Bethe-Peierls approximation for the triangular Ising antiferromagnet in a field

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We perform a Bethe-Peierls approximation to calculate the phase diagram of a nearest-neighbor triangular Ising antiferromagnet in a uniform field. The conjectured ground state is taken into account by the consideration of three interpenetrating sublattices. We find a set of self-consistent equations of state and an associated thermodynamic potential. Besides a disordered paramagnetic solution, at higher temperatures and fields, there are also low-temperature antiferromagnetic solutions. We show that the associated free energy is always smaller for the paramagnetic solution, which indicates that the antiferromagnetic ordering is just a local minimum of the thermodynamic potential. Similar results can still be obtained from a reaction-field approximation. [S0163-1829(97)01034-5]

I. INTRODUCTION

The Ising antiferromagnet on a triangular lattice is a highly frustrated spin system with a large residual entropy and no phase transition at finite temperatures.[1] Although there are no exact solutions in a field, there is an old conjecture about the existence of a threefold degenerate ordered state.[2] At sufficiently low temperatures (T), and in the presence of an external field ($H>0$), the system is expected to split into three equivalent sublattices (A, B, and C), with spin magnetizations $m_A=m_1\neq 0$, and $m_B=m_C=m_2\neq m_1$, with $m_2>m_1$. In this $\sqrt{3}\times\sqrt{3}$ structure of the degenerate ground state, the spins are aligned against the field on one of the sublattices and along the field on the other two sublattices (see Fig. 1).

The general picture of a threefold degenerate ordered state has been supported by Monte Carlo simulations,[3] and by a variety of real-space renormalization-group[4] and scaling arguments.[5] More recently, however, the mapping of the ground state onto a *solid-on-solid* model[6] has given support to suggestions of a much more complex field-temperature phase diagram.[7] In the past, the transition had been predicted to belong to the universality classes of the XY model, for smaller fields, and of the three-state Potts model, for sufficiently strong fields. In addition, in the upper critical field the ground state can be mapped onto the hard-hexagons problem.[8]

It is not surprising that the conventional mean-field approach leads to incorrect results for this highly frustrated antiferromagnet. If we consider a mean-field Ising Hamiltonian on a triangular lattice, in which the spins of a sublattice interact antiferromagnetically with all the spins of the two remaining sublattices, there is a (spurious) continuous phase transition even in the absence of an external field. Direct applications of the Bethe-Peierls[9] or the Kikuchi[10] self-consistent approximations, which are usually suitable for spin systems on lattices of low dimensionality, lead to equally wrong results in zero field. Actually, the Bethe approximations reported in the literature[9] always refer to just two sublattices and thus do not distinguish between a (loose-packed) simple cubic and a (close-packed) planar triangular lattice, which have the same coordination. In a very recent publication, however, Kabakçioğlu and co-workers[11] used a proposed Monte Carlo mean-field method[12] to reanalyze the field-temperature phase diagram of the triangular Ising antiferromagnet. It is interesting to point out some conclusions of this investigation: (i) there is no phase transition in zero field; (ii) the resulting phase diagram is in fair agreement with the reported results of the literature; (iii) even if the calculations begin with different values of the spin magneti-

