wave number, the frequency, and the speed of light in vacuum), and $\varepsilon(x) = \varepsilon_0 + \varepsilon'(x) + i\varepsilon''$ is the complex dielectric constant, $\varepsilon'(x)$ being the spatially random part and ε'' a constant nonrandom imaginary part describing absorption (if $\varepsilon'' > 0$) or coherent amplification (if $\varepsilon'' < 0$) of the wave field.[8,9] We have defined $\eta(x) = \eta'(x) + i\varepsilon''/\varepsilon_0$ and assume $\eta'(x) = \varepsilon'(x)/\varepsilon_0$ to be a Gaussian white noise with zero mean:

$$\langle \eta'(x)\eta'(x')\rangle = g\,\delta(x-x'), \quad \langle \eta'(x)\rangle = 0. \tag{2}$$

On the right and on the left of the random medium, the wave field has the form

$$E(x) = e^{-ik_0(x-L)} + R(L)e^{ik_0(x-L)}, \quad x > L, \tag{3}$$

$$E(x) = T(L)e^{-ik_0 x}, \quad x < 0, \tag{4}$$

where $R(L) = \sqrt{r}e^{i\Theta_R}$ and $T(L) = \sqrt{t}e^{i\Theta_T}$ are the complex reflection and transmission amplitudes, respectively, with $r \equiv r(L)$ and $t \equiv t(L)$ the reflection and transmission coefficients and $\Theta_R \equiv \Theta_R(L)$ and $\Theta_T \equiv \Theta_T(L)$ the phases of the reflection and transmission amplitudes.

In the invariant imbedding approach, the wave equation (1) with the proper boundary conditions is transformed so as to yield evolution equations for the emergent quantities $R(L) \equiv R$ and $T(L) \equiv T$ as a function of L:[3,4]

$$\frac{dR}{dL} = 2ik_0 R + \frac{ik_0}{2}\,\eta(L)(1+R)^2, \tag{5}$$

$$\frac{dT}{dL} = ik_0 T + \frac{ik_0}{2}\,\eta(L)(1+R)T, \tag{6}$$

with

$$\langle \eta'(L)\eta'(L')\rangle = g\,\delta(L-L'), \quad \langle \eta'(L)\rangle = 0. \tag{7}$$

In the Appendix we present a simple derivation of Eqs. (5) and (6) which we believe to be more transparent than traditional derivations.[3,4] This derivation also highlights the basic assumption involved in the invariant imbedding method.

Equations (5) and (6) are conveniently rewritten by separating out the phase factors of $R(L)$ and $T(L)$ associated with the free propagation of a wave in a passive medium ($\varepsilon''=0$), in the absence of randomness. Thus we define $R(L) = Q(L)\exp(2ik_0L)$ and $T(L) = S(L)\exp(ik_0L)$ and obtain, from Eqs. (5) and (6),

$$\frac{dQ}{dL} = \frac{ik_0}{2}\,e^{-2ik_0L}\eta(L)[1+e^{2ik_0L}Q]^2 \tag{8}$$

and

$$\frac{dS}{dL} = \frac{ik_0}{2}\,\eta(L)[1+e^{2ik_0L}Q]S. \tag{9}$$

As discussed in Sec. I, we are interested in the domain where

$$2k_0 L \ll 1, \tag{10}$$

that is, lengths scales small compared to the wavelength $\lambda = 2\pi/k_0$. Since the elastic backscattering mean free path (equal to the localization length) has a minimum value of the order of k_0^{-1},[15] Eq. (10) corresponds to a short length range within the weak localization domain $L \ll L_c$. The domain of longer lengths,

$$(2k_0)^{-1} \lesssim L \ll L_c, \tag{11}$$

which also exists within the weak localization regime for weak disorder (as compared to disorder where $L_c \sim k_0^{-1}$), will be briefly discussed in Sec. IV. For reduced lengths (10), the exact solutions of Eqs. (8) and (9) satisfying the boundary conditions $Q(0)=0$ and $S(0)=1$ are

$$Q=\frac{-iz(L)}{1+iz(L)}, \tag{12}$$

$$S=\frac{1}{1+iz(L)}, \tag{13}$$

where

$$z(L)=u(L)+iv(L), \tag{14}$$

with

$$u(L)=-\frac{k_0}{2}\int_0^L dL'\,\eta'(L'), \tag{15}$$

$$v(L)=-\frac{\varepsilon''}{2\varepsilon_0}\,k_0L\equiv v. \tag{16}$$

Note that the unitarity property $r+t=1$ which is obeyed by the reflection coefficient, $r=|R(L)|^2$, and the transmission coefficient, $t=|T(L)|^2$, in a passive system is now replaced by

$$r+t=\left[1+\frac{\varepsilon''}{\varepsilon_0}\,\frac{k_0L}{1+[(\varepsilon''/2\varepsilon_0)k_0L]^2+u(L)^2}\right]^{-1}, \tag{17}$$

as shown by Eqs. (12)–(16). This expression defines wavelength-dependent absorption or amplification coefficients, $\sigma=1-r-t$. For $\varepsilon''>0$ we have $r+t<1$ so that the total scattered intensity is attenuated (absorption), while for $\varepsilon''<0$, $r+t>1$, which shows that in this case the total scattered intensity is amplified.

