Inclusion of Landau damping in a time-dependent effective theory for a weak-coupling superconductor at finite temperature

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The effective propagator for the Goldstone mode (phase of the order parameter) is calculated for a neutral BCS system in the long-wavelength/low-frequency limit, with inclusion of Landau damping terms, for temperatures between $T=0$ and $T=0.6T_c$. The Landau terms are first evaluated numerically, and then accurate closed-form expressions are found for them. The resulting propagator is shown to be well approximated by the product of two simple poles at complex energy, corresponding to a damped mode with linear (and T-dependent) dispersion for both the real and imaginary parts. Damping is only significant for $T \geqslant 0.4T_c$. By considering the Fourier transform of the inverse of this pole-dominated propagator, an effective local equation of motion for the phase degree of freedom is obtained, which includes a specific damping term. The damping may be phenomenologically included in the equivalent time-dependent nonlinear Schrödinger equation by giving the pair mass a small temperature-dependent positive imaginary part. [S0163-1829(97)06937-3]

I. INTRODUCTION

The effective Lagrangian of Ginzburg and Landau[1] (GL) successfully describes static superconducting phenomena, and was derived[2] from the microscopic BCS theory[3] soon after its introduction. It has proved much more difficult to obtain an effective theory of GL type for time-dependent phenomena. At finite temperature, one reason for this is the existence of Landau damping terms in the effective action. These terms are singular at the origin of energy-momentum space, and consequently they cannot be expanded as a Taylor series about the origin. Equivalently, these terms do not have a well-defined expansion in terms of spatiotemporal gradients of the order parameter, and hence they cannot be represented directly as a contribution to a local effective Langrangian. At zero temperature, however, the Landau damping terms vanish and a local effective time-dependent theory is in principle possible. Even in this case it is only quite recently that a derivation[4,5] of the expected theory—a time-dependent nonlinear Schrodinger Lagrangian—has been given from the microscopic theory (see also Refs. 6 and 7).

The most recent study of time-dependent GL theory at finite temperature is that of Stoof[8] who, however, neglected the Landau damping terms entirely. The aim of the present paper is to make a detailed quantitative study of these terms within the framework set up by Stoof, and to examine what sort of effective theory can be derived when they are included.

Our basic strategy is to remain in momentum space for as long as possible, where the terms we are interested in are well defined. We work towards obtaining a simple approximate form for the momentum-space propagator of the Bogoliubov-Anderson-Goldstone (BAG) mode, including the effects of Landau damping. Only after this stage do we consider the (resulting) effective theory in coordinate space.

In Sec. II we briefly describe the effective action formalism, and present the momentum-space expressions for the quadratic fluctuations. These are of two types, ''regular'' and ''Landau,'' the former having a well-defined energy-momentum expansion about the origin, the latter not. We carry out the expansion of the regular terms up to second order in energy $k_0/|\Delta_0|$ and momentum $v_F|\mathbf{k}|/|\Delta_0|$ (where Δ_0 is the zero-temperature gap) and obtain the effective momentum-space inverse propagators for the two expected modes: one, S_G^{-1}, for fluctuations in the phase of the order parameter (the BAG mode), the other for fluctuations in the modulus of the order parameter. The latter is only excited at energies higher than the range of validity of our second-order expansion, so we concentrate on the BAG mode. In Sec. III we study the contribution of the Landau damping terms to S_G^{-1}, which we denote by F_L. We first evaluate F_L numerically. Then for the imaginary part of F_L, we are able to find an approximate, but quite accurate, closed form expression. For the real part, we fit a simple function to the numerical data. These calculations are valid for $k_0/|\Delta_0|$ and $v_F|\mathbf{k}|/\Delta_0$ less than unity and for temperatures T up to about $0.6T_c$, beyond which the gap begins to vary appreciably with T. In Sec. IV we include these approximate expressions for F_L in the propagator S_G, and show that this approximate propagator is very well represented by the product of two simple poles at complex energy, corresponding to a damped BAG mode with a linear dispersion relation for the real and imaginary parts. By considering the Fourier transform of (the inverse of) this pole-dominated propagator, we finally obtain an effective local equation of motion for the phase degree of freedom, which includes specific damping.

II. MOMENTUM-SPACE ACTION FOR QUADRATIC FLUCTUATIONS

Stoof[8] has given explicit expressions for the momentum-space contributions of the ''diagonal'' (see below) fluctuations, using a functional approach to the Keldysh formalism.[9] For our purposes, it is clearer to remain in momentum space throughout, so we shall give here a brief account of an alternative derivation based on the Matsubara formalism,[10] giving the corresponding expressions for the ''nondiagonal'' term also, for completeness.

We start with the BCS action for s-wave pairing and in the absence of external fields:

$$S=\int d^4x\left\{\sum_\sigma \psi_\sigma^*\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu\right)\psi_\sigma+g\psi_\uparrow^*\psi_\downarrow^*\psi_\downarrow\psi_\uparrow\right\}, \tag{1}$$

where $\hbar=1$, ψ_σ describes electrons of mass m and spin $\sigma=(\uparrow,\downarrow)$, and μ is the chemical potential (which we shall take to be a temperature-independent constant, equal to the Fermi energy $k_F^2/2m$). Introducing the auxiliary pair fields $\Delta(x)$ and $\Delta^*(x)$, and integrating out the electron fields, one obtains the effective action

$$S[\Delta^*,\Delta]=-i\mathrm{Tr}[\ln G^{-1}]-\frac{1}{g}\int d^4x|\Delta(x)|^2, \tag{2}$$

where

$$G^{-1}=\begin{pmatrix} \frac{i\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu & \Delta(\mathbf{x},t) \\ \Delta^*(\mathbf{x},t) & \frac{i\partial}{\partial t}-\frac{\nabla^2}{2m}+\mu \end{pmatrix}. \tag{3}$$

We now follow Stoof[8] and expand $S[\Delta^*,\Delta]$ about its minimum $S[\Delta_0^*,\Delta_0]$ by writing $\Delta(\mathbf{x},t)=\Delta_0+\Delta'(\mathbf{x},t)$. The terms quadratic in Δ' are given by

$$\frac{i}{2}\mathrm{Tr}[G_0\Sigma G_0\Sigma], \tag{4}$$

where G_0^{-1} is defined to be G^{-1} with Δ replaced by Δ_0 and Δ^* by Δ_0^*, and where

$$\Sigma(\mathbf{x},t)=\begin{pmatrix} 0 & -\Delta'(\mathbf{x},t) \\ -\Delta'^*(\mathbf{x},t) & 0 \end{pmatrix}\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{5}$$

In momentum space, Eq. (4) reads

$$\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\mathrm{tr}(G_0(p)\Sigma(k)G_0(p-k)\Sigma(-k)), \tag{6}$$

where $\Sigma(k)$ is the four-dimensional transform of $\Sigma(\mathbf{x},t)$, tr stands for the trace in the 2×2 Nambu space, and where

$$G_0(p)=\frac{1}{p_0^2-E^2(\mathbf{p})}\begin{pmatrix} p_0+\epsilon(\mathbf{p}) & -\Delta_0 \\ -\Delta_0^* & p_0-\epsilon(\mathbf{p}) \end{pmatrix} \tag{7}$$

with $\epsilon(\mathbf{p})=\mathbf{p}^2/2m-\mu$ and $E(\mathbf{p})=[|\Delta|^2+\epsilon^2(\mathbf{p})]^{1/2}$.