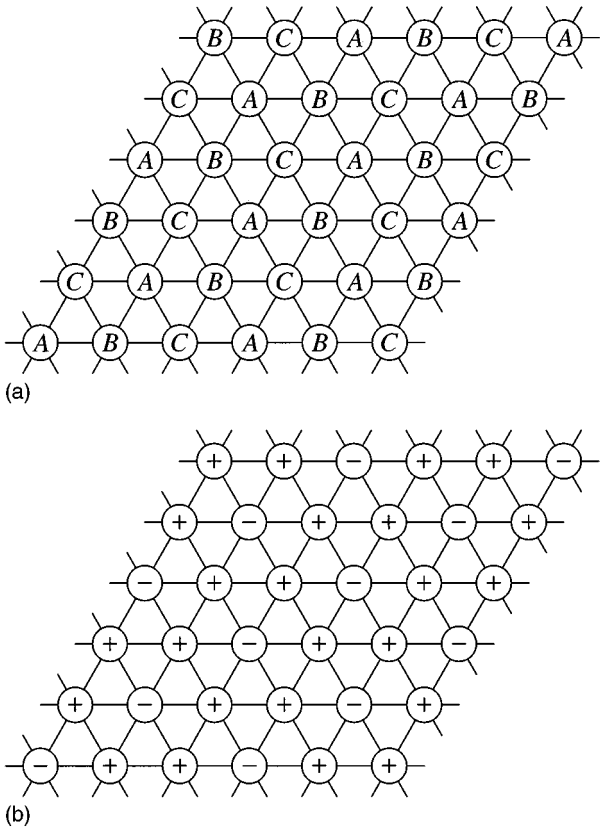

FIG. 1. (a) The three interpenetrating sublattices (A, B, and C) of the triangular lattice; (b) Spin configuration in one of the threefold degenerate ground states represented by the $\sqrt{3}\times\sqrt{3}$ structure (giving rise to an antiferromagnetic ordering with spin magnetization 1/3).

zation for a large number of sublattices, the self-consistent solutions in the ordered region are defined on three distinct sublattices only; (iv) the free energy displays a (nonphysical) discontinuity between the paramagnetic and the ordered phases (which is claimed to indicate a first-order transition). It is also interesting to remark that, at least for the simple Ising ferromagnet on a square lattice, the method described in Ref. 12 gives similar results as a standard Bethe-Peierls approximation.[13]

The general interest on the behavior of frustrated spin models, as well as the conclusions of the calculations of Kabakçioglu and co-workers,[11] provided the motivation for revisiting the problem of the field-temperature phase diagram of the nearest-neighbor Ising antiferromagnet on the triangular lattice. If the transitions display an XY character as pointed out by some authors,[7] they will hardly be detected by a (self-consistent) Bethe-Peierls calculation. If they are first-order transitions, as claimed by Kabakçioglu and co-workers,[11] there should be coexisting (Bethe-Peierls) solutions with the same free energy. We then decided to perform a more elaborate Bethe-Peierls approximation, taking into account the existence of three distinct sublattices as indicated by the simulations, to obtain a set of equations of state which should come from the minimization of an associated functional form of the free energy. Consistent treatments of this sort, which have been performed for a two-sublattices Ising antiferromagnet,[14] but which have not been performed for the three-sublattices model investigated by Kabakçioglu and co-workers, are essential to analyze the stability of the ordered phases and the nature of the phase transitions.

The layout of this paper is as follows. In Sec. II we formulate the problem and present the solution in the Bethe-Peierls approximation with three distinct sublattices. As we mentioned above, the division into three sublattices is crucial to make contact with the conjectured antiferromagnetic ordering at low temperatures. We show, however, that the free energy associated with the ordered state is always larger than the corresponding free energy associated with the paramagnetic solution. This result is particularly striking, as it indicates that this Bethe approximation gives rise to an ordered state that is a mere local minimum of the free energy, without any possibility of a true phase transition (which has indeed been mentioned by Campbell and Schick[15] in the context of a triangular lattice gas model). In Sec. III, we present the so-called reaction-field approximation, which can be obtained by expanding the equations of state of the Bethe approximation up to terms of order J^2 (and which gives rise to the well-known Thouless-Anderson-Palmer equations[16] in the case of an Ising spin-glass model). In this approximation, the associated free-energy functional can be easily obtained by a straightforward integration of the equations of state, providing a further confirmation of our findings. Some conclusions are presented in Sec. IV. In the Appendix we give some details of the calculations for the ground state in the Bethe-Peierls approximation.