The reflection and transmission amplitudes obtained from Eqs. (12) and (13) depend on the random variable $u\equiv u(L)$ whose probability distribution $P_u(u,L)$ is obtained from the characteristic function after evaluating the moments $\langle u^m\rangle$, $m=1,2,\ldots$. The odd moments vanish and, from Eqs. (7) and (15) one obtains

$$\langle u^{2n}\rangle=(2n-1)!!\,l^n, \quad n=1,2,3,\ldots,$$

where $L=L/L_c$ and the characteristic scaling length $L_c=4/(gk_0^2)$ is the localization length. We thus obtain the Gaussian distribution

$$P_u(u,l)=\frac{1}{\sqrt{2\pi l}}\exp\!\left(-\frac{u^2}{2l}\right). \tag{18}$$

The probability distribution $P_w(w,L)$ of any of the variables $w\equiv r,t,\Theta_R,\Theta_T$, depending on $u, w=f(u)$, is then given by

$$P_w(w,L)=\int_{-\infty}^{\infty}du\;\delta[w-f(u)]P_u(u,l). \tag{19}$$

From Eqs. (12)–(14) we have, successively,

$$r=\frac{u^2+v^2}{(1-v)^2+u^2}, \tag{20}$$

$$t=\frac{1}{(1-v)^2+u^2}, \tag{21}$$

$$\Theta_R=\mathrm{Tan}^{-1}\!\left(\frac{u}{v(v-1)+u^2}\right), \tag{22}$$

$$\Theta_T=\mathrm{Tan}^{-1}\!\left(\frac{u}{v-1}\right), \tag{23}$$

where in Eqs. (22) and (23) we choose the branches $\theta_R=\tan^{-1}\{u/[v(v-1)+u^2]\}$ and $\theta_T=\tan^{-1}[u/(v-1)]$, with $0\leqslant\theta_{R,T}\leqslant\pi$ as our principal branch when u varies between $-\infty$ and ∞. The phases $\Theta_{R,T}$ comprised between 0 and π are thus given by $\Theta_{R,T}=\theta_{R,T}$ and those lying between π and 2π by $\Theta_{R,T}=\pi+\theta_{R,T}$. Here and in the following sections, we disregard the additional phases $2k_0L$ and k_0L acquired, respectively, by the reflected and transmitted waves in propagating across a passive ($\varepsilon''=0$) disorder-free system. The reflection and transmission coefficients (20) and (21) are restricted to values less than 1 in the presence of absorption, while their values may exceed unity when amplification is present ($\varepsilon''<0$).

In obtaining the explicit probability distribution (19) for the quantities (20)–(23), we use the identity

$$\delta(w-f(u))=\sum_i |f'(u_i)|^{-1}\delta(u-u_i), \tag{24}$$

where the u_i are the real roots of algebraic equations $f(u)-w=0$. From Eqs. (18), (20), and (21), we then obtain the following exact results for the probability distributions of the reflection and transmission coefficients. First, the normalized distribution of the reflection coefficient is given by

$$P_r(r,L)=\frac{|1-2v|}{\sqrt{2\pi l}|1-r|^{3/2}}\,\frac{e^{(-|(1-v)^2r-v^2|/2l|1-r|)}}{|(1-v)^2r-v^2|^{1/2}}, \tag{25}$$

in the restricted domains

$$1<r<\frac{v^2}{(1-v)^2}\quad\text{for }v(L)>\frac{1}{2}, \tag{25a}$$

$$\frac{v^2}{(1-v)^2}<r<1\quad\text{for }0<v(L)<\frac{1}{2}\quad\text{or }v(L)<0, \tag{25b}$$

and

$$P_r(r,L)=0, \tag{26}$$

outside the domain (25a) for $v>\frac{1}{2}$ and outside (25b) for $0<v<\frac{1}{2}$ or $v<0$. It follows, in particular, that the reflection coefficient is larger than unity with probability 1 only for amplification parameters $v(L)>\frac{1}{2}$. Below this threshold r is less than unity with probability 1, both for absorption $[v(L)<0]$ and for amplification. The parameter $v(L)$ is interpreted in terms of absorption or amplification lengths in

Sec. III. On the other hand, the normalized probability distribution of the transmission coefficient reads

$$P_t(t,L)=\frac{1}{\sqrt{2\pi l}}\frac{e^{(-1/2l)[1/t-(1-v)^2]}}{t^2[t^{-1}-(1-v)^2]^{1/2}} \tag{27}$$

for

$$0<t<(1-v)^{-2}, \tag{27a}$$

$$P_t(t,L)=0 \quad \text{for } t>(1-v)^{-2}, \tag{28}$$

The domain (27a) includes values of $t>1$ if

$$0<v<2, \tag{27b}$$

while lying entirely below $t=1$ outside this range. The probability of transmission coefficients larger than unity for amplification parameters satisfying Eq. (27b) is less than 1. One may also wish to know in what range of amplification parameters $v(L)>0$ both r and t are amplified to values larger than unity with finite probability. For this purpose one has to obtain the joint probability distribution $P(r,t,L)$ of r and t. From Eq. (18), (20), and (21) we find

$$P(r,t,L)=\frac{1}{\sqrt{2\pi l}}\frac{e^{(-1/2l)[1/t-(1-v)^2]}}{t^2[t^{-1}-(1-v)^2]^{1/2}}\,\delta(r+t-1-2tv) \tag{29}$$

for $t<(1-v)^{-2}$ and $P(r,t,L)=0$ otherwise. From the δ factor in Eq. (29), it then follows that r values larger than unity exist only for $v(L)>\frac{1}{2}$, and hence both r and t are larger than 1 with finite probability for v lying in the range $\frac{1}{2}<v<2$. On the other hand, the total scattered intensity, $r+t$ is larger than 1 for any $v>0$.