After performing the trace in Eq. (6), one obtains two types of term: those involving $\Delta'(k)\Delta'^*(-k)$ and $\Delta'(-k)\Delta'^*(k)$ which are called ''diagonal,'' and those involving $\Delta'(k)\Delta'(-k)$ and $\Delta'^*(k)\Delta'^*(-k)$ which are ''nondiagonal.'' It is easy to check that the two diagonal terms are in fact identical, their combined contribution being

$$i\int\frac{d^4k}{(2\pi)^4}\Delta'(k)\Delta'^*(-k)\int\frac{d^4p}{(2\pi)^4}\frac{1}{p_0^2-E^2(\mathbf{p})}\times\frac{1}{(p_0-k_0)^2-E^2(\mathbf{p}-\mathbf{k})}[p_0+\epsilon(\mathbf{p})]\times[p_0-k_0-\epsilon(\mathbf{p}-\mathbf{k})]. \tag{8}$$

We proceed with the finite temperature extension of Eq. (8); the calculations for the nondiagonal terms are very similar.

We rotate to Euclidean space via $p_0\to ip_E$ and $k_0\to ik_E$, and replace the integral over p_E by a sum over ''fermionic'' frequencies $(2m+1)\pi/\beta$ $(\beta=1/k_BT)$ and that over k_E by a sum over ''bosonic'' frequencies $2n\pi/\beta$. The sum over m can be performed by contour integration, and the result continued back to real k_0 via $ik_E\to k_0+i\epsilon\equiv k_0^+$. As usual, one must be careful with terms involving $\exp(\pm i\beta k_E)$, which must be set equal to unity before continuing back. After some further manipulations we obtain, for the finite-temperature version of Eq. (8),

$$\int\frac{d^4k}{(2\pi)^4}\Delta(k)\Delta^*(-k)[F_{D,R}(k)+2F_{D,L}(k)], \tag{9}$$

where the *diagonal regular* term $F_{D,R}$ is

$$F_{D,R}(k_0,\mathbf{k})=-\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[1-N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|u(\mathbf{k}'+\mathbf{k}/2)|^2|u(\mathbf{k}'-\mathbf{k}/2)|^2}{k_0^+-E(\mathbf{k}'+\mathbf{k}/2)-E(\mathbf{k}'-\mathbf{k}/2)}+\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[1-N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|v(\mathbf{k}'+\mathbf{k}/2)|^2|v(\mathbf{k}'-\mathbf{k}/2)|^2}{k_0^++E(\mathbf{k}'+\mathbf{k}/2)+E(\mathbf{k}'-\mathbf{k}/2)} \tag{10}$$

and the *diagonal Landau damping* term $F_{D,L}$ is

$$F_{D,L}(k_0,\mathbf{k})=\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|u(\mathbf{k}'+\mathbf{k}/2)|^2|v(\mathbf{k}'-\mathbf{k}/2)|^2}{k_0^+-E(\mathbf{k}'+\mathbf{k}/2)+E(\mathbf{k}'-\mathbf{k}/2)}. \tag{11}$$

Here $N(\mathbf{k})$ is the thermal distribution function

$$N(\mathbf{k})=\frac{1}{\exp(\beta E(\mathbf{k}))+1} \tag{12}$$

and $u(\mathbf{k})$ and $v(\mathbf{k})$ are the coherence factors obeying

$$|u(\mathbf{k})|^2=\tfrac{1}{2}[1+\epsilon(\mathbf{k})/E(\mathbf{k})], \quad |v(\mathbf{k})|^2=\tfrac{1}{2}[1-\epsilon(\mathbf{k})/E(\mathbf{k})].$$

For the nondiagonal terms we find

$$\int \frac{d^4k}{(2\pi)^4}\left(\frac{\Delta_0^{*2}\Delta'(k)\Delta'(-k)}{|\Delta_0|^2}+\frac{\Delta_0^2}{|\Delta_0|^2}\Delta'^*(k)\Delta'^*(-k)\right)[F_{N,R}(k)+F_{N,L}(k)], \tag{13}$$

where the *nondiagonal regular* term $F_{N,R}$ is

$$\begin{aligned}F_{N,R}(k_0,\mathbf{k})=&\frac{1}{2}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[1-N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|\Delta_0|^2}{4E(\mathbf{k}'+\mathbf{k}/2)E(\mathbf{k}'-\mathbf{k}/2)[k_0^+-E(\mathbf{k}'+\mathbf{k}/2)-E(\mathbf{k}'-\mathbf{k}/2)]}\\&-\frac{1}{2}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[1-N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|\Delta_0|^2}{4E(\mathbf{k}'+\mathbf{k}/2)E(\mathbf{k}'-\mathbf{k}/2)[k_0^++E(\mathbf{k}'+\mathbf{k}/2)+E(\mathbf{k}'-\mathbf{k}/2)]}\end{aligned} \tag{14}$$

and the *nondiagonal Landau damping term* $F_{N,L}$ is

$$F_{N,L}(k_0,\mathbf{k})=\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{[N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]|\Delta_0|^2}{4E(\mathbf{k}'+\mathbf{k}/2)E(\mathbf{k}'-\mathbf{k}/2)[k_0^+-E(\mathbf{k}'+\mathbf{k}/2)+E(\mathbf{k}'-\mathbf{k}/2)]}. \tag{15}$$

The F_R terms have an obvious interpretation as amplitudes for the creation of two quasiparticles or two quasiholes, and develop imaginary parts only at values of $|k_0|$ greater than $2|\Delta_0|$. By contrast, the F_L terms develop an imaginary part when the condition $k_0=E(\mathbf{k}'+\mathbf{k}/2)-E(\mathbf{k}'-\mathbf{k}/2)$ is satisfied. Since we are interested in values of k_0 and $v_F|\mathbf{k}|$ substantially less than $|\Delta_0|$, we may write this condition as

$$k_0\approx\frac{\epsilon(\mathbf{k}')}{mE(\mathbf{k}')}\mathbf{k}\cdot\mathbf{k}'. \tag{16}$$

In evaluating this imaginary part we need to perform the integral over $\mathbf{k}'$ in Eqs. (11) and (15), so that Eq. (16) appears to be a rather complicated condition. However, it is an excellent approximation to replace $|\mathbf{k}'|$ by k_F, while $E(\mathbf{k}')$ is roughly of order Δ_0. Finally, we shall see in Sec. III that $\epsilon(\mathbf{k}')$ is effectively constrained to lie in a small range

$$|\epsilon(\mathbf{k}')|\leq|\Delta_0|(T/T_c)^{1/2}.$$

Then condition (16) becomes

$$|k_0|\leq v_F|\mathbf{k}|(T/T_c)^{1/2}|\cos\theta|, \tag{17}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Equation (17) may be interpreted as a ''Cerenkov'' condition[11] for the (real radiation) process quasiparticle ↔ quasiparticle + thermally excited fluctuation quantum. It is clear that this process will cause the F_L amplitudes to have a cut in k_0 extending between $\pm v_F|\mathbf{k}|(T/T_c)^{1/2}$, approximately, so that for $T\neq 0$ the F_L amplitudes will not be analytic at the origin of k space. This is what prevents a straightforward expansion in k_μ, and hence a local effective Langrangian.