II. BETHE APPROXIMATION

The Bethe-Peierls approximation for the Ising antiferromagnet consists in obtaining the spin magnetizations of a cluster with a central spin, $S_0=\pm 1$, surrounded by q nearest neighbors, $S_i=\pm 1$, $i=1,2,\ldots,q-1,q$, with energy given by the Hamiltonian

$$\mathcal{H}=-JS_0\sum_{i=1}^{q} S_i - HS_0 - \sum_{i=1}^{q} \eta_i S_i , \tag{1}$$

where $J<0$ is an (antiferromagnetic) exchange parameter, and $H>0$ is the external magnetic field. The spins at the bordering sites are in the effective fields $\{\eta_i\}$, that should be determined by the consistence equations of the approximation.

As shown in Fig. 1, the triangular lattice can be divided into three equivalent sublattices, identified by the letters A, B, and C. For a lattice with coordination q, and taking into account three sublattices, the canonical partition function associated with a cluster centered on sublattice A is given by

$$\begin{aligned} Z_A=\sum_{S_0,\cdots,S_q} \exp(-\beta\mathcal{H}) &= \exp(\beta H_A)[4\cosh(\beta\eta_{BA} \\ &+\beta J)\cosh(\beta\eta_{CA}+\beta J)]^{q/2}+\exp(-\beta H_A) \\ &\times[4\cosh(\beta\eta_{BA}-\beta J)\cosh(\beta\eta_{CA}-\beta J)]^{q/2}, \end{aligned} \tag{2}$$

where $\beta=1/(k_BT)$, H_A is the external magnetic field in sublattice A (to allow the integration of the equations of state, we introduce distinct external fields in each sublattice), and η_{BA} (η_{CA}) is the effective field in sublattice B (C). The magnetizations per site for each one of the three sublattices are given by

$$\begin{aligned} m_A=m_A(\eta_{BA},\eta_{CA},H_A)=\langle S_0\rangle &= \frac{1}{\beta}\left(\frac{\partial}{\partial H_A}\ln Z_A\right)_{\eta_{BA},\eta_{CA}} \\ &= \tanh\left\{\beta H_A+\frac{q}{2}\tanh^{-1}[\tanh(\beta\eta_{BA})\tanh(\beta J)]\right. \\ &\left.+\frac{q}{2}\tanh^{-1}[\tanh(\beta\eta_{CA})\tanh(\beta J)]\right\}, \end{aligned} \tag{3}$$

$$\begin{aligned} m_B=m_B(\eta_{BA},\eta_{CA},H_A)=\langle S_j\rangle_{j\in B} &= \frac{2}{\beta q}\left(\frac{\partial}{\partial\eta_{BA}}\ln Z_A\right)_{H_A,\eta_{CA}} \\ &= \frac{1}{2}(1+m_A)\tanh(\beta\eta_{BA}+\beta J) \\ &+\frac{1}{2}(1-m_A)\tanh(\beta\eta_{BA}-\beta J), \end{aligned} \tag{4}$$

and

$$\begin{aligned} m_C=m_C(\eta_{BA},\eta_{CA},H_A)=\langle S_k\rangle_{k\in C} &= \frac{2}{\beta q}\left(\frac{\partial}{\partial\eta_{CA}}\ln Z_A\right)_{\eta_{BA},H_A} \\ &= \frac{1}{2}(1+m_A)\tanh(\beta\eta_{CA}+\beta J) \\ &+\frac{1}{2}(1-m_A)\tanh(\beta\eta_{CA}-\beta J). \end{aligned} \tag{5}$$

We now write the remaining set of equations for the clusters centered on sublattices B and C. It is then convenient to use the notation

$$m_\mu = \tanh\left[\beta H_\mu + \frac{q}{2}\tanh^{-1}(v x_{\nu\mu}) + \frac{q}{2}\tanh^{-1}(v x_{\lambda\mu})\right], \tag{6}$$

where $v=\tanh(\beta J)=-\tanh(\beta|J|)$, the labels μ,ν,λ are permutations of the symbols A,B,C, and the new variables, $x_{\mu\nu}=\tanh(\beta\eta_{\mu\nu})$, are obtained from the consistence conditions,

$$\begin{aligned} m_\mu(\eta_{BA},\eta_{CA},H_A) &= m_\mu(\eta_{AB},\eta_{CB},H_B) \\ &= m_\mu(\eta_{AC},\eta_{BC},H_C). \end{aligned} \tag{7}$$