Turning now to the probability distributions of the reflection and transmission phases, we obtain, from Eqs. (18), (19), (22), and (23), the following exact expressions for the principal values distributions:

$$P_{\theta_R}(\theta_R,L)=\frac{1}{2\sqrt{2\pi l}}\frac{1}{\sin^2\theta_R}\left[\left(1+\frac{1}{q}\right)\exp\left(-\frac{(1+q)^2}{8\tan^2\theta_R}\right)+\left(1-\frac{1}{q}\right)\exp\left(-\frac{(1-q)^2}{8\tan^2\theta_R}\right)\right], \tag{30}$$

where

$$q\equiv q(\theta_R,L)=[1+4v(L)(1-v(L))\tan^2\theta_R]^{1/2}, \tag{30a}$$

for the domains

$$0\leqslant\theta_R\leqslant\pi \quad \text{if } 0<v<1 \tag{30b}$$

and

$$\tan^2\theta_R<-\frac{1}{4v(1-v)} \quad \text{if } v<0 \text{ or } v>1, \tag{30c}$$

$$P_{\theta_R}(\theta_R,L)=0, \quad \text{for } \tan^2\theta_R>-\frac{1}{4v(1-v)}, \tag{31}$$

if

$$v<0 \quad \text{or} \quad v>1,$$

and, finally,

$$P_{\theta_T}(\theta_T,L)=\frac{|1-v|}{\sqrt{2\pi l}}\frac{e^{(-(1-v)^2/2l)\tan^2\theta_T}}{\cos^2\theta_T}, \quad 0\leqslant\theta_T\leqslant\pi. \tag{32}$$

The expressions (30) and (32) are normalized in the interval $(0,\pi)$ of the principal values, as required. The normalization is a direct consequence of the definition (19) and of the the normalization of $P_u(u,l)$. The distributions of the actual phases $\Theta_{R,T}$ in the interval $(0,2\pi)$ are then given by

$$P_{\Theta_{R,T}}(\Theta_{R,T},L)=\frac{1}{2}[P_{\theta_{R,T}}(\Theta_{R,T},L)h(\pi-\Theta_{R,T})+P_{\theta_{R,T}}(\Theta_{R,T}-\pi,l)h(\Theta_{R,T}-\pi)], \tag{33}$$

where $h(x)=1$ for $x>0$ and zero otherwise. The probability distributions (25), (27), (30), and (32), which are exact for length scales (10), are studied in detail in the following section.

III. STATISTICAL PROPERTIES OF REFLECTION AND TRANSMISSION PARAMETERS AT SHORT LENGTH SCALES

A. Reflection and transmission coefficients

The characteristic length scales for the exponential evolution of the transmittance as a function of system length L are defined naturally by studying the average over the disorder of ln$t$. Using Eq. (27), the nth-order moment is, after a change of integration variable,

$$\langle(\ln t)^n\rangle=-\frac{1}{\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}}\,(\ln[2ly+(1-v)^2])^n, \tag{34}$$

which may be readily evaluated in two important limits of our treatment, namely, $|v(L)|\ll 1$ and $|v(L)|\gg 1$.[20] For these cases we obtain, successively,

$$\langle\ln t\rangle=-\left(\frac{1}{L_c}+\frac{\varepsilon'' k_0}{\varepsilon_0}\right)L, \tag{35}$$

to lowest order for $|v|\ll 1$, and

$$\langle\ln t\rangle=-2\ln|1-v|-\frac{l}{(1-v)^2}, \tag{36}$$

to lowest order for $|v|\gg 1$.

It follows from Eq. (35) that in the absence of disorder ($L_c\to\infty$) the intensity of a wave which has propagated over a distance L in the medium is amplified (for $\varepsilon''<0$) or attenuated (for $\varepsilon''>0$) by a factor $\exp(-\varepsilon'' k_0 L/\varepsilon_0)$. This leads to defining the absorption length

$$L_{\mathrm{ab}}=\frac{\varepsilon_0}{\varepsilon'' k_0}, \tag{37}$$

for $\varepsilon''>0$ and the amplification length

$$L_{\rm am}=-\frac{\varepsilon_0}{\varepsilon'' k_0}, \tag{38}$$

for $\varepsilon''<0$. Note that in terms of these lengths $v(L)=-L/(2L_{\rm ab})$ and $v(L)=L/(2L_{\rm am})$ for $\varepsilon''>0$ and $\varepsilon''<0$, respectively. The limits $|v(L)|\ll 1$ and $|v(L)|\gg 1$ defined above and considered repeatedly in the following are thus equivalent to $L\ll 2L_{\rm ab}$ or $L\ll 2L_{\rm am}$, and $L\gg 2L_{\rm ab/am}$, respectively. In the presence of disorder, Eq. (35) (for $|v|\ll 1$) leads to exponential decay rates of the transmission coefficient defined by an inverse length scale

$$\frac{1}{L_c'}=\frac{1}{L_c}+\frac{\varepsilon'' k_0}{\varepsilon_0} \tag{39}$$