In what follows we shall take Δ_0 to be real, and also independent of temperature which should be a reasonable approximation for $0\leq T\leq 0.6T_c$. The regular terms $F_{D,R}$ and $F_{N,R}$ then have well-defined Taylor expansions about the origin in k space, since their denominators never vanish for k_0 and $v_F|\mathbf{k}|$ less than Δ_0. Performing these expansions up to order k_0^2 and $\mathbf{k}^2$ we find

$$F_{D,R}/\mathcal{N}(0)=F_1(k_0,\mathbf{k})+O(k^4), \tag{18}$$

where

$$F_1(k_0,\mathbf{k})=A(T)(k_0/\Delta_0)^2+B(T)(v_F|\mathbf{k}|/\Delta_0)^2-\tfrac{1}{2}E(T) \tag{19}$$

with

$$A(T)=\frac{1}{8}\int_{-\infty}^{\infty}dy\left[\frac{1}{(1+y^2)^{3/2}}-\frac{1}{2(1+y^2)^{5/2}}\right][1-2N(y)], \tag{20}$$

$$B(T)=-\frac{1}{18}-\frac{1}{12}\int_{-\infty}^{\infty}dy\left[\frac{1}{(1+y^2)^{3/2}}-\frac{13}{2(1+y^2)^{5/2}}+\frac{5}{(1+y^2)^{7/2}}\right]N(y), \tag{21}$$

and

$$E(T)=\frac{1}{2}\int_{-\infty}^{\infty}dy[1-2N(y)]/(1+y^2)^{3/2}, \tag{22}$$

where $\mathcal{N}(0)=mk_F/2\pi^2$ is the density of states at the Fermi surface, and where

TABLE I. The values of the quantities $A(T)$, $B(T)$, $C(T)$, $D(T)$, $E(T)$, and $J(T)$ as defined in the text, at the indicated values of T/T_c.

T/T_c	$A(T)$	$B(T)$	$C(T)$	$D(T)$	$E(T)$	$J(T)$
0.2	0.167	−0.0555	−0.0416	0.0139	1.000	0.0001
0.3	0.166	−0.0554	−0.0415	0.0138	0.997	0.0013
0.4	0.165	−0.0548	−0.0410	0.0134	0.998	0.0043
0.5	0.162	−0.0535	−0.0400	0.0128	0.968	0.0070
0.6	0.158	−0.0518	−0.0385	0.0120	0.941	0.0109

$$N(y)=\frac{1}{\exp(\beta\Delta_0(1+y^2)^{1/2})+1}. \tag{23}$$

In obtaining these expressions, we have rewritten the integrals over $|\mathbf{k}'|$ as being over $y=\epsilon'/\Delta_0$, and used the fact that the behavior of the distribution function N is such that the resulting y integral is dominated by the region $|y|\leqslant(T/T_c)^{1/2}$. This immediately implies that isolated factors of $|\mathbf{k}'|$ may be replaced by k_F. Since $\Delta_0\ll\mu$, we have also extended the lower limit of integration to $-\infty$.

For the regular nondiagonal terms we find, similarly,

$$F_{N,R}(k_0,\mathbf{k})/\mathcal{N}(0)=F_2(k_0,\mathbf{k})+O(k^4), \tag{24}$$

where

$$F_2(k_0,\mathbf{k})=\left[C(T)\left(\frac{k_0}{\Delta_0}\right)^2+D(T)\left(\frac{v_F|\mathbf{k}|}{\Delta_0}\right)^2-\frac{E(T)}{4}\right] \tag{25}$$

with

$$C(T)=-\frac{1}{32}\int_{-\infty}^{\infty}dy\frac{[1-2N(y)]}{(1+y^2)^{5/2}} \tag{26}$$

and

$$D(T)=\frac{1}{72}+\frac{1}{96}\int_{-\infty}^{\infty}dy\left(\frac{14}{(1+y^2)^{5/2}}-\frac{20}{(1+y^2)^{7/2}}\right)N(y). \tag{27}$$

Where comparison is possible, these expressions agree with those given by Stoof.[8] The dimensionless quantities A, B, C, D, and E are tabulated as a function of T in Table I, for values of T/T_c from 0.2 to 0.6, and setting $\Delta_0=1.75k_BT_c$. It can be seen that they vary slowly in this temperature range.

At this stage our effective action takes the form

$$S_{\rm eff}(\Delta'^*,\Delta')=\mathcal{N}(0)\int\frac{d^4k}{(2\pi)^4}\left[\left(F_1+\frac{2F_{D,L}}{\mathcal{N}(0)}\right)\times\Delta'(k)\Delta'^*(-k)+\left(F_2+\frac{F_{N,L}}{\mathcal{N}(0)}\right)\times[\Delta'(k)\Delta'(-k)+\Delta'^*(k)\Delta'^*(-k)]\right]. \tag{28}$$

This can be readily diagonalized in terms of the real and imaginary parts of Δ', $\Delta'=\Delta_r'+i\Delta_i'$:

$$S_{\rm eff}(\Delta_r,\Delta_i)=\mathcal{N}(0)\int\frac{d^4k}{(2\pi)^4}\{\Delta_r'^2[F_1+2F_2+(2F_{D,L}+2F_{N,L})/\mathcal{N}(0)]+\Delta_i'^2[F_1-2F_2+(2F_{D,L}-2F_{N,L})/\mathcal{N}(0)]\}. \tag{29}$$

Consider first the coefficient of $\Delta_r'^2$, and let us ignore the Landau damping terms. This coefficient is

$$S_r^{-1}\equiv[A(T)+2C(T)]\left(\frac{k_0}{\Delta_0}\right)^2+[B(T)+2D(T)]\left(\frac{v_F|\mathbf{k}|}{\Delta_0}\right)^2-E(T), \tag{30}$$

which can be interpreted as the inverse propagator (in momentum space) for the mode corresponding to oscillations in the real part of Δ'. The approximate dispersion relation for this mode is obtained from the equation $S_r^{-1}=0$. Since $E\approx1$ (from Table I) it is clear that this is a *massive* mode, the approximate dispersion relation being

$$k_0=\pm[(\Delta_0^2+0.028v_F^2\mathbf{k}^2)/0.085]^{1/2}. \tag{31}$$

At $\mathbf{k}=\mathbf{0}$ this mode will be excited only at energies equal to several times Δ_0, which lie far beyond the region of validity of our expansion. Consequently, we cannot trust the detailed numbers in Eq. (31), and in our regime this mode will be effectively ''frozen.'' The inclusion of the Landau damping terms in Eq. (29) does not alter this conclusion at the temperatures we are considering. We therefore turn our attention to the Δ_i' mode.

If we write $\Delta'=|\Delta'|e^{i\phi}\approx|\Delta'|(1+i\phi)$ we can identify Δ_r' with $|\Delta'|$ and Δ_i' with $|\Delta'|\phi$, for small oscillations in the phase. Taking $|\Delta'|$ to be constant, we can then interpret the coefficient of $\Delta_i'^2$ in Eq. (29) as the inverse propagator for the phase (or Bogoliubov-Anderson-Goldstone) mode:

$$S_G^{-1}\approx[A(T)-2C(T)]\left(\frac{k_0}{\Delta_0}\right)^2+[B(T)-2D(T)]\left(\frac{v_F|\mathbf{k}|}{\Delta_0}\right)^2+2(F_{D,L}-F_{N,L})/\mathcal{N}(0). \tag{32}$$

If the Landau terms are neglected, Eq. (32) represents a *massless* mode with propagation velocity

$$v_p=\left(\frac{2D(T)-B(T)}{A(T)-2C(T)}\right)^{1/2}v_F. \tag{33}$$

We find that $v_p\to v_F/\sqrt{3}$ as expected[12,13] at $T=0$, and that v_p increases slowly with T (see Table II). We must now examine the contribution of the Landau damping terms in Eq. (32).