To be more explicit, we have the set of equations of state,

$$m_A = \tanh\left[\beta H_A + \frac{q}{2}\tanh^{-1}(v x_{BA}) + \frac{q}{2}\tanh^{-1}(v x_{CA})\right], \tag{8}$$

$$m_B = \tanh\left[\beta H_B + \frac{q}{2}\tanh^{-1}(v x_{AB}) + \frac{q}{2}\tanh^{-1}(v x_{CB})\right], \tag{9}$$

and

$$m_C = \tanh\left[\beta H_C + \frac{q}{2}\tanh^{-1}(v x_{AC}) + \frac{q}{2}\tanh^{-1}(v x_{BC})\right], \tag{10}$$

where

$$x_{AB} = \tanh\left[\beta H_A + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{BA}) + \frac{q}{2}\tanh^{-1}(v x_{CA})\right], \tag{11}$$

$$x_{AC} = \tanh\left[\beta H_A + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{CA}) + \frac{q}{2}\tanh^{-1}(v x_{BA})\right], \tag{12}$$

$$x_{BA} = \tanh\left[\beta H_B + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{AB}) + \frac{q}{2}\tanh^{-1}(v x_{CB})\right], \tag{13}$$

$$x_{BC} = \tanh\left[\beta H_B + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{CB}) + \frac{q}{2}\tanh^{-1}(v x_{AB})\right], \tag{14}$$

$$x_{CA} = \tanh\left[\beta H_C + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{AC}) + \frac{q}{2}\tanh^{-1}(v x_{BC})\right], \tag{15}$$

and

$$x_{CB} = \tanh\left[\beta H_C + \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_{BC}) + \frac{q}{2}\tanh^{-1}(v x_{AC})\right]. \tag{16}$$

Given the temperature T and the external fields, H_A, H_B, and H_C, these equations may have more than one solution. We then have to explore all the possible solutions of this nonlinear system. However, as we are looking for an antiferromagnetic ordering in which

$$m_1 = m_A, \quad m_2 = m_B = m_C \neq m_1, \tag{17}$$

the analysis can be considerably simplified. In this case, if we assume $H_1=H_A$ and $H_2=H_B=H_C$, we can write

$$x_1 \equiv x_{AB} = x_{AC}, \quad x_2 \equiv x_{BC} = x_{CB}, \quad x_3 \equiv x_{BA} = x_{CA}, \tag{18}$$

where the last variable, x_3, satisfies the relation

$$\begin{aligned} x_3(x_1,x_2) &= \tanh[\tanh^{-1}x_2 + \tanh^{-1}(v x_2) - \tanh^{-1}(v x_1)] \\ &= \frac{(1+v)x_2 - v(1+v x_2^2)x_1}{1+v x_2^2 - v(1+v)x_1 x_2}, \end{aligned} \tag{19}$$

which is obtained by eliminating βH_B from Eqs. (13) and (14). We can thus rewrite the equations of state in terms of x_1 and x_2,

$$\begin{aligned} m_1(x_1,x_2) &= \tanh\{\tanh^{-1}x_1 + \tanh^{-1}[v x_3(x_1,x_2)]\} \\ &= \frac{(1+v)[v(1-x_1^2)x_2 + (1-v)(1+v x_2^2)x_1]}{(1-v^2x_1^2)(1+v x_2^2)}, \end{aligned} \tag{20}$$

$$m_2(x_1,x_2) = \tanh[\tanh^{-1}x_2 + \tanh^{-1}(v x_2)] = \frac{(1+v)x_2}{1+v x_2^2}, \tag{21}$$

$$\beta H_1(x_1,x_2) = \tanh^{-1}x_1 - (q-1)\tanh^{-1}[v x_3(x_1,x_2)], \tag{22}$$

$$\beta H_2(x_1,x_2) = \tanh^{-1}x_2 - \frac{q}{2}\tanh^{-1}(v x_1) - \left(\frac{q}{2}-1\right)\tanh^{-1}(v x_2). \tag{23}$$