for absorption and for amplification when $L_{\rm am}>L_c$. Thus the decay length L_c in the absence of dissipation gets replaced by a shorter length in the case of absorption and by a longer length in the case of amplification, as one might have expected intuitively. We note, however, that in the case of amplification the decay length (39) is distinct from the localization length ξ. Indeed, ξ, as defined for large L, is given by $\xi^{-1}=L_c^{-1}+L_{\rm am}^{-1}$,[10] which shows that localization is actually enhanced by the coherent amplification. On the other hand, for $L_{\rm am}<L_c$, Eq. (35) defines an effective amplification length

$$\frac{1}{L_{\rm am}'}=\frac{1}{L_{\rm am}}-\frac{1}{L_c}, \tag{40}$$

which implies that, in this case, the amplification of the wave in the absence of disorder more than compensates for the decrease of the transmission coefficient due to the disorder in the absence of amplification.

In the limiting case $|v(L)|\gg 1$ the dominant decay in Eq. (36) is given by the transmission coefficient in the absence of disorder, namely,

$$t_0\simeq\frac{1}{v^2}=\frac{4L_{\rm am}^2}{L^2}, \tag{41}$$

which yields the meaning of the finite threshold value in Eqs. (27a) and (28). Finally, for $|v(L)|\ll 1$ the variance of $\ln t$ obtained from Eq. (34) is, to lowest order,

$$\operatorname{var}\ln t=2l^2+O(v^3,l^3,v^2l,vl^2), \tag{42}$$

which indicates that $\ln t$ is not self-averaging $[\operatorname{var}\ln t/\langle\ln t\rangle^2=2l^2(l-2v)^{-2}]$, in contrast to the case of long systems, $L\gg L_c$, where $\ln t$ has an approximately Gaussian distribution with $\operatorname{var}\ln t/\langle\ln t\rangle^2=L^{-1}$.[10,12,21]

Next, we analyze the reflection and transmission coefficient moments themselves. From Eqs. (25) and (27), we obtain, successively,

$$\langle r^n\rangle=\frac{1}{\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}}\left[\frac{2ly+v^2}{2ly+(1-v)^2}\right]^n, \tag{43}$$

both for $v<1/2(r<1)$ and for $v>1/2(r>1)$, and

$$\langle t^n\rangle=\frac{1}{\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}[2ly+(1-v)^2]^n}, \tag{44}$$

for both signs of ε''. It is noteworthy that the moments $\langle r^n\rangle$, $\langle t^n\rangle$, and $\langle(\ln t)^n\rangle$ in Eq. (43), (44), and (34) are divergent for $v(L)=1$, i.e., at a characteristic length

$$L_a=2L_{\rm am}, \tag{45}$$

which defines the lasing threshold length.[11] The divergence of $\langle r^n\rangle$ and $\langle t^n\rangle$ is related to the presence of essential singularities at $r=1$ and at $t=0$ in the distributions (25) and (27) for $v(L)=1$,

$$P_r(r,L)=\frac{1}{\sqrt{2\pi l}}\frac{e^{-1/2l(r-1)}}{(r-1)^{3/2}},\quad 1\leqslant r\leqslant\infty, \tag{46}$$

$$P_t(t,L)=\frac{1}{\sqrt{2\pi l}}\frac{e^{-1/2lt}}{t^{3/2}},\quad 0\leqslant t\leqslant\infty, \tag{47}$$

which implies that realizations with values of r or of t much larger than one have relatively high probabilities.

For $|v(L)|\ll 1$ we obtain

$$\langle r\rangle=l-3l^2+2lv(L)+v^2(L), \tag{48}$$

$$\langle r^2\rangle=3l^2+2l(v^2(L)-6l^2+4lv(L)), \tag{49}$$

and

$$\langle t^n\rangle=1-nl+2nv(L)+2n(n+1)\left(\tfrac{3}{4}l^2+v^2(L)-lv(L)\right)-nv^2(L), \tag{50}$$

up to terms of higher orders in L, v, and combinations thereof. Similarly, for $v(L)\gg 1$ we get, to lowest order,

$$\langle r^n\rangle=\left(\frac{v}{1-v}\right)^{2n}\left[1+nl\left(\frac{1}{v^2}-\frac{1}{(1-v)^2}\right)\right], \tag{51}$$

$$\langle t^n\rangle=\frac{1}{(1-v)^{2n}}\left(1-\frac{nl}{(1-v)^2}\right). \tag{52}$$

For $v(L)\ll 1$ it follows thus that absorption reduces the moments $\langle r^n\rangle$ and $\langle t^n\rangle$, while amplification enhances them. Our results for $\langle v\rangle$ agree with earlier findings for small L (Refs. 9, 11, and 22) and, those for $\langle t\rangle$, with results briefly mentioned by Paasschens *et al.*[12] in their paper mainly concerned with the large-L limit. From Eqs. (48) and (49) we have $\sqrt{\operatorname{var} r}/\langle r\rangle=\sqrt{2}+O(l)$ and, from Eq. (50), $\sqrt{\operatorname{var} t}/\langle t\rangle=\sqrt{2}l$, to lowest order. While r is thus (marginally) non-self-averaging, t is self-averaging for $L_c\to\infty$. On the other hand, Eq. (51) and (52) indicate that, for $v(L)\gg 1$, $\langle r\rangle$ and $\langle t\rangle$ are enhanced with respect to their (exact) zero-disorder values,

$$r_0=\frac{v^2}{(1-v)^2}, \tag{53}$$

and t_0 in Eq. (41), when disorder of the dielectric constant is included. On the other hand, we note that from Eqs. (48) and (50), as well as from Eqs. (51) and (52), one may verify that $\langle r\rangle+\langle t\rangle>1$ for amplification $[v(L)>0]$ and $\langle r\rangle+\langle i\rangle<1$ for absorption $[v(L)<0]$ as shown in Sec. II.