III. THE CONTRIBUTION OF THE LANDAU DAMPING TERMS TO S_G^{-1}

From Eqs. (11), (15), and (32), the contribution of $F_{D,L}$ and $F_{N,L}$ to S_G^{-1} is

TABLE II. Variation of v_p/v_F with temperature.

T/T_c	v_p/v_F
0.2	0.577
0.3	0.576
0.4	0.575
0.5	0.572
0.6	0.568

$$(S_G^{-1})_L \equiv \frac{2(F_{D,L}-F_{N,L})}{\mathcal{N}(0)}$$

$$= \frac{2}{\mathcal{N}(0)} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{[N(\mathbf{k}'+\mathbf{k}/2)-N(\mathbf{k}'-\mathbf{k}/2)]}{k_0^+ - E(\mathbf{k}'+\mathbf{k}/2)+E(\mathbf{k}'-\mathbf{k}/2)} \times \left(|u(\mathbf{k}'+\mathbf{k}/2)|^2 |v(\mathbf{k}'-\mathbf{k}/2)|^2 - \frac{\Delta_0^2}{4E(\mathbf{k}'+\mathbf{k}/2)E(\mathbf{k}'-\mathbf{k}/2)} \right). \quad (34)$$

In the spirit of our low-k development, we begin by replacing the singular denominator factor by

$$\left(k_0^+ - \frac{\mathbf{k}\cdot\mathbf{k}'}{m} \frac{\epsilon(\mathbf{k}')}{E(\mathbf{k}')} \right)^{-1}, \quad (35)$$

which must be treated exactly. The remaining factors in Eq. (34), however, all have well-defined expansions in powers of the momentum $|\mathbf{k}|$. The numerator term involving N is clearly odd in $(\mathbf{k}\cdot\mathbf{k}')$, and its leading term is of order $(\mathbf{k}\cdot\mathbf{k}')$. The other numerator factor contains terms which are both even and odd in $(\mathbf{k}\cdot\mathbf{k}')$, but we need only collect terms in the product which are odd in $(\mathbf{k}\cdot\mathbf{k}')$, since those which are even in $(\mathbf{k}\cdot\mathbf{k}')$ are found to vanish by particle-hole symmetry. Carrying out the expansions, we find that the leading contribution to $(S_G^{-1})_L$ is given by

$$(S_G^{-1})_L \approx F_L \equiv \frac{1}{4\mathcal{N}(0)} \frac{\Delta_0^2}{m^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{(\mathbf{k}\cdot\mathbf{k}')^3 \epsilon'}{E'^5} \frac{dN}{dE'} \times \left(k_0^+ - \frac{\mathbf{k}\cdot\mathbf{k}'}{m} \frac{\epsilon'}{E'} \right)^{-1}, \quad (36)$$

where $\epsilon' = \epsilon(\mathbf{k}'), E' = E(\mathbf{k}')$. The integral over the angles of $\mathbf{k}'$ can now be performed leaving a final integral over $|\mathbf{k}'|$. As before, we replace isolated factors of $|\mathbf{k}'|$ by k_F, and rewrite the integral as being over y. In this way we obtain the following expression for F_L:

$$F_L = J(T)\left(\frac{v_F|\mathbf{k}|}{\Delta_0}\right)^2 + H(T,a)\left(\frac{k_0}{\Delta_0}\right)^2, \quad (37)$$

where

$$J(T) = -\frac{\Delta_0}{12} \int_{-\infty}^{\infty} dy \frac{dN}{dE'} \frac{1}{(1+y^2)^2} \quad (38)$$

and where $H(T,a) = H_r(T,a) + iH_i(T,a)$ with

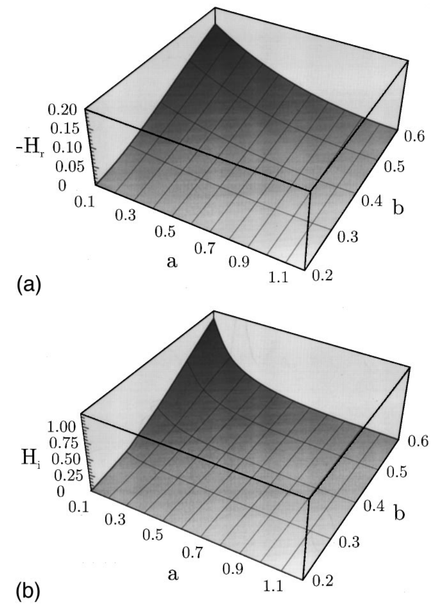

FIG. 1. (a) H_r as a function of $a = (k_0/v_F|\mathbf{k}|)$ and $b = (T/T_c)$. (b) H_i as a function of a and b.

$$H_r(T,a) = -\frac{\Delta_0}{4} \int_{-\infty}^{\infty} dy \frac{dN}{dE'} \frac{1}{y^2(1+y^2)} \times \left[1 - \frac{(1+y^2)^{1/2}}{2y} a \ln \left| \frac{y + a(1+y^2)^{1/2}}{y - a(1+y^2)^{1/2}} \right| \right] \quad (39)$$

and

$$H_i(T,a) = -\frac{\pi a \Delta_0}{8} \int_{-\infty}^{\infty} dy \frac{dN}{dE'} \frac{1}{(1+y^2)^{1/2}} \frac{1}{(y^2|y|)} \times \theta\left(\frac{|y|}{(1+y^2)^{1/2}} - |a| \right). \quad (40)$$

Note that $H(T,a)$ is a function only of T and the dimensionless variable $a = (k_0/v_F|\mathbf{k}|)$. The function H_i is nonzero only for $-v_F|\mathbf{k}| < k_0 < v_F|\mathbf{k}|$, or $-1 < a < 1$, which defines the extent of the branch cut in $H(T,a)$ as a function of k_0, for fixed $|\mathbf{k}|$.

We have evaluated the dimensionless quantities H_r, H_i, and J numerically, taking $\Delta_0 = 1.75\ T_c$ as before so that the expressions are functions of the dimensionless variables a and $b = T/T_c$. $J(T)$ is tabulated in Table I, while H_r and H_i are shown in Figs. 1(a) and 1(b), respectively, for $a > 0$.