Along the lines of the calculations for the Ising antiferromagnet on a bipartite lattice,[14] we are now prepared to obtain an expression for a thermodynamic potential. It is not difficult to integrate the parametric forms of the equations of state to write the free energy,

$$\begin{aligned} \beta g = &-\ln 2 + \frac{q}{4}\ln(1-v^2) + \frac{1}{6}\ln[(1-x_1^2)(1-x_2^2)^2] \\ &+ \frac{1}{3}\left(\frac{q}{2}-1\right)[\ln(1-v^2x_1^2) - \ln(1-v^2x_2^2) \end{aligned}$$

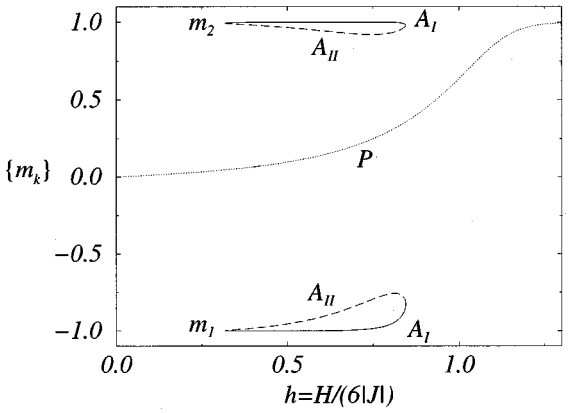

FIG. 2. Sublattice magnetizations per spin in the Bethe-Peierls approximation at dimensionless temperature $t=0.1$. The letters identify the various solutions: P is the paramagnetic solution; A_I and A_{II} are two distinct antiferromagnetic solutions.

$$+3\ln(1+vx_2^2)]-\frac{1}{6}(q-1)\ln[(1+v^2x_1)(1+vx_2^2)$$

$$-v(1+v)(1+x_1)x_2]-\frac{1}{6}(q-1)\ln[(1-v^2x_1)$$

$$\times(1+vx_2^2)+v(1+v)(1-x_1)x_2]. \tag{24}$$

For the triangular lattice ($q=6$), there is a region of the phase diagram, in terms of the temperature, $t=k_BT/(6|J|)$, and the field, $h=H/(6|J|)$, for $H_1=H_2=H$, in which there are two antiferromagnetic solutions, with $x_1\neq x_2$. In Figs. 2 and 3, we show the solutions of the equations of state and the associated free energies for a typical value of the temperature. As we can see from these figures, the two solutions corresponding to an antiferromagnetic ordering are always associated with a larger free energy as compared with the free energy of the paramagnetic solution ($x_1=x_2=x$ and $m_1=m_2=m$).

To calculate the boundary of existence of these ordered solutions, we expand the functions defined by Eqs. (22) and (23) around the critical values $\{x_k^c\}$. Therefore, in the $t-h$ phase diagram this boundary comes from the conditions for the existence of nontrivial solutions of the linear system

$$\begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{pmatrix}_{\{x_k\}=\{x_k^c\}} \begin{pmatrix} x_1-x_1^c \\ x_2-x_2^c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{25}$$