Finally, it is interesting to observe that for special domains of $v(L)$ considered above, namely, $|v(L)|\ll 1$ and $v(L)\sim 1$, the probability distribution of r reduces to normalized forms similar to the stationary distributions for $L \gg L_c$ derived by Pradhan and Kumar[9] from a Fokker-Planck equation, using the uniform random phase assumption (URPA) for Θ_R. Thus, for $v(L) \ll 1$, Eq. (25) becomes

$$P_r(r,L) = \frac{e^{-r/2L(1-r)}}{(2\pi l)^{1/2}(1-r)^{3/2} r^{1/2}}, \quad r<1, \tag{54}$$

while for $v(L) \sim 1$ we have [Eq. (46)]

$$P_r(r,L) = \frac{e^{-1/2l(r-1)}}{(2\pi l)^{1/2}(r-1)^{3/2}}, \quad r>1. \tag{55}$$

Now the distribution of Pradhan and Kumar involves a parameter $|D|$, which may be written as $|D| = 2L_c/L_{\mathrm{ab}}$ for $\varepsilon'' > 0$ and $|D| = 2L_c/L_{\mathrm{am}}$ for $\varepsilon'' < 0$. With the identification $(2l)^{-1} = |D|$, Eqs. (54) and (55) are very similar to the Eqs. (4b) and (4a) of Ref. 9, respectively. We note, however, that the identification $(2l)^{-1} = |D|$ corresponds to a value $v(L) = \frac{1}{8} \ll 1$, which is compatible with the range of the distribution (54) for $r<1$, but which departs strongly from the range $v(L) \sim 1$ where Eq. (55) is valid. Thus Eq. (55) is only qualitatively similar to Eq. (4a) of Ref. 9 for $r>1$. For the sake of completeness we also call attention to the limiting form of $P_r(r,L)$ for $v(L) \gg 1$ given by

$$P_r(r,L) = \left(\frac{2L_c}{\pi L}\right)^{1/2} \frac{e^{-L/L_c'}}{(1-r)^2}, \quad L_c' = \frac{8L_{\mathrm{am}}^2}{L_c}, \tag{56}$$

which is valid only near the upper limit in Eq. (25a). On the other hand, the limiting forms of the distribution of the transmission coefficient for $v(L) \ll 1$, $v(L) \sim 1$, and $v(L) \gg 1$ are readily read off from Eq. (27).

B. Reflection and transmission phases

For $\varepsilon'' = 0$, the distribution of principal reflection phases, Eq. (30), reduces to the expression

$$P_{\theta_R}(\theta_R, L) = \frac{1}{\sqrt{2\pi l}} \frac{e^{-1/2l \, \tan^2 \theta_R}}{\sin^2 \theta_R}. \tag{57}$$

which has been discussed earlier for electronic motion in a random potential.[23] The latter has a sharp peak at $\theta_R = \pi/2$, corresponding to peaks at $\Theta_R = \pi/2$ and $\Theta_R = 3\pi/2$ of the actual phase distribution (33). $P_{\Theta_R}(\Theta_R, L)$ could in fact be approximated asymptotically by the sum of two Gaussians of half-width L, centered at $\Theta_R = \pi/2$ and $\Theta_R = 3\pi/2$, respectively.[23]

For $\varepsilon'' < 0$ (amplification), the peaks at $\Theta_R = \pi/2$ and at $\Theta_R = 3\pi/2$ are attenuated by a factor $\exp(-L_c/4L_{\mathrm{am}})$ for $v(L) \ll 1$, while for $\varepsilon'' > 0$ (absorption) these peaks are replaced by dips where $P_{\Theta_R}(\Theta_R, L) = 0$. Furthermore, $P_{\Theta_R}(\Theta_R, L) = 0$ for $\Theta_R = 0$ and $\Theta_R = \pi$ for both signs of ε''. Actually, the peaks at $\pi/2$ and $3\pi/2$ for $\varepsilon'' < 0$ can no longer be approximated by Gaussians because the second derivatives of the exponents in the square brackets of Eqs. (30) and (33) are infinite at $\pi/2$ and $3\pi/2$. Thus a nonzero ε'', while smoothening the distribution of the reflection phase in short samples in the case of absorption, renders it more structured in the case of amplification. In any case, neither the results for short systems for $\varepsilon'' = 0$ in Ref. 23 nor the above results for $\varepsilon'' \neq 0$ allow one to justify the URPA, which is inherent to the studies of reflection coefficients in one-dimensional and quasi-one-dimensional systems[9,11] for $L \gg L_c$.