Since our aim is to investigate how the inclusion of F_L modifies the Goldstone propagator S_G [cf. Eq. (32)], we would like to find simple closed-form approximations for H_r and H_i, rather than having to interpolate from a table of numerical values. The key to such approximations lies in the behavior of the function dN/dE', which—like N itself—is sharply peaked around $y=0$ with a width of order $b^{1/2}$. This suggests replacing dN/dE' by a ''square-wave'' profile centered on $y=0$ with a width $2(fb)^{1/2}$, where f is a numerical factor, of order unity, to be determined by comparison with

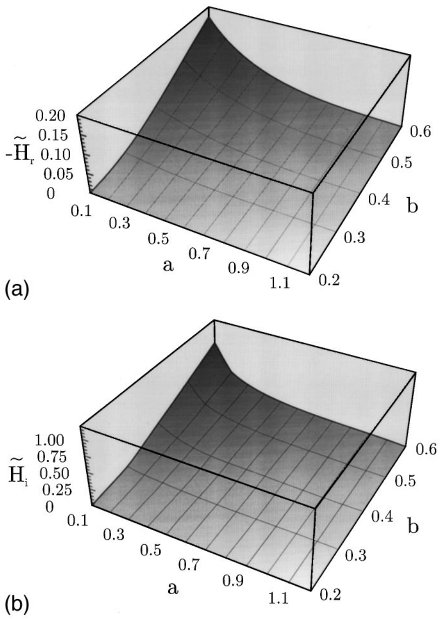

FIG. 2. (a) $\widetilde{H}_r$ as a function of a and b. (b) $\widetilde{H}_i$ as a function of a and b.

the exactly evaluated integrals. As a second approximation, we may try replacing $1+y^2$ in H_r and H_i by 1.

Applying both approximations to H_i allows us to evaluate the integral analytically, and leads to the approximate expression

$$H_i(T,a)\approx\widetilde{H}_i(T,a)=-\frac{\pi\Delta_0}{8}\left(\frac{dN}{dE}\right)_{\epsilon=0}\frac{3}{4}\left(\frac{1}{a}-\frac{a}{fb}\right)\times\theta((fb)^{1/2}-|a|). \tag{41}$$

We find that for $f\approx 0.8$, and with the factor of 3/4 in Eq. (41) to adjust the normalization, $\widetilde{H}_i(T,a)$ provides a good representation of $H_i(T,a)$. $\widetilde{H}_i$ is shown in Fig. 2(b) which may be compared with Fig. 1(b). Note that $\widetilde{H}_i$ vanishes at $|a|=(fb)^{1/2}$, and is prevented from going negative for $|a|>(fb)^{1/2}$ by the θ function. Thus $\widetilde{H}_i$ is continuous at $|a|=(fb)^{1/2}$, with a discontinuous derivative there.

One important feature of $\widetilde{H}_i$ deserves immediate comment, namely, that it is only nonzero for $|a|<(fb)^{1/2}$, or $|k_0|<v_F|\mathbf{k}|(fT/T_c)^{1/2}$. This is in contrast to the exact H_i, which is nonzero inside the temperature-independent region $|k_0|<v_F|\mathbf{k}|$ (i.e., $|a|<1$). The choice of the parameter f has to be made so as to effect a compromise between ensuring that $\widetilde{H}_i$ is indeed small for $|a|>(fb)^{1/2}$ (which requires a ''larger'' f), while at the same time providing a reasonable fit to the $|a|<(fb)^{1/2}$ region (which tends to favor ''smaller'' f's). In practice, $\widetilde{H}_i$ is indeed very small for $(fb)^{1/2}<|a|<1$, and the simple concept of a temperature-dependent effective cutoff for the imaginary part is a very useful one, as we shall see in the next section. The conclusion that the imaginary part is effectively nonzero only for $|k_0|<v_F|\mathbf{k}|(fT/T_c)^{1/2}$ is fully consistent with our qualitative discussion in Sec. II [cf. Eq. (17)]. We shall call the region $-(fT/T_c)^{1/2}v_F|\mathbf{k}|<k_0<(fT/T_c)^{1/2}v_F|\mathbf{k}|$ the ''effective Cerenkov region.'' This region is shown shaded in Fig. 3, on which are also drawn the lines $a=\pm 1$ bounding the full Cerenkov region in which the exact H_i of Eq. (40) is nonzero.

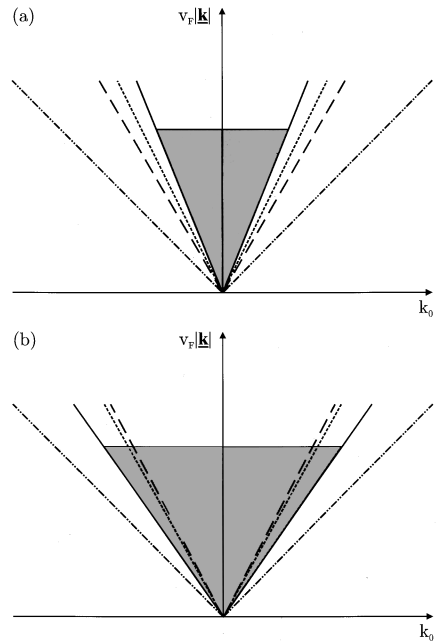

FIG. 3. The $k_0-v_F|\mathbf{k}|$ plane for (a) $T=0.2T_c$ and (b) $T=0.6T_c$, showing the lines $|k_0|=v_F|\mathbf{k}|$ (dash-dot), $|k_0|=v_p^*|\mathbf{k}|$ (dash), $|k_0|=f_1v_F|\mathbf{k}|$ (dot) and $|k_0|=(fT/T_c)^{1/2}v_F|\mathbf{k}|$ (solid). The effective Cerenkov region $-(fT/T_c)^{1/2}v_F|\mathbf{k}|\leqslant k_0\leqslant(fT/T_c)^{1/2}v_F|\mathbf{k}|$ is shown shaded.

Turning now to H_r, we found that while it is a good approximation to replace dN/dE' by $(dN/dE)_{\epsilon=0}$ outside the integral, replacing powers of $1+y^2$ by 1 is *not* satisfactory. However, this is of little importance, since even if both approximations are made the remaining integral still cannot be performed analytically. Making only the first approximation (for dN/dE'), we find that the integral can be quite well represented by a simple function of a, via

$$H_r(T,a)\approx\widetilde{H}_r(T,a)=\Delta_0\left(\frac{dN}{dE}\right)_{\epsilon=0}\frac{1.4}{1+14a^2}. \tag{42}$$

$\widetilde{H}_r$ is shown in Fig. 2(a), which may be compared with Fig. 1(a).

Our final explicit form for $(S_G^{-1})_L$ is therefore

$$(S_G^{-1})_L=J(T)\left(\frac{v_F|\mathbf{k}|}{\Delta_0}\right)^2+[\widetilde{H}_r(T,a)+i\widetilde{H}_i(T,a)](k_0/\Delta_0)^2. \tag{43}$$

We make the two observations about Eq. (43). First, in the static limit ($k_0=0$) the first term survives, and it provides a correction to the $(v_F|\mathbf{k}|/\Delta_0)^2$ term in Eq. (32). This reflects the fact that when $k_0=0$ the Landau terms do have a well-defined expansion in powers of $|\mathbf{k}|$. On the other hand, for any nonzero k_0, the $\widetilde{H}$ terms in Eq. (43) have to be included (if numerically significant—a point we shall discuss in the following section). Furthermore, the numerical value of the $\widetilde{H}$ terms depends on the way the origin in energy-momentum space is approached, via the ratio $a=k_0/(v_F|\mathbf{k}|)$; this is expected from the nonanalyticity, at the origin, of the F_L terms.

TABLE III. Variation of $(fT/T_c)^{1/2}$ and of v_p^*/v_F with temperature.