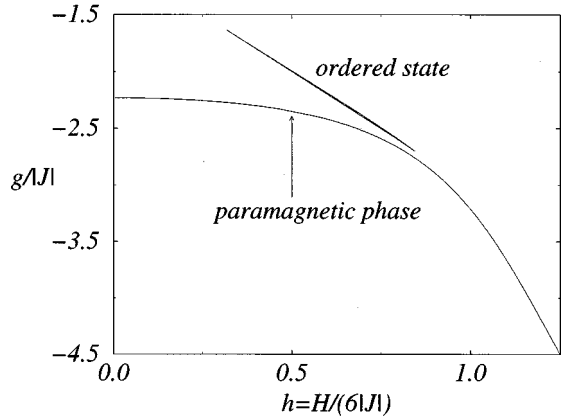

FIG. 3. Free energies associated with the solutions of the Bethe-Peierls approximation at the temperature $t=0.1$. The value of the free energy is always larger for the ordered solutions (as compared with the paramagnetic state).

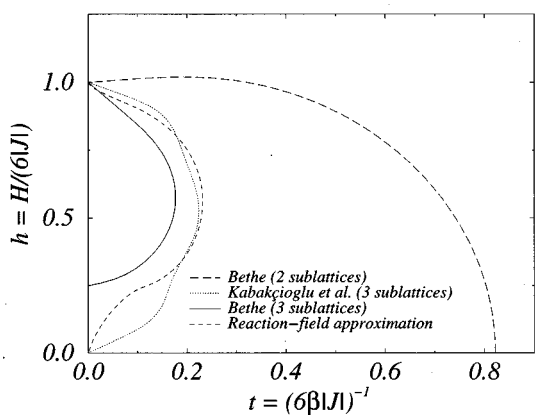

FIG. 4. Regions of existence of the ordered solutions of the equations of state obtained from various approximations. The full line corresponds to the Bethe approximation with three sublattices. We also show the results from the reaction-field approximation (dashed line), the work of Kabakçioglu *et al.* (Ref. 11) (dotted line), and from a Bethe approximation with two sublattices (long-dashed line).

In Fig. 4, we show the region of existence of the ordered solutions. It is interesting to mention the existence of a finite lower critical field at zero temperature, $H_c(T=0)=3|J|/2$, which is not predicted by the early Monte Carlo simulations. As shown by the asymptotic expressions given in the Appendix, this result of the Bethe-Peierls approximation can be confirmed by a careful expansion of the equations of state and of the free energy in the $T\to 0$ limit.

III. REACTION-FIELD APPROXIMATION

The reaction-field approximation corresponds to an expansion of the Bethe-Peierls equations of state about $t=k_BT/(6|J|)\to\infty$, up to t^{-2} terms. From the previous expressions for the spin magnetizations of the three types of clusters, it is not difficult to find the reaction-field forms,

$$m_A=\tanh[\beta H_A+3\beta J(m_B+m_C)-6\beta^2J^2m_A$$
$$+3\beta^2J^2m_A(m_B^2+m_C^2)], \tag{26}$$

$$m_B=\tanh[\beta H_B+3\beta J(m_A+m_C)-6\beta^2J^2m_B$$
$$+3\beta^2J^2m_B(m_A^2+m_C^2)], \tag{27}$$

and

$$m_C=\tanh[\beta H_C+3\beta J(m_A+m_B)-6\beta^2J^2m_C+3\beta^2J^2m_C(m_A^2+m_B^2)]. \quad (28)$$

Now it is straightforward to integrate these equations of state to write the thermodynamic potential

$$g=f-\frac{1}{3}(m_AH_A+m_BH_B+m_CH_C), \quad (29)$$

where

$$3f=\frac{1}{\beta}\int \tanh^{-1}m_A dm_A+\frac{1}{\beta}\int \tanh^{-1}m_B dm_B+\frac{1}{\beta}\int \tanh^{-1}m_C dm_C-3J(m_Am_B+m_Am_C+m_Bm_C)+3\beta J^2(m_A^2+m_B^2+m_C^2)-\frac{3}{2}\beta J^2(m_A^2m_B^2+m_A^2m_C^2+m_B^2m_C^2)+3f_0(T), \quad (30)$$