Consider now the distribution of the phase of the complex transmission amplitude. In contrast to $P_{\Theta_R}(\Theta_R, L)$, $P_{\Theta_T}(\Theta_T, L)$ in Eqs. (32) and (33) has peaks at $\Theta_T = 0$, π, and 2π, where it can be represented approximately by the normalized sum of half Gaussians at $\Theta_T = 0$ and $\Theta_T = 2\pi$ and a full Gaussian centered at $\Theta_T = \pi$,

$$\begin{aligned} P_{\Theta_T}(\Theta_T, L) = \frac{|1-v|}{2\sqrt{2\pi l}} \Bigg[& \frac{1}{2} \exp\left(-\frac{(1-v)^2}{2l}\,\Theta_T^2\right) \\ & + \exp\left(-\frac{(1-v)^2}{2l}\,(\Theta_T - \pi)^2\right) \\ & + \frac{1}{2} \exp\left(-\frac{(1-v)^2}{2l}\,(\Theta_T - 2\pi)^2\right) \Bigg], \\ & 0 \le \Theta_T \le 2\pi. \end{aligned} \tag{58}$$

In order to analyze the effects of absorption and amplification on the fluctuations of Θ_T, it is useful to obtain the mean and the variance from Eqs. (32) and (33):

$$\begin{aligned} \langle \Theta_T \rangle &= \int_0^{2\pi} d\Theta_T \Theta_T P_{\Theta_T}(\Theta_T, L) \\ &= \frac{\pi}{2} + \int_0^{\pi} d\theta_T \theta_T P_{\theta_T}(\theta_T, L) = \frac{\pi}{2} + \langle \theta_T \rangle. \end{aligned} \tag{59}$$

and

$$\mathrm{var}\Theta_T = \langle \Theta_T^2 \rangle - \langle \Theta_T \rangle^2 = \frac{\pi^2}{4} + \mathrm{var}\theta_T. \tag{60}$$

Using Eq. (32), we have, explicitly,

$$\begin{aligned} \langle \theta_T^n \rangle = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{dy}{\sqrt{y}} e^{-y} \Bigg[& \left(\tan^{-1} \frac{\sqrt{2ly}}{|1-v|} \right)^n \\ & + \left(\pi - \tan^{-1} \frac{\sqrt{2ly}}{|1-v|} \right)^n \Bigg], \end{aligned} \tag{61}$$

whose evaluation for $\sqrt{2l}/|1-v| \ll 1$ yields

$$\langle \Theta_T \rangle = \pi \quad \left(\langle \theta_T \rangle = \frac{\pi}{2} \right) \tag{62}$$

and

$$\mathrm{var}\Theta_T = \frac{\pi^2}{2} - \frac{\sqrt{2l}}{|1-v|} \left(\sqrt{\pi} - \frac{1}{2} \frac{\sqrt{2l}}{|1-v|} + \cdots \right). \tag{63}$$

Thus we find that absorption acts to suppress fluctuations of Θ_T in short samples, while amplification enhances them for $v(L) < 2$. We note that Freilikher and Pustilnik[19] found that absorption also suppresses fluctuations of θ_T in the strong localization regime $L \gg L_c$. They also found that the transmission phase for $L \gg L_c$ is approximately Gaussian as in Eq. (58), which exhibits, however, three peaks at intervals of π

within the domain $(0,2\pi)$. However, we observe that their results are obtained by using the URPA for the reflection phase θ_R.

Finally, the results (62) and (63) illustrate the deviations of the transmission phase distribution (32) and (33) from a uniform distribution between 0 and 2π, for which $\langle\Theta_T\rangle=\pi$ and $\mathrm{var}\Theta_T=\pi^2/3$.

IV. REFLECTION AND TRANSMISSION IN THE WEAK LOCALIZATION REGIME AT LONGER LENGTH SCALES

Here we briefly analyze effects of absorption or amplification on simple statistical moments related to the reflection or transmission of light for sample lengths larger than the wavelength, but small compared to the localization length. More precisely, we consider the range

$$(2k_0)^{-1}\lesssim L\ll L_c, \tag{64}$$

whose existence requires the disorder to be sufficiently weak, i.e., $k_0L_c\gg 1$, as discussed above. Thus, in the range (64), the disorder may be treated as a small perturbation (assuming also $|\varepsilon''|\ll\varepsilon_0$) as done earlier for electronic conduction ($\varepsilon''=0$).[18] In the electronic case it has been shown,[18] using the invariant imbedding method, that the distribution of the reflection amplitude phase Θ_R tends towards a uniform distribution for large reduced length $2k_0L$, within the domain (64). This provided a direct justification for the uniform random phase assumption (URPA) in the scaling theories of resistance fluctuations,[1,2] at least for the weak localization regime. Since the URPA is also implicit in the recent studies of reflection and transmission in absorbing or amplifying random systems,[9,11,12] a reexamination of the reflection phase distribution in the domain (64), when $\varepsilon''\neq 0$, would be of interest. While this problem is left for the future, we confine ourselves here to discussing analytical results for the mean values $\langle r\rangle$, $\langle \ln t\rangle$, and $\langle\Theta_T\rangle$.