T/T_c	$\left(\frac{fT}{T_c}\right)^{1/2}$	$\frac{v_p^*}{v_F}$
0.2	0.400	0.577
0.3	0.490	0.573
0.4	0.566	0.560
0.5	0.632	0.542
0.6	0.693	0.525

Formula (43) may be phenomenologically useful, for example, for making a simple estimate of the likely importance of the Landau terms.

IV. SINGLE, DAMPED, MODE APPROXIMATION TO THE FULL EFFECTIVE GOLDSTONE PROPAGATOR

Assembling the results of the previous section, we now write the full Goldstone propagator, in our approximate form, as

$$S_G^{-1}=\left(\frac{|\mathbf{k}|v_F}{\Delta_0}\right)^2\{-[2D(T)-B(T)-J(T)] +[A(T)-2C(T)]a^2+[\widetilde{H}_r(T,a)+i\widetilde{H}_i(T,a)]a^2\}. \tag{44}$$

As it stands, Eq. (44) is still a rather complicated function of k_0 and $|\mathbf{k}|$; in particular, if we were to seek some kind of effective Lagrangian in coordinate space, by Fourier transforming, the result would be so unwieldy as to be impracticable. The difficulty lies, of course, in the Landau terms, even in their simplified form $\widetilde{H}$. If $\widetilde{H}$ were absent, S_G would have a simple pole structure, and the Fourier transform of S_G^{-1} would be the wave operator, leading to the usual local effective Lagrangian for the Goldstone mode. Since the major effect of $\widetilde{H}$ is to introduce damping (via $\widetilde{H}_i$) it is then natural to ask whether the effect of $\widetilde{H}$ can be approximately included by representing S_G in terms of a *complex* pole (or poles). Then a Fourier transform should be feasible. We therefore study S_G^{-1} as a function of complex a, which we shall take to mean complex k_0, keeping $|\mathbf{k}|$ real. We shall show that such an approximation is indeed a very good one.

We begin by observing that, if the $\widetilde{H}$ terms in Eq. (44) are neglected, S_G^{-1} has simple zeros at the real values $a=\pm v_p^*/v_F$, where

$$v_p^*=\left(\frac{2D(T)-B(T)-J(T)}{A(T)-2C(T)}\right)^{1/2}v_F, \tag{45}$$

which is just the original Goldstone mode speed Eq. (33), modified by the inclusion of $J(T)$. The quantity v_p^* is tabulated in Table III. The fact that v_p^* is reduced (with respect to v_p) by the inclusion of $J(T)$ is significant, because we must now ask how v_p^*/v_F compares with the quantity $(fT/T_c)^{1/2}$, which sets the boundary of the effective Cerenkov region (in the real variable a) in which $\widetilde{H}_i\neq 0$. If v_p^*/v_F were always greater than $(fT/T_c)^{1/2}$ we would effectively have an undamped mode—but while this is obviously true for small enough T, it is not so in general. This is illustrated in Fig. 3; see also Table III. Indeed, we find that for $T\geqslant 0.4T_c$, v_p^*/v_F is less than $(fT/T_c)^{1/2}$, so that propagation occurs inside the effective Cerenkov region, and damping (via $\widetilde{H}_i$) must be included for consistency. In addition, there is the effect of $\widetilde{H}_r$ to consider.

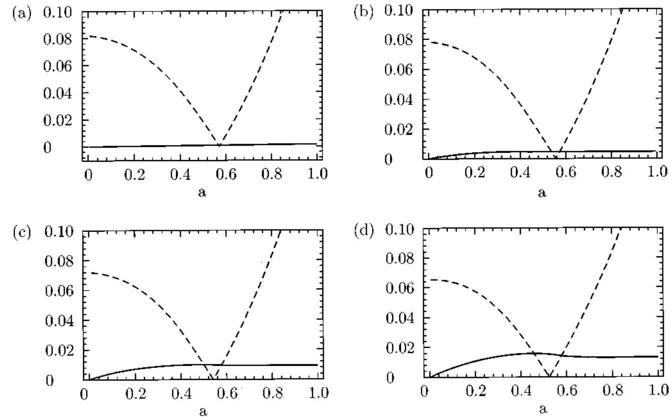

FIG. 4. $|R|$ [dash, see Eq. (46)] and $|S|$ [solid, see Eq. (47)] versus a for (a) $T=0.3T_c$, (b) $T=0.4T_c$, (c) $T=0.5T_c$, and (d) $T=0.6T_c$.

Of course, it might be that the magnitude of the $\widetilde{H}$ terms in Eq. (44) is actually very small. In Fig. 4 we compare $|R|$ with $|S|$ where [cf. Eq. (44)]

$$R=-(2D-B-J)+(A-2C)a^2 \tag{46}$$

and

$$S=(\widetilde{H}_r+i\widetilde{H}_i)a^2. \tag{47}$$

We see that the singular Landau term $|S|$ is non-negligible only for $T\geqslant 0.4T_c$. Below this temperature we do have effectively undamped propagation, while above it we have damping, which increases with T.

The simplest form of damping corresponds to a zero of S_G^{-1} at a complex value of a. Consider how this may arise in Eq. (44). For $a=f_1+i\epsilon$ with $f_1>0$, we see from Eq. (41) that $\widetilde{H}_i$ is positive. Since (from Table I) the quantity $A-2C$ is positive, we might expect the imaginary part of Eq. (44) to vanish for some $a=f_1-if_2$, where $f_2\ll f_1$. The real part of Eq. (44) will continue to vanish if $f_1\approx v_p^*/v_F$.

To search properly for this complex zero of S_G^{-1} we need to extend to complex values of a our approximations for H_r and H_i, which were valid for a approaching the real axis from above, via $a=a^+=k_0^+/v_F|\mathbf{k}|$, with $k_0^+=k_0+i\epsilon$ and k_0 real. We denote complex a by $\hat{a}=a_r-ia_i$, and examine first the case $a_r>0,a_i>0$. To reach such values, we must analytically continue the Landau terms F_L [Eq. (36)] from a^+ to $\hat{a}$. Since F_L has a cut in $\hat{a}$ for $-(fT/T_c)^{1/2}<\hat{a}<(fT/T_c)^{1/2}$, we must perform the continuation carefully for a_r in the range $0<a_r<(fT/T_c)^{1/2}$.

The correct continuation can be obtained by returning to the expression for $H(T,a)$ before the angular integral leading to Eqs. (39) and (40) is performed, namely

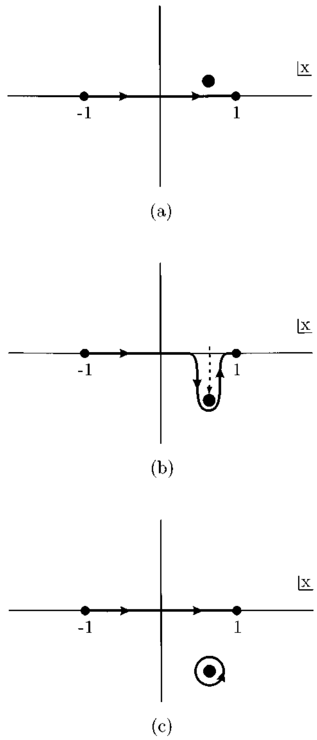

FIG. 5. The complex x plane, with the contour of integration in Eq. (49) with $0<a_r<y/[(1+y^2)^{1/2}]$ and (a) $a_i=\epsilon>0$, (b) $a_i<0$, with contour distortion (c) $a_i<0$, after rearrangement of contour.