and f_0 comes from the high-temperature limit of the free energy. Given the temperature and the fields (H_A, H_B, and H_C), we can perform a detailed analysis of the minima of the functional g in terms of the spin magnetizations, m_A, m_B, and m_C. In the case of interest, $m_A=m_1$, $m_B=m_C=m_2$, with $H_A=H_B=H_C=H$, we have the equations of state

$$m_1=\tanh[\beta H+6\beta Jm_2-6\beta^2J^2m_1(1-m_2^2)], \quad (31)$$

and

$$m_2=\tanh[\beta H+3\beta J(m_1+m_2)-3\beta^2J^2m_2(2-m_1^2-m_2^2)], \quad (32)$$

which are associated with the thermodynamic potential

$$\frac{1}{|J|}g=2m_1m_2+m_2^2-2h(m_1+2m_2)+t\,\ln(1-m_1^2)+2t\,\ln(1-m_2^2)+2tm_1\tanh^{-1}m_1+4tm_2\tanh^{-1}m_2+\frac{1}{6t}m_1^2(1-m_2^2)+\frac{1}{12t}m_2^2(4-m_2^2)-6t\,\ln 2-\frac{1}{4t}, \quad (33)$$

where $t=k_BT/(6|J|)$, and $h=H/(6|J|)$, for $H_1=H_2=H$. There is again a region of ordered antiferromagnetic solutions in the field-temperature phase diagram (see Figs. 4, 5, and 6). Again, these ordered solutions are associated with larger values of the free energy as compared with the free energy of the disordered paramagnetic solution.

IV. CONCLUSIONS

We have performed a Bethe-Peierls self-consistent calculation to investigate the field-temperature phase diagram of a nearest-neighbor Ising antiferromagnet on a triangular lattice. To represent the presumed ground state of the system, we have considered the existence of three distinct sublattices, with different sublattice magnetizations. We obtain the equations of state together with an associated thermodynamic potential. Besides the disordered paramagnetic solution, we show the existence of ordered antiferromagnetic solutions at sufficiently low temperatures (and for a certain range of field values). These ordered solutions, however, are local minima of the free energy, well above the minimum of the free energy associated with the paramagnetic solution. The same conclusions can be reached from a reaction-field approximation associated with an expansion of the Bethe-Peierls solutions up to terms of order J^2 (and which is amenable to much easier manipulations).

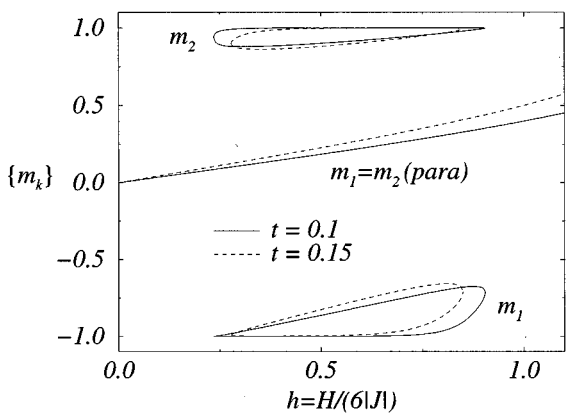

FIG. 5. Sublattice magnetizations per spin in the reaction-field approximation at $t=0.1$ and $t=0.15$.

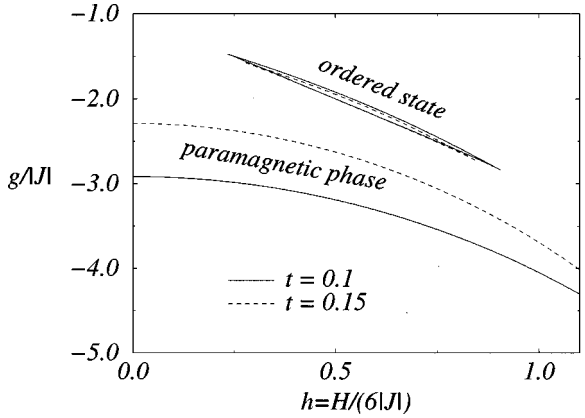

FIG. 6. Free energies associated with the solutions of the reaction-field approximation at $t=0.1$ and $t=0.15$. The value of the free energy is always larger for the ordered solutions (as compared with the paramagnetic state).