To second order in $\eta(L)$ the solutions of Eqs. (8) and (9) obeying $Q(0)=0$ and $S(0)=1$ are

$$Q(L)=Q_1(L)+Q_2(L) \tag{65}$$

and

$$\ln S(L)=\frac{ik_0}{2}\int_0^L dL'\,\eta(L') + ik_0\int_0^L dL'\,\eta(L')e^{2ik_0L}Q_1(L'), \tag{66}$$

where

$$Q_1(L)=\frac{ik_0}{2}\int_0^L dL' e^{-2ik_0L'}\eta(L'), \tag{65a}$$

$$Q_2(L)=ik_0\int_0^L dL'\,\eta(L')Q_1(L'). \tag{65b}$$

The second-order terms in Eqs. (65) and (66) are needed in order to capture the effect of the interplay of absorption or amplification and disorder, as well as the effect of absorption or amplification on the transmission phase,

$$\Theta_T=\mathrm{Im}\,\ln S(L)h(\pi-\Theta_T)+[\pi+\mathrm{Im}\,\ln S(L)]h(\Theta_T-\pi). \tag{67}$$

Here the imaginary parts of logarithmic functions denote principal values between 0 and π, and we have omitted the phase k_0L acquired by the wave when propagating from one end of the sample to the other, in the absence of disorder and absorption or amplification. The mean values of $r=[\mathrm{Re}(Q_1+Q_2)]^2+[\mathrm{Im}(Q_1+Q_2)]^2$, $\ln t=2\,\mathrm{Re}\,\ln S$, and Θ_T are calculated explicitly from Eqs. (65)–(67), using Eq. (7). Restricting ourselves to contributions of lowest order in the strength of the correlation (7), we obtain, after some amount of algebra,

$$\begin{aligned}\langle r\rangle=&\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\left\{\left[1-\frac{\varepsilon''}{\varepsilon_0}k_0L+\frac{1}{4}\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\right](1-\cos 2k_0L)\right.\\&\left.-\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\frac{k_0L}{2}(\sin 2k_0L-k_0L)\right\}+l\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\right.\\&\left.-\frac{\varepsilon''}{\varepsilon_0}k_0L+\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\frac{k_0^2L^2}{3}\right]+\frac{1}{4}\left(\frac{\varepsilon''}{\varepsilon_0}\right)^2\left\{\left(l-\frac{1}{k_0L_c}\right)\right.\\&\times(\cos 2k_0L-1)+\frac{l}{2}\cos 4k_0L-\frac{1}{2}\left(l-\frac{1}{4k_0L_c}\right)\\&\left.\times\sin 4k_0L-\frac{1}{k_0L_c}(\cos 2k_0L+\sin 2k_0L-1)\right\},\end{aligned} \tag{68}$$

$$\langle \ln t\rangle=-\frac{\varepsilon''}{\varepsilon_0}k_0L-2l+\left(\frac{\varepsilon''}{2\varepsilon_0}\right)^2(1-\cos 2k_0L), \tag{69}$$

$$\langle\Theta_T\rangle=\frac{\pi}{2}+\left(\frac{\varepsilon''}{2\varepsilon_0}\right)^2\left(k_0L-\frac{1}{2}\sin 2k_0L\right), \tag{70}$$

Note that the disorder has no effect on the mean transmission phase at lowest order in the correlator g.

We first observe that for reduced lengths $2k_0L$ larger than 1, Eq. (69) no longer gives a characteristic exponentially linear L dependence (with effective absorption or amplification length scales) for the typical transmission coefficient, $\exp(\langle \ln t\rangle)$; instead, it displays oscillations superimposed to a monotonic linear variation of the exponent. Similar oscillatory behavior is also shown in Eqs. (69) and (70) for the mean reflection and transmission coefficients, respectively. The next step in the analysis of the domain (64) would be to use the weak disorder solutions (65), (65a), (65b), and (66) for studying the probability distributions, of which the simplest are the distribution of $\ln t=2\,\mathrm{Re}\,\ln S(L)$ and of the transmission phase Θ_T.

V. CONCLUDING REMARKS

We have presented an exact treatment of the effect of absorption or amplification on the probability distributions of the reflection and transmission coefficients for reduced sample widths $2k_0L\ll 1$. These lengths correspond to the weak localization regime $L \ll L_c$, for any strength of disorder since in one dimension k_0^{-1} is the lower limit of the optical localization length.[15] We have analyzed these results analytically for limiting values of the ratio $v(L) = -L/2L_{\rm ab}$ (for $\varepsilon'' > 0$) or $v(L) = L/2L_{\rm am}$ (for $\varepsilon'' < 0$) of length L and absorption or amplification lengths $L_{\rm ab}$ and $L_{\rm am}$. In particular, we found that the reflection coefficient gets amplified to values larger than unity for $v(L) > 1/2$ and that $v(L) = 1$ defines a lasing threshold length $L_a = 2L_{\rm am}$, where all reflection moments diverge. Near $v(L) = 1$, where optical pumping above threshold leads to self-sustained laser oscillations, the linear treatment with a constant ε'' breaks down and it would then be necessary to include the dependence of ε'' on the wave amplitude in Eq. (1).