$$H(T,a^+)\frac{k_0^2}{\Delta_0^2}=\left\{-\frac{\Delta_0}{4}\int_{-\infty}^{\infty}dy\,\frac{1}{(1+y^2)y^2}\,\frac{dN}{dE'}\right.$$
$$\left.\times\left[1+\frac{\sqrt{1+y^2}}{2y}a^+\int_{-1}^{1}\frac{dx}{x-(\sqrt{1+y^2}a^+)/y}\right]\right\}$$
$$\times\frac{k_0^{+2}}{\Delta_0^2}. \tag{48}$$

All dependences on k_0 or a on the right-hand side of Eq. (48) are trivial to continue, except for the singular term $I(a^+)$ where

$$I(a^+)=\int_{-1}^{1}\frac{dx}{x-(\sqrt{1+y^2}a^+)/y}. \tag{49}$$

Figure 5(a) shows the integration contour of Eq. (49) in the x plane, with the position of the pole indicated for the case $a_r<y/\sqrt{1+y^2}$ and $y>0$. We may separate the real and imaginary parts of I via

$$I(a^+)=I_r(a_r)+i\pi(y/|y|)\,\theta\!\left(\frac{|y|}{\sqrt{1+y^2}}-|a_r|\right), \tag{50}$$

from which it is clear that H_r and H_i of Eqs. (39) and (40) are obtained by inserting Eq. (50) into Eq. (48). Now consider continuing k_0 (or a) down into the lower half plane $a_r+i\epsilon\to a_r-ia_i$. For $a_r<y/\sqrt{1+y^2}$ we must deform the x contour smoothly away from the advancing pole, as shown in Fig. 5(b). We may replace Fig. 5(b) by Fig. 5(c), in which we have our original integral along the real x axis (but with a negative imaginary part for $\hat{a}$), together with a contribution of $2\pi i$ from the pole.

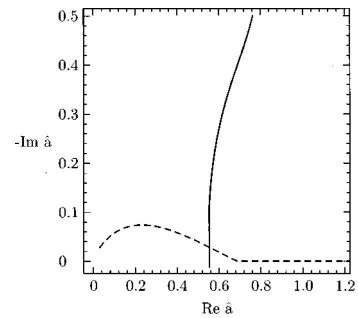

FIG. 6. The loci of Re$F=0$ (solid) and Im$F=0$ (dotted) in the complex $\hat{a}$ plane at $T=0.6T_c$, where $F=S_G^{-1}(v_F|\mathbf{k}|/\Delta_0)^2$.

Now we have seen that damping is negligible for $T\leqslant 0.4T_c$, and is expected to be significant but still small in the region $0.4T_c\leqslant T\leqslant 0.6T_c$ where our approximations hold: so we expect $a_i\ll a_r$. In that case, it seems reasonable to expect our previous approximate expressions for the real part of the integral along the x axis in Fig. 5(a) to carry over to the similar integral in Fig. 5(c), but with a replaced by $\hat{a}$. The imaginary part of the integral in Figure 5(c) will, however, have the opposite sign from the $i\pi$ term in Eq. (50) (for small a_i). Finally, we must include the contribution from the residue at the pole in Fig. 5(c). These considerations lead to the following approximation to Eq. (48) for $a^+\to\hat{a}=a_r-ia_i$ (always for $a_r>0$):

$$\left(\frac{v^2|\mathbf{k}|^2}{\Delta_0^2}\right)\hat{a}^2\left\{\widetilde{H}_r(T,\hat{a})-i\left[\widetilde{h}_i(T,\hat{a})-2\frac{\hat{a}}{a_r}\widetilde{h}_i(T,a_r)\right]\right.$$
$$\left.\times\,\theta((fb)^{1/2}-a_r)\right\}, \tag{51}$$

where $\widetilde{h}_i$ is the same as $\widetilde{H}_i$ of Eq. (41), but without the θ function.

Expression (51) now replaces the $\widetilde{H}$ terms in Eq. (44), and we are free to set $a^2\to\hat{a}^2$ in the rest of Eq. (44) and explore the possibility of a zero at some value of $\hat{a}$. In Fig. 6 we show the loci of $\mathrm{Re}[S_G^{-1}/(v_F|\mathbf{k}|/\Delta_0)^2]=0$ and $\mathrm{Im}[S_G^{-1}/(v_F|\mathbf{k}|/\Delta_0)^2]=0$, for $T=0.6T_c$: where these loci cross we have a complex zero of S_G^{-1}. We find that the position of the zero, as a function of T, can be well fitted by

$$\hat{a}=f_1(T)-if_2(T), \tag{52}$$

where

$$f_1(T)=0.580-0.0007\exp(5T/T_c) \tag{53}$$

and

$$f_2(T)=\left(\frac{0.176T}{T_c}-0.071\right)\theta\!\left(\frac{T}{T_c}-0.4\right), \tag{54}$$

all for $0<T/T_c\leqslant 0.6$.

At this stage, therefore, we have identified $Z_+=[k_0-v_F|\mathbf{k}|f_1+iv_F|\mathbf{k}|f_2\theta(\check{C}-k_0)\theta(k_0)]$ as an approximate factor of S_G^{-1}, where $\check{C}=v_F|\mathbf{k}|(fT/T_c)^{1/2}$. In view of the quadratic behavior in k_0 and $|\mathbf{k}|$ of the dominant terms in S_G^{-1}, there must clearly be a second, ''conjugate,'' factor also. For $f_2\to 0$, this will have the form $(k_0+v_F|\mathbf{k}|f_1)$, and it therefore corresponds to $a_r<0$. If a^+ is continued from $a_r>0$ to $a_r<0$, keeping the $+i\epsilon$ unchanged, it is easy to see that S_G^{-1} will not develop a zero, since H_i (or $\tilde{H}_i$) is then negative [see Eqs. (40),(41)]. To find the zero with $a_r<0$, it is necessary first to continue in $\hat{a}$ around the branch point $\hat{a}=1$ from the upper to the lower side of the cut $-1<\hat{a}<1$. This changes $I(a^+)$ of Eq. (50) to $I(a^-)$. H_i is then positive once more, for $a_r<0$. Keeping $a_r<0$, one then continues in $\hat{a}$ up to the point $\hat{a}=-|a_r|+ia_i$ with $a_i>0$, in a similar fashion to the continuation described in Eqs. (48)–(50). In this way, we find the second approximate factor of S_G^{-1}, namely $Z_-=[k_0+v_F|\mathbf{k}|f_1-iv_F|\mathbf{k}|f_2\theta(\check{C}+k_0)\theta(-k_0)]$. The approximate dispersion relation $|k_0|=v_F|\mathbf{k}|f_1$ is shown in Fig. 3.