In Fig. 4 we present the results for the Bethe-Peierls and the reaction-field approximations. We also include the results from an application of the scheme of Kabakçioglu and co-workers,[11] as well as the phase diagram obtained from a Bethe-Peierls approximation for an antiferromagnetic Ising model on a bipartite lattice of the same coordination, $q=6$ (which gives a finite critical temperature in zero field). In the Bethe approximation with three sublattices there is a lower critical field, which vanishes in the reaction-field approximation.

The antiferromagnetic regions in the phase diagram of Kabakçioglu and co-workers are in qualitative correspondence with the regions of local minima of the Bethe-Peierls and reaction-field approximations. As we do have expressions for a thermodynamic potential, which comes from the integration of the equations of state, we have been able to avoid nonphysical assertions (as a jump of the free energy through the transition) and to show that, within the self-consistent approximations, there is no room for a thermodynamically stable ordered region in the phase diagram. The present calculations might raise some questions. The results may be an artifact of the cluster approximations, which are usually inadequate to deal with a Kosterlitz-Thouless transition, as predicted for this antiferromagnet by some investigators.[7] In any case, the existence of local minima of the free energy should pose many problems for a reliable numerical simulation of this system. It should be interesting to look again at the simulations, in particular to check the existence of a lower critical field.

ACKNOWLEDGMENTS

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APPENDIX: ASYMPTOTIC EXPRESSIONS OF THE BETHE-PEIERLS SOLUTIONS AT ZERO TEMPERATURE

At the $T\rightarrow 0$ limit, we can perform asymptotic expansions of the equations of state and the free energy in the Bethe-Peierls approximation to show the existence of a finite lower critical field, $H_c(T=0)=3|J|/2$, which has not been predicted by the early Monte Carlo simulations.

The paramagnetic solution is given by

$$\text{(i)}\quad m=0,\ \text{and}\ \ g=3J,$$

for $0<H/(6|J|)<1$, and

$$\text{(ii)}\quad m\rightarrow 1-2\ \exp(-2\beta H-12\beta J),\ \text{and}\ \ g=-H-3J,$$

for $H/(6|J|)>1$.

The antiferromagnetic solution I, associated with the free energy

$$g=-\frac{1}{3}H+J,$$

is given by

$$\text{(i)}\quad m_1\rightarrow -1+2\ \exp(-10\beta H-12\beta J),$$

$$m_2\rightarrow 1-2\ \exp(-2\beta H),$$

for $1/4<H/(6|J|)<1/3$,

$$\text{(ii)}\quad m_1\rightarrow -1+2\ \exp(2\beta H+12\beta J),$$

$$m_2\rightarrow 1-2\ \exp(-2\beta H),$$

for $1/3<H/(6|J|)<2/3$, and

$$\text{(iii)}\quad m_1\rightarrow -1+2\ \exp(2\beta H+12\beta J),$$

$$m_2\rightarrow 1-2\ \exp(4\beta H+24\beta J),$$

for $2/3<H/(6|J|)<1$.

The antiferromagnetic solution II is given by

$$m_1\rightarrow -1+2\ \exp\left(\frac{2}{3}\beta H+4\beta J\right),$$

$$m_2\rightarrow 1-2\ \exp\left(\frac{2}{3}\beta H+4\beta J\right),$$

for $1/4<H/(6|J|)<1$, with the same free energy as solution I, $g=-H/3+J$.

For $3/2<H/|J|<6$, we see that there are two antiferromagnetic solutions, associated with a limiting free energy, $g=-H/3+J$, that is always larger than the free energy of the paramagnetic state.

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