On the other hand, within the domain $v(L) > \frac{1}{2}$, where $r > 1$, we have examined in detail the limit $v(L) \ll 1$, where the linear treatment with a constant $\varepsilon'' < 0$ is again valid. From the point of view of the feasability of so-called ''random lasers,'' one is interested in the change of coherent amplification which is due specifically to the disorder. For $v(L) \gg 1$ this effect is precisely given by Eq. (51) for $\langle r \rangle$ where the first term is the exact zero-disorder reflection coefficient $r_0 > 1$. From this expression it follows that the disorder does further enhance r [disorder also yields an enhancement of the transmission coefficient in Eq. (52), but the support of $P_t(t,L)$ in Eq. (27) remains confined to the domain $t < 1$ for $v(L)$ outside the range (27b)]. Finally, we recall that the disorder-specific coherent enhancement of the reflection coefficient in the case $L \gg L_c$ has been attributed to confinement by Anderson localization of the light[9] rather than just to increased optical path lengths due to light diffusion in the random amplifying system. Since in one dimension the backscattering mean free path is equal to the localization length, it does not seem possible to separate the respective effects of Anderson localization and of diffusion in the disorder-enhanced amplified reflection coefficient at the length scales $L \ll L_c$ considered in this paper.

Our analysis of the reflection and transmission phase distributions has revealed that these distributions are quite structured and rather sensitive to absorption or amplification. It would be quite interesting in future work to study the marginal phase distributions in the strong localization regime $L \gg L_c$ and, more generally, to analyze the effects of the strong correlations between the phases and the reflection and transmission coefficients, starting from the stochastic differential equations (5) and (6). The limiting stationary reflection phase distribution for $L \gg L_c$ has been studied previously for $\varepsilon'' = 0$ and was shown to approach a uniform distribution for weak disorder such that $k_0 L_c \gg 1$.[17] This suggests, in particular, that the limiting stationary distribution of the reflection coefficient derived by Pradhan and Kumar[9] may be valid only for weak disorder.

APPENDIX: DERIVATION OF INVARIANT IMBEDDING EQUATIONS

We first transform Eq. (1) into a first-order differential equation for the logarithmic derivative of the wave amplitude,

$$y(x) = \frac{1}{E(x)} \frac{dE(x)}{dx} \equiv y(x,L). \tag{A1}$$

This yields

$$\frac{dy(x)}{dx} + y^2(x) + k_0^2[1 + \eta(x)] = 0, \quad 0 \leq x \leq L, \tag{A2}$$

By expressing the continuity of the wave field and its derivative at the boundaries of the disordered medium, we have, using Eqs. (3) and (4),

$$\begin{aligned} E(L) &= 1 + R(L), \\ E'(L) &= -ik_0[1 - R(L)], \\ E(0) &= T(L), \\ E'(0) &= -ik_0 T(L). \end{aligned} \tag{A3}$$

The boundary conditions for $y(x)$ are thus

$$y(L) = -ik_0 \left(\frac{1 - R(L)}{1 + R(L)} \right), \tag{A4}$$

$$y(0) = -ik_0. \tag{A5}$$

The invariant imbedding equation for $R(L)$ now follows by identifying the partial derivative of $y(x)$ just inside the random medium, namely,

$$\left. \frac{dy(x)}{dx} \right|_{x=L-0^+} \equiv \left. \frac{\partial y(x,L)}{\partial x} \right|_{x=L-0^+} \quad \text{i.e.}$$
$$\left. \frac{dy(x)}{dx} \right|_{x=L-0^+} = -y^2(L) - k_0^2[1 + \eta(L)], \tag{A6}$$

with the total derivative of $y(L,L)$, namely, the derivative of the right-hand side of Eq. (A4),

$$\frac{dy(L,L)}{dL} = \frac{2ik_0}{[1 + R(L)]^2} \frac{dR}{dL}. \tag{A7}$$

This yields the evolution equation (5) in terms of the imbedding parameter L.

The identification of Eq. (A6) with Eq. (A7), which is also implicit in the derivations of, e.g., Ref. 4, may be viewed as the basic assumption of the imbedding procedure. The random system of length L is assumed to be imbedded invariantly in its extension, which implies, in particular, that the assumed Gaussian randomness of $\eta'(x)$ within the system, i.e., for $0 \leq x \leq L$, is maintained invariantly in its extension to values L' slightly larger than L. More precisely, the values $\eta'(L)$ and $\eta'(L')$ at the end points $x = L$ and $x' = L' > L$ have the same Gaussian correlation as that assumed at points x and x' within the dielectric system of length L.

Although we could not find a corresponding simple derivation of the invariant imbedding equation obeyed by $T(L)$, the latter can actually be inferred from the differential equation for $R(L)$. Indeed from Eq. (5) we obtain, for arbitrary complex $\eta(L)$,

$$\frac{d|R|^2}{dL}=\frac{ik_0}{2}[\eta(L)R^*(1+R)^2-\text{c.c.}],\quad R\equiv R(L),\quad (A8)$$

Now, for a passive medium ($\varepsilon''=0$), R and T obey the unitarity property

$$|R|^2+|T|^2=1,\quad \varepsilon''=0.$$

For $\varepsilon''=0$, Eq. (A8) can thus be rewritten as

$$\frac{d|T|^2}{dL}=-\frac{ik_0}{2}\eta(L)[R^*(1+R)^2-\text{c.c.}]$$
$$=\frac{ik_0}{2}\eta(L)[(1+R)|T|^2-\text{c.c.}],\quad \varepsilon''=0.\quad (A9)$$

From the generalization of Eq. (A9) to complex $\eta(L)$, by analogy with Eq. (A8), on the one hand, and the comparison of Eqs. (5) and (A8), on the other hand, it follows that the natural form for the evolution equation for $T(L)$, leading to the generalized form of Eq. (A9), is indeed Eq. (6).

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