The upshot of these considerations is that the following pole form for S_G^{-1} is suggested:

$$S_G^{-1}\approx\left(\frac{A-2C}{\Delta_0^2}\right)Z_+Z_-, \tag{55}$$

which can be written as

$$S_G\approx\left(\frac{\Delta_0^2}{(A-2C)}\right)[k_0^2-v_F^2\mathbf{k}^2f_1^2+2iv_F^2\mathbf{k}^2f_1f_2\theta(\check{C}-|k_0|)]^{-1}, \tag{56}$$

neglecting f_2^2. The form of Eq. (56) shows that, as we would expect from the properties of $\tilde{H}_i$, the imaginary part is nonzero only inside the effective Cerenkov region. However, we must recall from the remark after Eq. (41) that $\tilde{H}_i$ vanishes at $|k_0|=\check{C}$, so that it has no discontinuity at this point. Nor, of course, does the true function H_i. Expression (56), on the other hand, does have such a discontinuity, and this suggests that Eq. (56) should be replaced by a function which is smooth at $|k_0|=\check{C}$. For small f_2 it is an excellent approximation (except at the actual point of discontinuity) to replace $|k_0|$ in the quantity $\theta(\check{C}-|k_0|)$ of Eq. (56) by $v_F|\mathbf{k}|f_1$, so that the θ function reduces to $\theta((fT/T_c)^{1/2}-f_1(T))\approx\theta(T/T_c-0.4)$, as in Eq. (54). This now removes the discontinuity at $|k_0|=\check{C}$, and provides a smooth function to be compared with Eq. (44). Incidentally, the θ function would present complications in the next section when we seek the approximate equation of motion for the Goldstone mode, by Fourier transforming S_G^{-1}. These complications are not insuperable (the imaginary part becomes nonlocal in space-time), but they are unnecessary, we believe, in the light of the above arguments.

Finally, therefore, defining $\hat{S}_G$ by

$$\hat{S}_G^{-1}=\left(\frac{\mathbf{k}^2v_F^2(A-2C)}{\Delta_0^2}\right)S_G^{-1}, \tag{57}$$

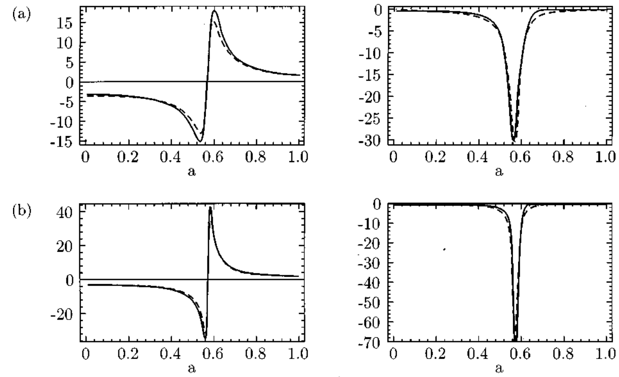

FIG. 7. (a) Left box: ReS_G (solid) and Re$S_{G,\text{pole}}$ (dotted) at $T=0.6T_c$. Right box: ImS_G (solid) and Im$S_{G,\text{pole}}$ (dotted) at $T=0.6T_c$. (b) Left box: ReS_G (solid) and Re$S_{G,\text{pole}}$ (dotted) at $T=0.5T_c$. Right box: ImS_G (solid) and Im$S_{G,\text{pole}}$ (dotted) at $T=0.5T_c$.

we arrive at the approximation

$$\hat{S}_G\approx\hat{S}_{G,\text{pole}}=(a^2-f_1^2+2if_1f_2)^{-1}. \tag{58}$$

Figure 7 shows the comparison between $\hat{S}_G$ [using Eqs. (57) and (44)] and $\hat{S}_{G,\text{pole}}$. We see that the simplified pole approximation represented by $\hat{S}_{G,\text{pole}}$ is indeed quite satisfactory, keeping track of both the shift in the resonance position and the broadening of the peak as T increases. Having established the usefulness of the simple form (58) we turn, finally, to the question of its Fourier transform, and the effective Lagrangian.

V. APPROXIMATE LOCAL EFFECTIVE THEORY FOR THE GOLDSTONE MODE

The preceding calculations amount to saying that the nonanalyticity of the Landau terms around the origin in k space, which taken at face value prevents an expansion in powers of energy momenta and hence also a local effective action, is in fact not numerically significant, at least for the range of parameters we have investigated. In essence, the full propagator S_G, which has indeed a complicated analytic structure in k_0 including a branch cut between $-v_F|\mathbf{k}|$ and $v_f|\mathbf{k}|$, can be very well approximated by a function which is analytic except for two complex poles in k_0. The inverse quantity S_G^{-1}, which appears in the effective action in momentum space, is a simple quadratic function of k_0 and $|\mathbf{k}|$. Hence its Fourier transform is trivial and we have, after all, recovered a simple local effective action.

Indeed, from Eqs. (29), (32), (57), and (58) the approximate effective action for the phase degree of freedom is proportional to

$$S_{\text{eff}}(\phi)=\int\frac{d^4k}{(2\pi)^4}\phi(k)[k_0^2-(f_1^2-2if_1f_2)v_F^2\mathbf{k}^2]\phi(-k), \tag{59}$$

which becomes

$$S_{\rm eff}(\phi)=\int d^4x\{[\dot\phi(\mathbf{x},t)^2-(f_1^2-2if_1f_2)v_F^2(\nabla\phi(\mathbf{x},t))^2\}\tag{60}$$

in coordinate space, which is the effective local theory. It is now simple to include electromagnetic interactions via the usual minimal coupling procedure.

The effective equation of motion for ϕ which follows from Eq. (60) is, of course, just

$$\frac{\partial^2\phi}{\partial t^2}=(f_1^2-2if_1f_2)v_F^2\nabla^2\phi.\tag{61}$$

As remarked in the Introduction, it has recently been shown[4] that the effective theory for the Lagrangian of Eq. (1) at $T=0$ can (in the long-wavelength approximation) be written as a nonlinear Schrödinger theory. It is then natural to ask how the imaginary term in Eqs. (60) or (61) can be incorporated into this picture. In terms of a Schrödinger wave function $\psi=\rho e^{i\phi}$ and a potential $V(\rho)=(\rho-\rho_0)^2/2\mathcal{N}(0)$, the equations of motion derived in Ref. 4 are

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0\tag{62}$$

and

$$\rho=\rho_0-\mathcal{N}(0)[\dot\phi+(\nabla\phi)^2/4m]\tag{63}$$

to leading order in derivatives, and where $\mathbf{j}=\rho\nabla\phi/2m$. Substituting Eq. (63) into Eq. (62), we find

$$\ddot\phi\approx[\rho_0/2m\mathcal{N}(0)]\nabla^2\phi=\frac{v_F^2}{3}\nabla^2\phi,\tag{64}$$

as expected for the undamped Goldstone mode. It follows that the only modification we need make to Eqs. (62) and (63), in order to reproduce Eq. (61), is to replace the parameter m in the expressions for ρ and $\mathbf{j}$ by

$$m_c=m/[3(f_1^2-2if_1f_2)],\tag{65}$$

while leaving $\mathcal{N}(0)$ unchanged. This means that the mass parameter in the equivalent Schrödinger theory is replaced by one which is T dependent, and which has a small positive T-dependent imaginary part.

In conclusion, we note that it would clearly be desirable not to have to make the ''small ϕ'' approximation, as we did above Eq. (32), but rather to perform a gauge transformation so as to remove ϕ from the complete gap function Δ from the start, as was done in Ref. 4, and then expand in derivatives of ϕ. This is presumably particularly important in regard to questions of vortex dynamics. Unfortunately, the algebraic complexity of the finite-temperature case (and the attendant absence of Galilean invariance) have so far prevented us from making much progress with such a program.

ACKNOWLEDGMENTS

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